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EDITORIAL

I enjoy a variety of hand crafts and, being a mathematician, I ponder about math in each of them. I scrapbook and realize how much proportional thinking goes into photography and design work. I knit and connections to modular arithmetic are obvious each time I have to follow an intricate pattern. I sew and wonder how it is possible that my grandmother was such an amazing natural topologist. Recently, I also picked up embroidery. I am originally from Belarus, where national clothing and linens are known to be adorned with beautiful red and white embroidered designs. Various mathematical elements in standard patterns are clear: rotational and reflection symmetries, parity details, features occurring in multiples and so on. Different elements mean different things, such as beauty, land, sun, soul, spring.

But having finished my first piece of non-traditional embroidery (I figured I'll practice on something more playful before moving onto patterns of cultural significance), I discovered a deeper connection to the approach and the presentation of mathematics. Indulge me while I philosophize.

The front of the finished piece is a neatrly "drawn" pattern; like a finished mathematical solution or proof, it looks well thoughtout and nicely laid out. But the reverse side of the embroidery tells a different story: stitches overlap in a disorganized fashion, there are knots for each thread's beginning and end, there are straight lines connecting seemingly unrelated points on the pattern. The overall pattern is still recognizable, but I wouldn't boast about my skills if this was the final product. This side is the messy story of creation that we see so seldom in math. This is unsuccessful solution attempts, trial and error approaches, small examples, ideas that never worked out, proof write-ups that were deemed not elegant enough; this is draft work that noone but the craftsperson themself sees.

Every presented solution in ***Crux*** makes the work look effortless and immediate, like the person knew what they were doing from the very first to the last step with no additional investigations, miscalculations, wrong turns. But do not be fooled: there is the reverse side to each solution, where the author might have laboured for hours before arriving at the idea and producing a clean write-up. Noone's first draft is their final draft. Practice will make this process faster: expert problem solvers recognize familiar patterns, know shortcuts and are more proficient in writing. But no matter how expert, the reverse side is never as perfect as the front. Though it is necessary: the imperfections make the pattern possible.

Kseniya Garaschuk

MATHEMATTIC

No. 28

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **December 30, 2021**.

MA136. *Proposed by Ed Barbeau.*

Determine all sets consisting of an odd number $2m+1$ of consecutive positive integers, for some integer $m \geq 1$ such that the sum of the squares of the smallest $m+1$ integers is equal to the sum of the squares of the largest m integers.

MA137.

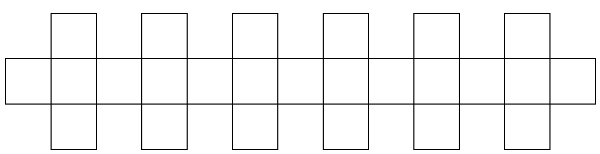
Triangle ABC has area 1. X, Y are points on the side AB and Z a point on the side AC such that $XY = 2AX$, XZ is parallel to YC and YZ is parallel to BC. Determine the area of triangle XYZ.

MA138. *Proposed by Aravind Mahadevan.*

Prove that $\csc 6^\circ + \csc 78^\circ - \csc 42^\circ - \csc 66^\circ = 8$.

MA139.

The shape below was created by pasting together 25 unit squares. When a similar shape is created with n squares, its perimeter is 100 units. Determine n.



MA140.

If $f(x) = 1 - x - x^3$, what are all real values of x which satisfy

$$1 - f(x) - (f(x))^3 > f(1-5x)?$$

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 decembre 2021**.

MA136. *Proposé par Ed Barbeau.*

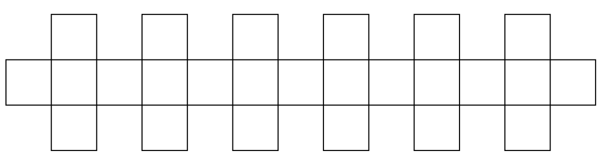
Déterminer tous les ensembles comprenant un nombre impair $2m + 1$ d'entiers positifs consécutifs, où m est un entier, $m \geq 1$, sachant qu'en plus la somme des carrés des $m + 1$ plus petits éléments de l'ensemble égale la somme des carrés des m plus grands éléments.

MA137. Le triangle ABC a une surface de 1. Les points X et Y se trouvent sur le côté AB, tandis que le point Z se trouve sur le côté C, de façon à ce que $XY = 2AX$, que XZ soit parallèle à YC et que YZ soit parallèle à BC. Déterminer la surface du triangle XYZ.

MA138. *Proposé par Aravind Mahadevan.*

Démontrer que $\csc 6^\circ + \csc 78^\circ - \csc 42^\circ - \csc 66^\circ = 8$.

MA139. La forme présentée ci-bas a été créée à l'aide de 25 carrés de côté unitaire. Lorsqu'une forme similaire est créée à l'aide de n carrés, son périmètre est de 100 unités. Déterminer n.



MA140. Si $f(x) = 1 - x - x^3$, déterminer toutes les valeurs réelles x telles que

$$1 - f(x) - (f(x))^3 > f(1 - 5x)?$$

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(3), p. 121–122.

MA111. Peter and Steve had a 50-metre race. When Peter crosses the finish line, Steve was 5 metres behind. A rematch is set up, with Peter handicapped by being 5 metres behind the starting line. Assuming that the boys run at the same speed as before, who wins the race (or is it a tie) and by how much?

Originally from 2000 ESSO-CMS Math Camp, Problem Set 4, problem 5.

We received 6 submissions, of which 5 were correct and complete. We present the solution by Dominique Mouchet, Lycée Evariste de Parny, St Paul, France.

Notons v_p et v_s les vitesses de Peter et Steve. Soit t la durée de la première course. Alors $v_p = 50/t$ et $v_s = 45/t$. D'où $v_p/v_s = 10/9$. Lors de la 2^{e} course, Steve a couru pendant une durée $t_s = 50/v_s$ et Peter une durée

$$t_p = \frac{55}{v_p} = \frac{55}{\frac{10}{9} \cdot v_s} = \frac{49,5}{v_s} < t_s.$$

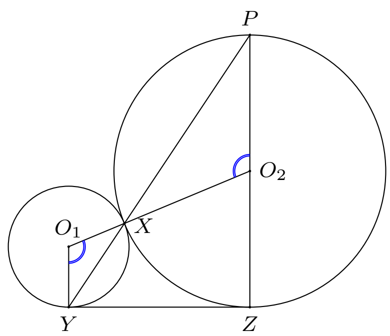
Donc Peter arrive encore le premier.

Pendant le temps t_p, Steve a parcouru $v_s \times t_p = 49,5$ m. Peter a donc 50 cm d'avance.

MA112. Circle C_1 has radius 13 and is tangent to line ℓ at Y. Circle C_2 has radius 23, is tangent to ℓ at Z and is externally tangent to C_1 at X. The line through X and Y intersects C_2 at P. Determine the length of PZ.

Inspired by 1983 Euclid contest, essay question 2b.

We received 5 submissions, all of which were correct. We present the solution by Miguel Amengual Covas, modified by the editor.



Let O_1 and O_2 be centres of C_1 and C_2, respectively. The line O_1O_2, joining the centres of the touching circles, goes through the point X. Observe that ΔO_1YX and ΔO_2PX are isosceles triangles. As $\angle O_2XP = \angle O_1XY$, we have that $\angle XO_1Y = \angle XO_2P$ and PO_2 is parallel to O_1Y.

As O_1Y is perpendicular to l, so is PO_2. As O_2Z is also perpendicular to l, it follows that the length of PZ is the diameter of C_2. Therefore, $PZ = 46$.

MA113. For any positive number t, $\lfloor t \rfloor$ denotes the integer part of t and $\{t\}$ denotes the "decimal" part of t. If $x + \{y\} = 7.32$ and $y + \lfloor x \rfloor = 8.74$, then determine $\{x\}$.

Originally from 2000 ESSO-CMS Math Camp, Problem Set 3, problem 3.

We received 8 correct solutions for this problem. We present the solution by the Missouri State University Problem Solving Group.

There are integer n and m such that $n \le x < n+1$ and $m \le y < m+1$. Then $\lfloor x \rfloor = n$ and $\{y\} = y - m$. The two given equations can be written as

$$\begin{aligned} x + y - m &= 7.32, \\ y + n &= 8.74. \end{aligned}$$

Subtracting, we get

$$x - n - m = -1.42, \quad \text{or} \quad x + 1.42 = n + m.$$

Since $n + m$ is an integer, we must have $\{x\} = 0.58$.

Remark. Since $0 \le \{y\} < 1$, we have $6.32 \le x < 7.32$, hence $x = 6.58$ and $y = 2.74$.

MA114. Let p, q, and r be positive constants. Prove that at least one of the following equations has real roots.

$$\begin{aligned} px^2 + 2qx + r &= 0 \\ rx^2 + 2px + q &= 0 \\ qx^2 + 2rx + p &= 0 \end{aligned}$$

Originally from 1983 Descartes Contest, problem 6b.

We received 9 correct solutions and one incorrect solution. Both solutions below assume only that p, q, r are real.

Solution 1, by Vikas Kumar Meena and Dominique Mouchet (done independently).

Adding the discriminants of the three equations, we get

$$4(q^2 - pr) + 4(p^2 - rq) + 4(r^2 - qp) = 2[(p-q)^2 + (q-r)^2 + (r-p)^2] \ge 0,$$

whence at least one of the discriminants is nonnegative. The corresponding equation has real roots.

Solution 2, by Corneliu Manescu-Avram, Daniel Vacaru, and the Missouri State University Problem Solving Group (done independently).

If the discriminants of all three equations are negative, then

$$q^2 < pr, \quad p^2 < rq, \quad r^2 < qp,$$

whence

$$p^2q^2r^2 < (pr)(rq)(qp) = p^2q^2r^2,$$

which is a contradiction. Hence at least one equation must have real roots.

Comments from the editor. One solver assumed, wolog, that $p \le q \le r$. This is fine, but requires a bit of argument. His conclusion that $r^2 - pq \ge 0$ works when $p, q, r > 0$, but not in general. A better approach would be to assume, wolog, that r, say, has largest absolute value, whereupon $r^2 - pq \ge 0$ and we can identify which equation has real roots. Conversely, if p, q, r all have the same sign, a similar argument (or one along the lines of Solution 2), will ensure that at least one equation has nonreal roots. When p, q, r do not all have the same sign, then two equations must have real roots and either possibility obtains for the third.

MA115. I met a person the other day that told me they will turn x years old in the year x^2. What year were they born?

Inspired by 1982 Fermat Contest, problem 19.

We received 8 submissions, all correct. We present the solution by Bellamy Kastanya of Garth Webb Secondary School in Oakville, Ontario, Canada.

Since the birthday will happen in the future, we are interested in the year in the near future which is a square. There are two candidates: 2025 and 2116.

If it is 2025, then the person will turn 45. This means that the person was born in 1980. If it is 2116, then the person will turn 46. This means that the person will be born in 2070 (so, I cannot meet this person yet). So, the only logical solution is the first one and the person was born in 1980.

OLYMPIAD CORNER

No. 396

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **December 30, 2021**.

OC546. Let a, b, c be three real numbers. For each positive integer n, $a^n + b^n + c^n$ is an integer. Prove that there exist three integers p, q, r such that a, b, c are the roots of the equation $x^3 + px^2 + qx + r = 0$.

OC547. In a non-equilateral triangle ABC, I is the incentre and O is the circumcentre. Prove that $\angle AIO \leq 90^\circ$ if and only if $2BC \leq AB + AC$.

OC548. Each of the digits $1, 3, 7, 9$ occurs at least once in the decimal representation of some positive integer. Prove that one can permute the digits of this integer such that the resulting integer is divisible by 7.

OC549. In the triangle ABC we have $AB = AC$ and $\angle BAC = 90^\circ$. Consider points M and P on AB such that $AM = BP$. Let D be the midpoint of BC, R a point on CM, and Q a point on BC such that A, R, Q are collinear and the line AQ is perpendicular to CM. Show that:

(a) $\angle AQC \cong \angle PQB$;

(b) $\angle DRQ = 45^\circ$.

OC550. The real numbers x, y, and z are not all equal and satisfy:

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = k.$$

Determine all possible values of k.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 decembre 2021**.

OC546. Soient a, b et c trois nombres réels tels que $a^n + b^n + c^n$ est un entier pour tout n entier positif. Démontrer qu'il existe trois entiers p, q et r de façon à ce que a, b et c soient les trois racines de $x^3 + px^2 + qx + r = 0$.

OC547. Dans un triangle ABC qui n'est pas équilatéral, soit I le centre du cercle inscrit et O le centre du cercle circonscrit. Démontrer que $\angle AIO \leq 90^\circ$ si et seulement si $2BC \leq AB + AC$.

OC548. Chacun des chiffres 1, 3, 7 et 9 a lieu au moins une fois dans la représentation décimale d'un certain entier positif. Démontrer qu'il et possible de permuter les chiffres de cet entier de façon à ce que l'entier résultant soit divisible par 7.

OC549. Un triangle ABC est tel que $AB = AC$ et $\angle BAC = 90^\circ$. Soient M et P des points dans AB tels que $AM = BP$; soit aussi D le point milieu de BC; enfin, soit R un point dans CM et Q un point dans BC tels que A, R et Q sont allignés et que la ligne AQ est perpendiculaire à la ligne CM. Démontrer que:

(a) $\angle AQC \cong \angle PQB$;

(b) $\angle DRQ = 45^\circ$.

OC550. Les nombres réels x, y et z ne sont pas tous égaux. Ils vérifient

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = k.$$

Déterminer toute valeur possible de k.

OLYMPIAD CORNER SOLUTIONS

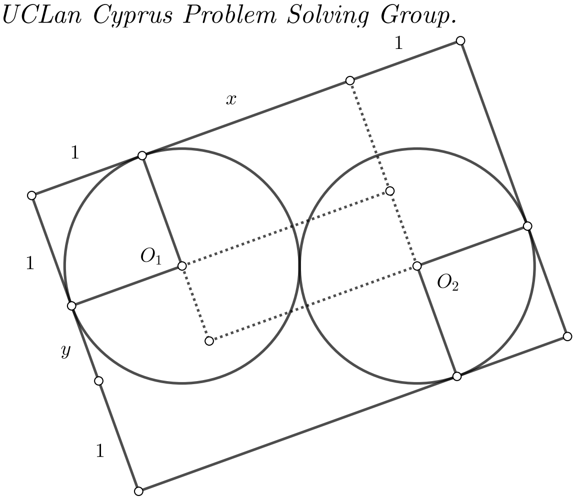
Statements of the problems in this section originally appear in 2021: 47(3), p. 135–136.

OC521. In the plane there are two identical circles with radius 1, which are tangent externally. Consider a rectangle containing both circles, each side of which touches at least one of them. Determine the largest and the smallest possible area of such a rectangle.

Originally from 2018 Czech-Slovakia Math Olympiad, 3rd Problem, Category A, First Round.

We received 8 *solutions. We present* 2 *solutions.*

Solution 1, by UCLan Cyprus Problem Solving Group.



From the figure above we see that the area is equal to $A = (2 + x)(2 + y)$ where x, y satisfy $x^2 + y^2 = (O_1O_2)^2 = 4$.

By Cauchy-Schwarz, we have

$$A = 4 + 2(x + y) + xy \leqslant 4 + 2\sqrt{2(x^2 + y^2)} + \frac{x^2 + y^2}{2} = 6 + 4\sqrt{2}$$

with equality if and only if $x = y = \sqrt{2}$.

Since also $x + y \geqslant \sqrt{x^2 + y^2} = 2$, then

$$A = 4 + 2(x + y) + xy = 2 + \frac{1}{2}(x + y)^2 + 2(x + y) \geqslant 8$$

with equality if and only if $x = 2, y = 0$ or $x = 0, y = 2$.

Both equality cases can actually be achieved but we omit the figures.

Solution 2, by the Missouri State University Problem Solving Group.

We will assume that the bounding rectangle is oriented so that its sides are parallel to the coordinate axes and its center is at the origin. Suppose the two circles are tangent at the origin, arranged as follows. One circle has center $(\cos t, \sin t)$, and the other circle has center $(-\cos t, -\sin t)$, for $0 \leq t \leq \pi/2$. The bounding rectangle has dimensions $2(1+\cos t)$ and $2(1+\sin t)$. We wish to find the largest and smallest values of the area function $A(t) = 4(1+\cos t)(1+\sin t)$ for $0 \leq t \leq \pi/2$. Now

$$\begin{aligned}(1+\cos t)(1+\sin t) &= 1+\sin t+\cos t+\sin t\cos t\\ &= 1+\sqrt{2}\sin(t+\pi/4)+\tfrac{1}{2}\sin(2t).\end{aligned}$$

The second and third terms increase from $t=0$ to $\pi/4$ then decrease from $t=\pi/4$ to $\pi/2$, so the maximum occurs at $t=\pi/4$. Since the values at $t=0$ and $t=\pi/2$ are equal, the minimum occurs here. This gives

$$4(1+\cos\pi/4)(1+\sin\pi/4) = 6+4\sqrt{(2)} \approx 11.6569$$

as the maximum value and

$$4(1+\cos 0)(1+\sin 0) = 8$$

as the minimum value.

OC522. Find the largest natural number n such that the sum

$$\lfloor\sqrt{1}\rfloor+\lfloor\sqrt{2}\rfloor+\cdots+\lfloor\sqrt{n}\rfloor$$

is a prime number.

Originally from 2018 Czech-Slovakia Math Olympiad, 4th Problem, Category A, First Round.

We received 13 *submissions, of which* 10 *were correct and complete. We present the solution by the Missouri State University Problem Solving Group.*

Let

$$f(n) = \lfloor\sqrt{1}\rfloor+\lfloor\sqrt{2}\rfloor+\cdots+\lfloor\sqrt{n}\rfloor.$$

Note that $f(47) = 197$, which is prime. We claim that this is the largest n such that $f(n)$ is prime.

If $k^2 \leq n \leq (k+1)^2-1$, then

$$\begin{aligned}f(n) = \left(\sum_{i=1}^{k-1}\sum_{j=i^2}^{(i+1)^2-1}\lfloor\sqrt{j}\rfloor\right)+\sum_{j=k^2}^{n}\lfloor\sqrt{j}\rfloor &= \left(\sum_{i=1}^{k-1}\sum_{j=i^2}^{(i+1)^2-1} i\right)+\sum_{j=k^2}^{n} k\\ &= \left(\sum_{i=1}^{k-1}(2i+1)i\right)+(n+1-k^2)k\\ &= \frac{k(k-1)(4k+1)}{6}+(n+1-k^2)k.\end{aligned}$$

Now $f(48) = 203 = 7 \cdot 29$ is not prime.

If $n \geq 49$, then $k > 6$. Let $m = \dfrac{k(k-1)(4k+1)}{6} \in \mathbb{Z}$. We claim $\gcd(m,k) > 1$. If not, then $k(k-1)(4k+1) = 6m$ and $\gcd(m,k) = 1$ would imply that k divides 6, which contradicts the fact that $k > 6$. If $d = \gcd(m,k)$, then d is a nontrivial factor of $f(n)$, since $d > 1$ and $d \leq k < m \leq f(n)$. Therefore $f(n)$ is not prime.

Editor's Comment. UCLan Cyprus Problem Solving Group observed that this is sequence A022554 in OEIS. In the comments for the sequence it is mentioned that "It seems that 197 is the largest prime in this sequence....". So, here it has been confirmed that this is indeed the case.

OC523. Let $(a_n)_{n\geq 1}$ be a sequence such that $a_n > 1$ and $a_{n+1}^2 \geq a_n a_{n+2}$ for all $n \geq 1$. Prove that the sequence $(x_n)_{n\geq 1}$ defined by $x_n = \log_{a_n} a_{n+1}$ for $n \geq 1$ is convergent and find its limit.

Originally from 2018 Romania Math Olympiad, 3rd Problem, Grade 11, District Round.

We received 5 *solutions. We present the solution by Oliver Geupel.*

Let $b_n = \log a_n$. By $a_n > 1$, we have $b_n > 0$. Hence,

$$x_n = \frac{\log a_{n+1}}{\log a_n} = \frac{b_{n+1}}{b_n} > 0.$$

From the hypothesis $a_n a_{n+2} \leq a_{n+1}^2$, we obtain $b_n + b_{n+2} \leq 2b_{n+1}$ by monotoneity of the logarithm. Multiplying both sides by $\frac{1}{b_{n+1}}$, we get

$$\frac{1}{x_n} + x_{n+1} \leq 2. \tag{1}$$

On the other hand, it holds

$$2 \leq \frac{1}{x_n} + x_n. \tag{2}$$

Therefore, $0 < x_{n+1} \leq x_n$ for $n \geq 1$. By the monotone convergence theorem, (x_n) is convergent. Putting $L = \lim_{n\to\infty} x_n$, we obtain from (1) and (2) that

$$L + \frac{1}{L} = 2;$$

whence $L = 1$.

OC524. Let $p \geq 2$ be a natural number and let $(M, \cdot)$ be a finite monoid such that $a^p \neq a$ for all $a \in M \setminus \{e\}$, where e is the identity element of M. Prove that $(M, \cdot)$ is a group.

Originally from 2018 Romania Math Olympiad, 2nd Problem, Grade 12, District Round.

We received 6 *solutions. We present the solution by the UCLan Cyprus Problem Solving Group.*

Take $x \in M$ and consider the sequence $x, x^2, x^3, \ldots$. Since M is finite, there are positive integers $k < \ell$ such that $x^k = x^\ell$. Letting $m = \ell - k$, by induction we can deduce that $x^k = x^{k+mr}$ for each $r \geqslant 0$. Multiplying both sides by x^{n-k} we also deduce that $x^n = x^{n+mr}$ for each $n \geqslant k$ and each $r \geqslant 0$.

In particular, letting $a = x^{km}$ and taking $n = km$ and $r = k$ above, we have $a = x^{km} = x^{km+km} = a^2$. By induction we now get $a = a^t$ for each $t \geqslant 1$. In particular $a = a^p$ and therefore $a = e$.

So for each $x \in M$ there is an $s \geqslant 1$ such that $x^s = e$. Then either $s = 1$ and $x = e$ or $s > 1$ and $x \cdot x^{s-1} = x^s = e = x^s = x^{s-1} \cdot x$. In both cases we deduce that x has an inverse. (I.e. an element which is both a left inverse and a right inverse.) Therefore M is a group.

Comment. Note that the fact that M is finite is needed. Otherwise $(\mathbb{N}, +)$ is a counterexample.

OC525. Consider the sequence $(a_1, a_2, \ldots, a_n)$ with terms from the set $\{0, 1, 2\}$. We will call a *block* a subsequence of the form $(a_i, a_{i+1}, \ldots, a_j)$, where $1 \leq i \leq j \leq n$, and $a_i = a_{i+1} = \cdots = a_j$. A block is called *maximal* if it is not contained in any longer block. For example, in the sequence $(1, 0, 0, 0, 2, 1, 1)$ the maximal blocks are (1), $(0, 0, 0)$, (2) and $(1, 1)$. Let K_n be the number of such sequences of length n with terms from the set $\{0, 1, 2\}$ in which all maximal blocks have odd lengths. Moreover, let L_n be the number of all sequences of length n with terms from the set $\{0, 1, 2\}$ in which the numbers 0 and 2 do not appear in adjacent positions. Prove that $L_n = K_n + \dfrac{1}{3}K_{n-1}$ for all $n > 1$.

Originally from 2018 Poland Math Olympiad, 4th Problem, First Round.

We received 2 *solutions. We present the solution by the UCLan Cyprus Problem Solving Group.*

We call a sequence *nice* if it is a sequence with terms from $\{0, 1, 2\}$ in which all maximal blocks have odd length. We also call a sequence *good* if it is a sequence with terms from $\{0, 1, 2\}$ in which the numbers 0 and 2 do not appear in adjacent positions.

We observe that there is a 1-1 correspondence between nice sequences of length n and nice sequences of length $n+2$ whose last maximal block has length at least 3. The correspondence is obtained by adding two elements to the sequence of length n at its end, both equal to the last term of the sequence.

We also observe that there are $2K_{n+1}$ nice sequences of length $n+2$ whose last maximal block has length 1. Indeed every such sequence is obtained from a nice sequence of length $n+1$ by adding at its end a term which is different from the last term of the sequence.

Thus $K_{n+2} = 2K_{n+1} + K_n$ for each $n \geqslant 1$. Defining $K_0 = 0$, then the condition holds for $n = 0$ as well.

Now let A_n, B_n, C_n be the number of good sequences of length n whose last term is equal to $0, 1, 2$ respectively. By symmetry we have $A_n = C_n$. Then $A_{n+1} = A_n + B_n$ and $B_{n+1} = A_n + B_n + C_n = 2A_n + B_n$ for each $n \geqslant 1$. Defining $A_0 = C_0 = 0$ and $B_0 = 1$, then the condition holds for $n = 0$ as well. Then

$$L_{n+2} = 2A_{n+2} + B_{n+2} = 4A_{n+1} + 3B_{n+1} = 2L_{n+1} + B_{n+1} = 2L_{n+1} + L_n$$

for each $n \geqslant 0$.

Defining $M_n = K_n + \frac{1}{3}K_{n-1}$ for $n \geqslant 1$, we observe that M_n satisfies the same recurrence relation as L_n since

$$\begin{aligned} 6M_{n+1} + 3M_n &= (6K_{n+1} + 2K_n) + (3K_n + K_{n-1}) \\ &= 6K_{n+1} + 5K_n + (K_{n+1} - 2K_n) \\ &= (6K_{n+1} + 3K_n) + K_{n+1} \\ &= 3K_{n+2} + K_{n+1} = 3M_{n+2} \end{aligned}$$

We now observe that $L_1 = M_1 = 3$ and $L_2 = M_2 = 7$, so $L_n = M_n = K_n + \frac{1}{3}K_{n-1}$ for each $n > 1$.

An introduction of the problem of finding the chromatic number of the plane (II)

Veselin Jungić

1 Introduction

In the first part of this two-part note [5], we introduced the chromatic number of the plane problem:

Problem 1 *What is χ, the smallest number of sets ("colours") with which we can cover the plane in such a way that no two points of the same set are a unit distance apart? [1]*

Through a series of examples and exercises, we guided the reader to establish that $4 \leq \chi \leq 7$. In particular, we used the unit distance graph called the Moser spindle, Figure 1, to show that $\chi \geq 4$ since 3 colours are not enough for a proper colouring that avoids adjacent vertices of the same colour. The Moser spindle was originally introduced by Canadian mathematicians Leo and William Moser in 1961. [6]

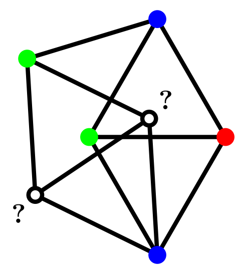


Figure 1: The Moser spindle.

Regardless of all efforts, until very recently, the lower bound for χ, established by an American mathematician Edward Nelson in the 1950s [12] remained unchanged.

2 de Grey's theorem

In 1979, the renowned Hungarian mathematician Paul Erdős wrote:

> It seems likely that the chromatic number [of the plane] is greater than four. By a theorem of de Bruijn and myself this would imply that there are n points $x_1, x_2, \ldots, x_n$ in the plane so that if we join any two of them whose distance is 1, then the resulting graph $G(x_1, x_2, \ldots, x_n)$

> has a chromatic number > 4. I believe such an n exists but its value may be very large. [3, p.156]

Almost forty years later, in 2018, an English author and biomedical gerontologist Aubrey de Grey proved that Erdős was correct on both accounts. de Grey constructed a unit distance graph with 20425 vertices and with its chromatic number greater than 4.

Therefore, de Grey proved the following:

Theorem 1 $\chi \geq 5$. *[2]*

de Grey's proof was a combination of new insights into some of the well-known facts and techniques and a computer assisted mathematical proof. In de Grey's words:

> In seeking graphs that can serve as [a unit distance graph with the chromatic number greater than four] in our construction, we focus on graphs that contain a high density of Moser spindles. The motivation for exploring such graphs is that a spindle contains two pairs of vertices distance $\sqrt{3}$ apart, and these pairs cannot both be monochromatic. Intuitively, therefore, a graph containing a high density of interlocking spindles might be constrained to have its monochromatic $\sqrt{3}$-apart vertex pairs distributed rather uniformly (in some sense) in any 4-colouring. Since such graphs typically also contain regular hexagons of side-length 1, one might be optimistic that they could contain some such hexagon that does not contain a monochromatic triple in any 4-colouring of the overall graph, since such a triple is always an equilateral triangle of edge $\sqrt{3}$ and thus constitutes a locally high density, i.e. a departure from the aforementioned uniformity, of monochromatic $\sqrt{3}$-apart vertex pairs. [2]

Initially, de Grey "developed a custom program" to test his original graph on 20425 vertices for the existence of monochromatic points with the unit distance under a 4-colouring. de Grey explained that "this algorithm was implemented in Mathematica 11 on a standard MacBook Air and terminated in only a few minutes."

In the following exercises, we will explore some terms from the above quote and construct three "tightly linked Moser spindles," one of the tools that de Grey used in his proof.

Exercise 1 *Use a graphing tool, Desmos or GeoGebra, for example, to construct the Moser spindle with equilateral triangles of side length 1. Let A be the vertex of degree four and let C and D be two vertices that are not adjacent to A. Let α be the measure of $\angle DAC$ in degrees. See Figure 2.*

i) Determine $|\overline{AC}|$ and $|\overline{AD}|$.

ii) Determine the value of α.

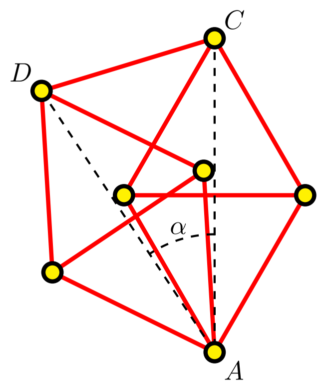


Figure 2: Moser spindle and the angle α

iii) Construct the Moser spindle by starting with drawing two concentric circles, one with radius 1 and the other with radius $|\overline{AC}|$.

Exercise 2 *Confirm de Grey's observation from the quote above that "a spindle contains two pairs of vertices distance* $\sqrt{3}$ *apart, and these pairs cannot both be monochromatic" in a proper* k*-colouring of the Moser spindle,* $k > 3$.

Exercise 3 *Use your graphing tool to rotate clockwise the Moser spindle through* α *about the point* A *to obtain another Moser spindle. Call the newly obtained graph, i.e. the graph that consists of the original Moser spindle and its image obtained by the rotation,* MS_2. *What do you observe? Is this a unit distance graph? Does* MS_2 *contain a hexagon, not necessarily regular, "of side-length 1?"*

Exercise 4 *Find a proper 4-colouring of* MS_2. *Do this in two ways, by hand and by writing a program that would search for proper* 4*-colourings of* MS_2.

Exercise 5 *Alter the drawing of* MS_2 *by rotating clockwise the original Moser spindle by* $\frac{\alpha}{2}$ *about the point* A. *Find a proper* 4*-colouring of the newly obtained graph.*

We would like to encourage the reader to explore other graphs, tools, and ideas that de Grey presents in his remarkable paper.

Already in his original paper, de Grey was able to reduce the size of the unit distance graph with the chromatic number greater than four to 1581. Again, in de Grey's words, "Happily, G has turned out to be within the reach of standard SAT solvers with which others have now confirmed its chromatic number to be 5 without the need to resort to using custom code or checking weaker properties of subgraphs."

On August 3, 2019, as part of the Polymath 16 project [9], Jaan Parts from Kazan, Russia, posted an image of a unit distance graph with 510 vertices and 2508 edges that confirms that $\chi \geq 5$.

3 Conclusion

The goals of the researchers involved in the Polymath 16 project included:

1. To further reduce the size of the "good" graph;
2. To find a human-verifiable proof that $\chi \geq 5$.

Clearly, the two goals are interrelated. A graph of relatively small size found by a computer may lead to "a human-verifiable proof that $\chi \geq 5$." And the other way around, a moment of human brilliance may provide a proof that would contain, if not the exact value, then a reasonable bound for the size of the minimal unit distance graph with a chromatic number greater than four.

It seems that the first scenario is more likely. For example, in their recent paper that includes some of the authors' contributions to the Polymath 16 project, Nóra Frankl, Tamaás Hubai, and Dömötör Pálvölgyi wrote:

> Note that G_{34} [a graph on 34 vertices that cannot be 4-coloured with the so-called "bichromatic origin"] was found by a computer search, and for finding other similar graphs one might rely on a computer program. Thus, the approach we propose, is human-verifiable, however it might be computer-assisted. [4]

In 2020, Parts contributed to both of the above listed goals. In [7], Parts described a set of ideas that led to a construction of a 5-chromatic graph with 509 vertices and 2442 edges. Parts also came up with a human-verifiable proof that $\chi \geq 5$:

> The proof presented here may leave a sediment of dissatisfaction. We tend to explain this by the fact that we expect not only a *human-verifiable* proof, but a really *human* proof. The difference is that in the latter case, the proof is created by a person, while in the first case it is not necessary: it is only important that the person is able to complete the verification in a reasonable number of steps. [8]

The recent developments in the search for the chromatic number of the plane have further highlighted the following two realities:

1. Mathematical research has become increasingly collaborative;
2. Computing has become an instrumental part of mathematical research.

The Polymath 16 project is winding down. A paper that will summarize the outcomes of the project is expected in the near future. In a post dated February 1, 2021, de Grey concluded:

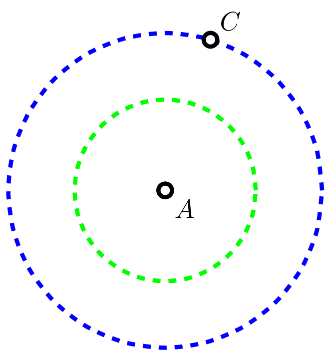
> Perhaps the most satisfying aspect of the project, to me, is its quite remarkable success in attracting amateur mathematicians like myself (of whom Jaan is one). At least half a dozen of us have been significant contributors over these three years, working side by side with numerous professionals including Terry Tao himself. [10]

Still, is $\chi = 5$? Or maybe it is $\chi = 6$? Or, perhaps, $\chi = 7$?

4 Hints and solutions

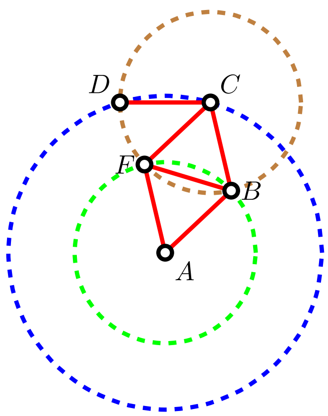
Exercise 1. i) $|\overline{AC}| = |\overline{AD}| = \sqrt{3}$. ii) $\alpha = \arccos\left(\frac{5}{6}\right) \approx 33.56^0$. iii) see below.

Step 1



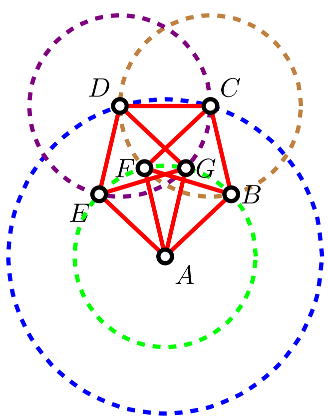
Start by choosing a point A in the plane and then draw a circle with the centre at A and radius 1. Denote this circle by c_1. Next, draw a circle with the centre at A and radius $\sqrt{3}$. Denote this circle by c_2. Choose a point C on the circle c_2.

Step 2



Draw a circle with the centre at C and radius 1. Denote this circle by c_3. Let D be a point of intersection of c_2 and c_3. Let B and F be the points of intersection of c_1 and c_3. Draw the line segments $\overline{AB}$, $\overline{AF}$, $\overline{BC}$, $\overline{BF}$, $\overline{CD}$, and $\overline{CF}$. Observe that those line segments are of length 1. How would you argue that $|\overline{BF}| = 1$?

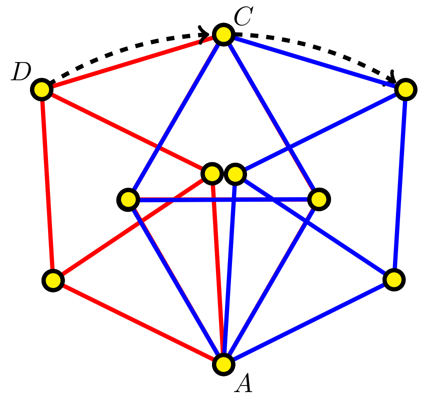
Step 3



Draw a circle with the centre at D and radius 1. Denote this circle by c_4. Let E and G be the points of intersection of c_1 and c_4. Draw the line segments $\overline{AE}$, $\overline{AG}$, $\overline{DE}$, $\overline{DG}$, and $\overline{EG}$. Observe that those line segments are of length 1. The Moser spindle appears!

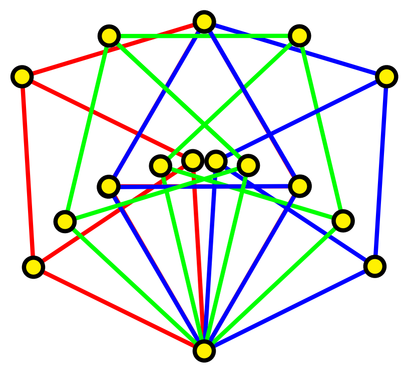
Exercise 2. If both (A, C) and (A, D) are monochromatic then the pair (C, D) is monochromatic as well.

Exercise 3. Observe that the two spindles share four vertices and five edges. This is what de Grey meant when saying "interlocking spindles." Also note that the outer polygon is a hexagon of side-length 1.

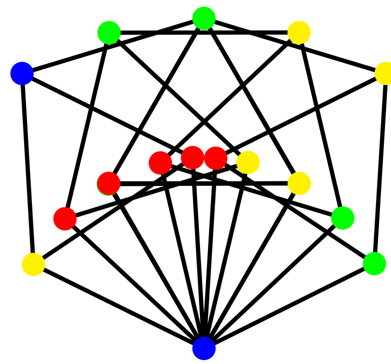


Exercise 4. Say that A is the vertex of degree 6. Colour A blue and then colour the vertices not adjacent to A by yellow and green alternatively. Use blue and red to colour the remaining vertices. To write a program you may use the following strategy. Denote the four colours by the numbers 1, 2, 3, and 4 . Use 0 to denote the vertex that is not coloured. Then go through each vertex that was not coloured, get all the numbers of the vertices around it, then colour the vertex by the lowest positive number that is not in that set. For example, if a vertex had vertices around it with the numbers 0, 1, 2, and 2, then this vertex would have to be given the number 3 (which means colour number three).

Exercise 5. An additional clockwise rotation by $\frac{\alpha}{2}$ produces, what de Grey calls,"three tightly linked Moser spindles."



Use the same idea as in Exercise 4 to find a proper 4-colouring by hand. Alternatively, you may write a program that generates proper 4-colourings. Here is a 4-colouring generated by a Python program written by Ewan Brinkman, a first-year student at Simon Fraser University:



Do you notice what de Grey calls "monochromatic $\sqrt{3}$-apart vertex pairs $(\ldots)$ in any 4-colouring"?

References

[1] Croft, H.T., Falconer, K.J., and Guy, R.K. *Unsolved Problems in Geometry.* Problem Books in Mathematics. Springer-Verlag, New York, 1991. Unsolved Problems in Intuitive Mathematics, II.

[2] de Grey, Aubrey D.N.J. (2018), The chromatic number of the plane is at least 5. *Geombinatorics*, 28, p. 5–18. arXiv:1804.02385.

[3] Erdős, P., Combinatorial problems in geometry and number theory. *Proc. of the Sympos. Pure Math.*, 34, Amer. Math. Soc., Providence, RI, (1979), p. 149–162.

[4] Frankl, N., Hubai, T., and Pálvölgyi, D. Almost-monochromatic sets and the chromatic number of the plane. *Proceedings of SoCG 2020.* arXiv:1912.02604.

[5] Jungić V., An introduction of the problem of finding the chromatic number of the plane (I), *Crux Mathematicorum*, 46 (8), p. 390–396.

[6] Moser, L., and Moser, W., Solution to Problem 10, *Can. Math. Bull.* 4 (1961), p. 187–189.

[7] Parts, Jaan. (2020), Graph minimization, focusing on the example of 5-chromatic unit-distance graphs in the plane, *Geombinatorics*, 29, p. 137–166. arXiv:2010.12665.

[8] Parts, Jaan. (2020), The chromatic number of the plane is at least 5 – a human-verifiable proof, *Geombinatorics*, 30, p. 11–102. arXiv:2010.12661.

[9] Polymath16 Project. https://dustingmixon.wordpress.com/2019/07/08/polymath16-thirteenth-thread-bumping-the-deadline/#comment-23999

[10] Polymath16 seventeenth thread: Declaring victory. https://dustingmixon.wordpress.com/2021/02/01/polymath16-seventeenth-thread-declaring-victory

[11] Soifer, A. *The mathematical coloring book*, Springer, New York, 2009

[12] Soifer, A. Chromatic number of the plane & its relatives, history, problems and results: an essay in 11 parts, *Ramsey theory*, Progr. Math., vol. 285, Birkhäuser/Springer, New York, 2011, p. 121–161.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **December 30, 2021**.

4671. *Proposed by Minh Ha Nguyen.*

Given a triangle ABC and a point M on the side BC, let r_1 and r_2 be inradii of triangles ABM and ACM, respectively. Determine the position of M which yields the maximum value of $r_1 + r_2$.

4672. *Proposed by Nguyen Viet Hung.*

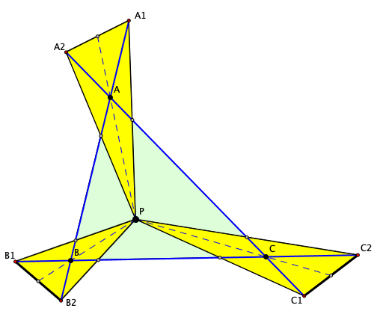
Let a, b, c be rational numbers such that

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} = \sqrt{a^2 + ac + c^2}.$$

Prove that $\sqrt{a^2 + b^2 + c^2}$ is a rational number.

4673. *Proposed by Todor Zaharinov.*

Let P be a point in the plane of triangle ABC not on any line joining two of its vertices. Let PA_1A_2 be the triangle with centroid A and vertices $A_1 \in BA$ and $A_2 \in AC$. By cyclically permuting A, B, C we similarly define triangle ΔPB_1B_2 with centroid B, and PC_1C_2 with centroid C. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ have equal areas.



4674. *Proposed by Michel Bataille.*

Find all monic polynomials $P(x) \in \mathbb{Z}[x]$, of positive degree $2n$, that have a complex root of multiplicity n and satisfy $P(0) = -1$ and $P(1) \cdot P(-1) \neq 0$.

4675. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$2(a^4 + b^4 + c^4) - (a^3 + b^3 + c^3) \geq 3abc.$$

4676. *Proposed by Lorian Saceanu.*

Let $ABCD$ be a convex quadrilateral with E at the intersection of diagonals. From E build the bisectors of the angles $\angle AEB$, $\angle BEC$, $\angle CED$, $\angle DEA$. We get an orthodiagonal quadrilateral $FGHI$ with $F \in [AB]$, $G \in [BC]$, $H \in [CD]$, $I \in [DA]$. Prove that:

$$\frac{[AF]}{[BF]} \cdot \frac{[BG]}{[CG]} \cdot \frac{[CH]}{[DH]} \cdot \frac{[DI]}{[AI]} = 1.$$

4677. *Proposed by Seán M. Stewart.*

Evaluate

$$\int_0^\infty \arctan\left(\frac{2x}{1+x^2}\right) \frac{x}{x^2+4}\, dx.$$

4678. *Proposed by Michel Bataille.*

Let n be a nonnegative integer. Evaluate in closed form

$$\frac{\displaystyle\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n+1}{n-2j} \left(\frac{1}{5}\right)^j}{\displaystyle\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{n-2j} \left(-\frac{1}{5}\right)^j}.$$

4679. *Proposed by Daniel Sitaru.*

Let $(x_n)_{n \geq 1}$ be a sequence of real numbers such that $x_1 = \frac{1}{7}, x_2 = \frac{1}{5}$ and for $n \geq 2$,

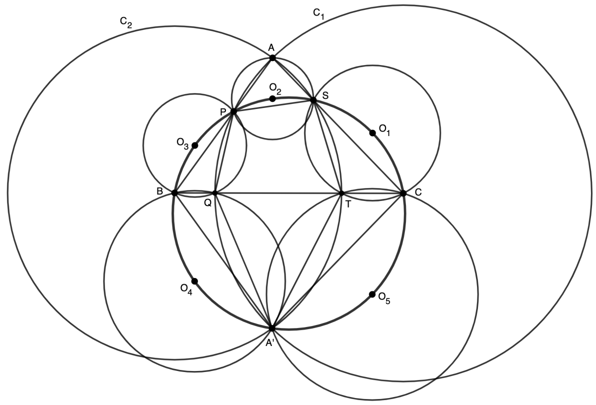
$$2nx_{n+1} \cdot x_{n-1} = (n+1)x_n \cdot x_{n-1} + (n-1)x_n \cdot x_{n+1}.$$

Find

$$\lim_{n \to \infty} \left(\frac{2}{3} + x_n\right)^{nx_n}.$$

4680. *Proposed by Pericles Papadopoulos.*

In triangle ABC with $|BC| > |AC| > |AB|$, the circle C_1, centered at C and with radius CA, meets the sides AB, BC at points P, Q, respectively and the circle C_2, centered at B and with radius BA, meets the sides AC, BC at points S, T, respectively. Let A' be the second intersection point of C_1 and C_2 and let O_1, O_2, O_3, O_4, O_5 be the circumcenters of triangles $CST, SAP, PBQ, BA'Q, A'CT$, respectively. Prove that the points $C, O_1, S, O_2, P, O_3, B, O_4, A', O_5$ are concyclic.



. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 decembre 2021**.

4671. *Proposé par Minh Ha Nguyen.*

Soit un triangle ABC et M un point sur le côté BC, puis soient r_1 et r_2 les rayons des cercles inscrits des triangles ABM et ACM respectivement. Déterminer où situer M afin d'obtenir la valeur maximale de $r_1 + r_2$.

4672. *Proposé par Nguyen Viet Hung.*

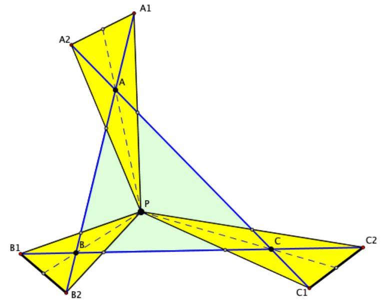
Soient a, b, c des nombres rationnels tels que

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} = \sqrt{a^2 + ac + c^2}.$$

Démontrer que $\sqrt{a^2 + b^2 + c^2}$ est un nombre rationnel.

4673. *Proposé par Todor Zaharinov.*

Soit P un point dans le plan du triangle ABC, mais ne se trouvant pas sur les lignes rejoignant deux de ses sommets. Soit PA_1A_2 le triangle de centroïde A et sommets $A_1 \in BA$ et $A_2 \in AC$. Par permutation cyclique, on définit les triangles PB_1B_2 et PC_1C_2 de centroïdes B et C respectivement. Démontrer que les triangles $A_1B_1C_1$ et $A_2B_2C_2$ ont la même surface.



4674. *Proposé par Michel Bataille.*

Déterminer tout polynôme unitaire $P(x) \in \mathbb{Z}[x]$, de degré positif $2n$, tel que $P(1) \cdot P(-1) \neq 0,\ P(0) = -1$ et ayant une racine complexe de multiplicité n.

4675. *Proposé par George Apostolopoulos.*

Soient a, b et c des nombres réels positifs tels que $a^2 + b^2 + c^2 = 3$. Démontrer que

$$2(a^4 + b^4 + c^4) - (a^3 + b^3 + c^3) \geq 3abc.$$

4676. *Proposé par Lorian Saceanu.*

Soit $ABCD$ un quadrilatère convexe et soit E le point d'intersection de ses diagonales. À partir de E, construire les bissectrices des angles $\angle AEB$, $\angle BEC$, $\angle CED$, $\angle DEA$. Il en suit un quadrilatère à diagonales orthogonales $FGHI$, où $F \in [AB]$, $G \in [BC]$, $H \in [CD]$, $I \in [DA]$. Démontrer que

$$\frac{[AF]}{[BF]} \cdot \frac{[BG]}{[CG]} \cdot \frac{[CH]}{[DH]} \cdot \frac{[DI]}{[AI]} = 1.$$

4677. *Proposé par Seán M. Stewart.*

Évaluer

$$\int_0^\infty \arctan\left(\frac{2x}{1+x^2}\right) \frac{x}{x^2+4}\, dx.$$

4678. *Proposé par Michel Bataille.*

Soit n un entier non négatif. Évaluer en forme close

$$\frac{\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n+1}{n-2j} \left(\frac{1}{5}\right)^j}{\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{n-2j} \left(-\frac{1}{5}\right)^j}.$$

4679. *Proposé par Daniel Sitaru.*

Soit $(x_n)_{n\geq 1}$ une suite de nombres réels telle que $x_1 = \frac{1}{7}, x_2 = \frac{1}{5}$ et, pour $n \geq 2$,

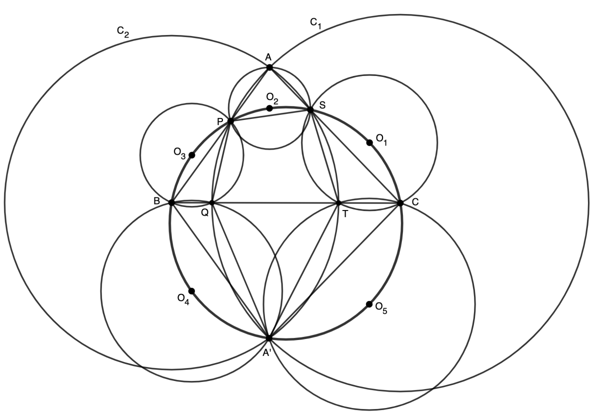
$$2nx_{n+1} \cdot x_{n-1} = (n+1)x_n \cdot x_{n-1} + (n-1)x_n \cdot x_{n+1}.$$

Déterminer

$$\lim_{n\to\infty} \left(\frac{2}{3} + x_n\right)^{nx_n}.$$

4680. *Proposé par Pericles Papadopoulos.*

Soit ABC un triangle où $|BC| > |AC| > |AB|$. Supposons que le cercle C_1, centré à C et de rayon CA, rencontre les côtés AB et BC en P et Q respectivement, puis que le cercle C_2, centré à B et de rayon BA, rencontre les côtés AC et BC en S et T respectivement. Soit A' le second point d'intersection de C_1 et C_2 et soient O_1, O_2, O_3, O_4, O_5 les centres des cercles circonscrits de CST, SAP, PBQ, $BA'Q$, $A'CT$, respectivement. Démontrer que les points C, O_1, S, O_2, P, O_3, B, O_4, A', O_5 sont cocycliques.



SOLUTIONS

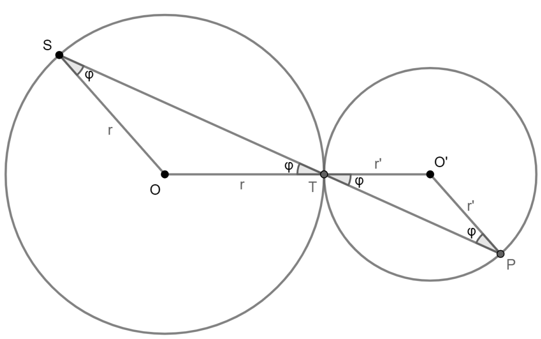
No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

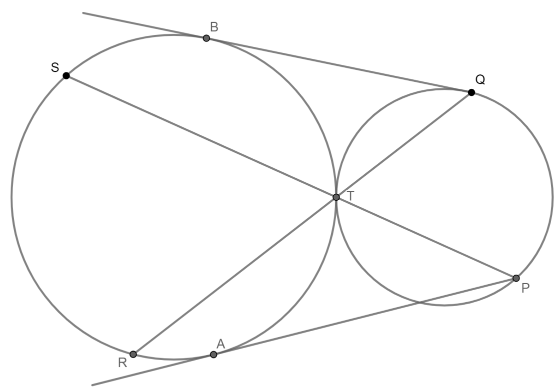
Statements of the problems in this section originally appear in 2021: 47(3), p. 151–154.

4621. *Proposed by Michel Bataille.*

In the plane, circles $\mathcal{C}$ and $\mathcal{C}'$ are externally tangent at T. Points P, Q of $\mathcal{C}'$ are such that $P \neq T$ and $\angle PQO = 90^\circ$ where O is the centre of $\mathcal{C}$. Points A, B of $\mathcal{C}$ are such that PA and QB are tangent to $\mathcal{C}$. If the line PT intersects $\mathcal{C}$ again at S, prove that $PS \cdot QT = PA \cdot QB$.

We received 7 submissions, all correct, and present the solution by Borche Joshevski.





The requirement that $\angle PQO = 90^\circ$ is not needed; instead, we let P and Q be arbitrary points of the circle $\mathcal{C}'$ with P distinct from Q and T. From the figure on the left, we see that $\Delta SOT = \Delta PO'T$ (where O and O' are the centers of $\mathcal{C}$ and $\mathcal{C}'$). It follows that

$$\frac{PT}{ST} = \frac{O'T}{OT} = \frac{r'}{r}. \tag{1}$$

If Q and T coincide, then $QT = 0 = QB$, and the condition $PS \cdot QT = PA \cdot QB$ is satisfied trivially. So we take $Q \neq T$ and define R to be the point where QT again intersects $\mathcal{C}$. Equation (1) implies that $\frac{PT}{ST} = \frac{r'}{r} = \frac{QT}{RT}$, or

$$\frac{PT}{QT} = \frac{TS}{TR} = \frac{PT + TS}{QT + TR} = \frac{PS}{QR}.$$

That is,

$$PT \cdot QR = PS \cdot QT. \tag{2}$$

The powers of P and Q with respect to $\mathcal{C}$ satisfy both $PT \cdot PS = PA^2$ and $QT \cdot QR = QB^2$. By multiplying these two equalities and applying equation (2) we get, in turn,

$$\begin{aligned} PT \cdot PS \cdot QT \cdot QR &= PA^2 \cdot QB^2, \\ (PT \cdot QR) \cdot (PS \cdot QT) &= (PA \cdot QB)^2, \\ (PS \cdot QT)^2 &= (PA \cdot QB)^2, \\ PS \cdot QT &= PA \cdot QB. \end{aligned}$$

4622. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle and AA', BB' and CC' be its medians. Let Γ be the circumcircle of $\triangle\ A'B'C'$ and let $\Gamma \cap AA' = \{A''\}$, $\Gamma \cap BB' = \{B''\}$, $\Gamma \cap CC' = \{C''\}$. Show that if $\overrightarrow{AA''} + \overrightarrow{BB''} + \overrightarrow{CC''} = \overrightarrow{0}$, then ABC is an equilateral triangle.

We received 9 submissions, but only five were complete and correct. We present the solution by Theo Koupelis.

Because A', B', C' are the midpoints of the sides of the triangle, the circle Γ is its Euler circle. As such, the feet of the altitudes also belong on Γ, and because the triangle is acute, these points are on the sides AB, BC, CA and not on their extensions; therefore, Γ cuts the medians at points A'', B'', C'' that are inside the triangle.

Let a, b, c be the lengths of the triangle's sides, let G be the centroid, and let D be the foot of the altitude from C to AB. Then the power of point A with respect to Γ is equal to $AD \cdot AC' = AA'' \cdot AA'$. But

$$AD = b \cdot \cos A = \frac{b^2 + c^2 - a^2}{2c}, \qquad \text{and} \qquad AC' = \frac{c}{2}.$$

Therefore, with $AA' = m_a$, where $m_a^2 = \dfrac{1}{4} \cdot (2b^2 + 2c^2 - a^2) = \dfrac{9}{4} \cdot AG^2$, we get $AA'' = \dfrac{b^2 + c^2 - a^2}{4m_a}$, and therefore

$$\overrightarrow{AA''} = AA'' \cdot \frac{\overrightarrow{AG}}{AG} = \frac{3(b^2 + c^2 - a^2)}{8m_a^2} \cdot \overrightarrow{AG} = \overrightarrow{AG} - \frac{a^2 + b^2 + c^2}{8m_a^2} \cdot \overrightarrow{AG}.$$

Adding similar expressions for $\overrightarrow{BB''}, \overrightarrow{CC''}$, and using the well-known expression $\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = \overrightarrow{0}$, we get

$$\begin{aligned} \overrightarrow{0} &= \overrightarrow{AA''} + \overrightarrow{BB''} + \overrightarrow{CC''} \\ &= (\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG}) - \frac{a^2 + b^2 + c^2}{8} \cdot \left(\frac{\overrightarrow{AG}}{m_a^2} + \frac{\overrightarrow{BG}}{m_b^2} + \frac{\overrightarrow{CG}}{m_c^2} \right) \\ &= -\frac{a^2 + b^2 + c^2}{8m_a^2} \cdot \left\{ \left[\left(\frac{m_a}{m_b} \right)^2 - 1 \right] \cdot \overrightarrow{BG} + \left[\left(\frac{m_a}{m_c} \right)^2 - 1 \right] \cdot \overrightarrow{CG} \right\}. \end{aligned}$$

But $\overrightarrow{BG}$ and $\overrightarrow{CG}$ are linearly independent vectors on the plane of the triangle and therefore we must have $m_a = m_b = m_c$, and thus $a = b = c$ and the triangle is equilateral.

Editor's comments. Michel Bataille observed that it is not necessary to suppose that ΔABC is acute. He showed that the proposer's result is equivalent to

Theorem. Suppose triangle ABC, with centroid G, is inscribed in the circle Ω while the medians from A, B, C intersect Ω again at A_1, B_1, C_1. Then the relation $\overrightarrow{GA_1} + \overrightarrow{GB_1} + \overrightarrow{GC_1} = \overrightarrow{0}$ implies that ΔABC is equilateral.

He argued as follows: If G is the centroid of ΔABC, then $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{0}$. It follows that $\overrightarrow{AA''} + \overrightarrow{BB''} + \overrightarrow{CC''} = \overrightarrow{0}$ is equivalent to $\overrightarrow{GA''} + \overrightarrow{GB''} + \overrightarrow{GC''} = \overrightarrow{0}$. Moreover, G is also the centroid of $\Delta A'B'C'$, and ΔABC is equilateral if and only if $\Delta A'B'C'$ is. In other words, by applying Bataille's result to prove that $\Delta A'B'C'$ is equilateral, it follows that the given triangle ABC is likewise equilateral.

4623. *Proposed by Nguyen Viet Hung.*

Let $p(x) = x^2 + 2ax - b - 1$ and $q(x) = x^2 + 2bx - a - 4$ be two polynomials with integer coefficients. Determine all pairs (a, b) of non-negative integers such that these two polynomials simultaneously have integer solutions.

We received 13 correct and 3 incorrect solutions. We present the solution of Corneliu Avram-Manescu and the proposer (done independently).

Since

$$p(x) = (x + a)^2 - (a^2 + b + 1) \quad \text{and} \quad q(x) = (x + b)^2 - (b^2 + a + 4)$$

have integer roots, both $a^2 + b + 1$ and $b^2 + a + 4$ are integer squares exceeding a^2 and b^2 respectively. Since

$$a^2 + b + 1 \geq (a + 1)^2 \quad \text{and} \quad b^2 + a + 4 \geq (b + 1)^2,$$

it follows that $b \geq 2a$ and $a + 3 \geq 2b \geq 4a$. Therefore $a = 0$ or $a = 1$.

If $a = 0$, then $b^2 + 4$ is a square, which implies that $b = 0$. In this case,

$$p(x) = x^2 - 1 = (x - 1)(x + 1)$$

and

$$q(x) = x^2 - 4 = (x - 2)(x + 2).$$

If $a = 1$, then $b^2 + 5$ is a square, which implies that $b = 2$. In this case,

$$p(x) = x^2 + 2x - 3 = (x - 1)(x + 3)$$

and

$$q(x) = x^2 + 4x - 5 = (x - 1)(x + 5).$$

4624. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers with $a+b+c=3$. Prove that

$$\sqrt{\frac{ab}{2a+b+c}}+\sqrt{\frac{bc}{2b+c+a}}+\sqrt{\frac{ca}{2c+a+b}} \leq \frac{3}{2}.$$

We received 24 submissions of which 23 were correct and complete. Most solutions made explicit use of one or more of Cauchy's inequality, Jensen's inequality, and the AM-HM inequality. We present two solutions.

Solution 1, by Antonio Garcia and Nikos Ntorvas (done independently).

Let S denote the left-hand side of the inequality. Since $a+b+c=3$, and by Cauchy's inequality we have

$$\begin{aligned} S=\sqrt{\frac{ab}{3+a}}+\sqrt{\frac{bc}{3+b}}+\sqrt{\frac{ca}{3+c}} &\leq \sqrt{b+c+a}\left(\sqrt{\frac{a}{3+a}}+\sqrt{\frac{b}{3+b}}+\sqrt{\frac{c}{3+c}}\right) \\ &= \sqrt{3}\cdot\sqrt{\frac{a}{3+a}+\frac{b}{3+b}+\frac{c}{3+c}}. \end{aligned}$$

Hence, it suffices to show that

$$\frac{a}{3+a}+\frac{b}{3+b}+\frac{c}{3+c} \leq \frac{3}{4}.$$

Next, define $f(x)=\frac{x}{3+x}$ for $x>0$ so that $f''(x)=-6(3+x)^{-3}<0$. Hence, f is concave on $(0,\infty)$ and by Jensen's inequality

$$f(a)+f(b)+f(c) \leq 3\cdot f(\frac{a+b+c}{3}) = 3\cdot f(1) = \frac{3}{4}.$$

Solution 2, by the proposer and numerous others.

By the HM-AM inequality, we have the following cyclic sums:

$$\begin{aligned} \sum\frac{a}{2a+b+c}=\sum\frac{a}{(a+b)+(a+c)} &\leq \frac{1}{4}\sum\left(\frac{a}{a+b}+\frac{a}{a+c}\right) \\ &= \frac{1}{4}\sum\left(\frac{a}{a+b}+\frac{b}{b+a}\right)=\frac{3}{4}. \end{aligned}$$

Next, define $S=\sum\sqrt{\frac{ab}{2a+b+c}}$. So, by Cauchy's inequality, and since $a+b+c=3$,

$$S^2 \leq (b+c+a)\sum\frac{a}{2a+b+c} \leq 3\cdot\frac{3}{4}.$$

Hence, $S \leq \frac{3}{2}$. In fact, the conclusion holds under the hypothesis that $a+b+c \leq 3$.

4625. *Proposed by Corneliu Manescu-Avram.*

Let a, m and n be positive integers greater than 1. Prove that $a^2 + a + 1$ divides $(a+1)^m + a^n$ if and only if m is odd and 3 divides $m + n$.

We received 18 submissions, 4 of which were incomplete having only proven one direction of the "if and only if" statement. We present the solution by the editorial board.

Let $r = a^2 + a + 1$. Then modulo r, we have $a^2 \equiv -(a+1)$, $a^3 \equiv 1$ and $a^k \equiv a^t$, where $k \equiv t \pmod 3$ for some t with $0 \leq t < 3$.

If m is even, then $(a+1)^m + a^n \equiv a^{2m} + a^n \pmod r$. Modulo r, $a^{2m} + a^n$ is of the form $a^u(1 + a^v)$, where $u, v \in \mathbb{Z}$ such that $0 \leq u, v < 3$. Since $0 < 1 + a^v < r$, r does not divide $1 + a^v$, so the divisibility is never possible when m is even.

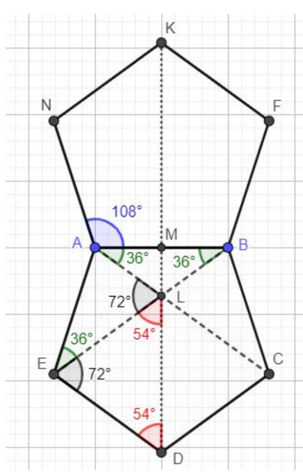
If m is odd, then $(a+1)^m + a^n \equiv -a^{2m} + a^n \pmod r$, which is divisible by r if and only if $a^m(-a^{2m} + a^n) = -a^{3m} + a^{m+n} \equiv -1 + a^{m+n} \pmod r$ is divisible by r. But $a^{m+1} \equiv 1 \pmod r$ if and only if $3|(m+n)$, so $r|((a+1)^m + a^n)$ if and only if m is odd and $3|(m+n)$, which completes the proof.

4626. *Proposed by Alpaslan Ceran.*

Let $ABCDE$ and $ABFKM$ be regular pentagons and suppose that AC intersects EB at a point L. Show that

$$\frac{|LK|}{|DL|} = \frac{\sqrt{5}+1}{2}.$$

We received 22 solutions, all of which were correct. We present the solution by Theo Koupelis.



Let a be the side length of the regular pentagon whose vertex angle is 108°. Triangles ABC and EAB are congruent and thus their base angles are equal to 36°.

Therefore $\angle EAC = \angle BED = 108° - 36° = 72°$ and thus triangle EAL is isosceles, with $\angle ELA = 72°$ and $EL = a$. As a result, triangle LED is also isosceles with base angles equal to $54°$. Therefore, if M is the midpoint of side AB, the points M, L, D are collinear and DM is the perpendicular bisector of AB. Similarly, because $ABCDE$ and $ABFKN$ are congruent, KM is the perpendicular bisector of AB, and thus the points K, M, L, D are collinear and $KM = MD$.

Triangles ALB and EAB are similar and thus $AB^2 = BL \cdot BE$ or $a^2 = BL \cdot (BL + a)$. Solving the quadratic in BL we find $BL = a \cdot \frac{\sqrt{5}-1}{2}$. From the right triangle BML we have $a = 2 \cdot BL \cdot \cos 36°$, and therefore $\cos 36° = \frac{\sqrt{5}+1}{4}$. Also, $LM = \frac{a}{2} \cdot \tan 36°$, and $DL = 2a \cdot \cos 54° = 2a \sin 36°$. Thus,

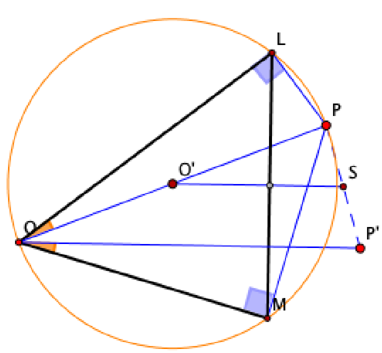
$$\frac{|LK|}{|DL|} = \frac{|LM| + |DM|}{|DL|} = 1 + 2 \cdot \frac{|LM|}{|DL|} = 1 + \frac{1}{2\cos 36°} = \frac{\sqrt{5}+1}{2}.$$

4627. *Proposed by Dong Luu.*

Let P be a point not on the circumcircle (O) of a given triangle ABC, nor on the extensions of any of its sides. Define U, V, W to be the projections of P on the lines BC, CA, AB, and X, Y, Z to be the vertices of the triangle formed by the perpendicular bisectors of the segments PA, PB, PC. Suppose that the circumcircle of ΔXYZ intersects (O) in two points, E and F. Prove that the foot of the perpendicular from P to the line EF is the circumcenter of triangle UVW.

The only submission came from the proposer himself. As you can see from his solution (which was only slightly modified by the editor), the problem is rather interesting, and so it is a mystery to this editor why it failed to attract more attention.

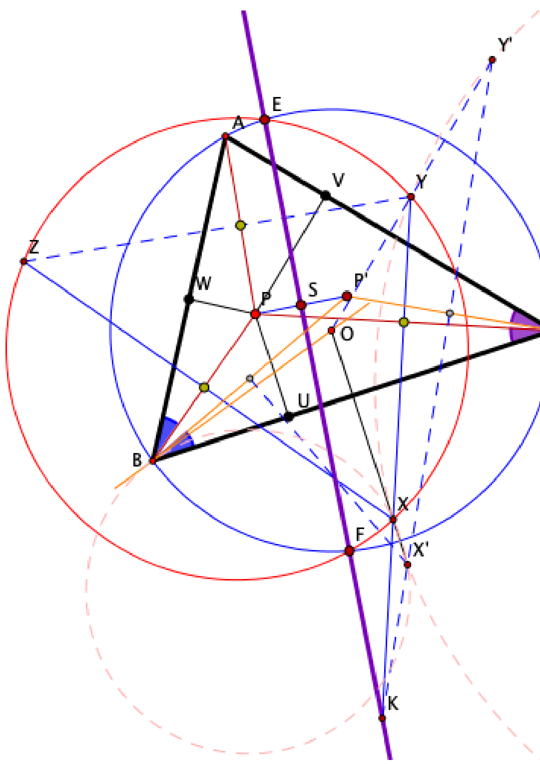
Recall that two lines through the vertex of a directed angle are isogonal with respect to that angle if they are symmetrical about the angle bisector; in other words, they share the same angle bisector.



Lemma. Given two lines, ℓ and m that intersect in a point O, and a second point P in their plane, define L and M to be the projections of P on ℓ and m. Then if

P' is an arbitrary point on the line isogonal to OP (with respect to ℓ and m), the midpoint S of PP' is equidistant from L and M.

Proof of the lemma. Because of the right angles at L and M, OP is the diameter of the circumcircle of triangle OLM. It is known (and easier to prove than to find in a reference) that the line OP (that joins the vertex O to the circumcenter, call it O') and the altitude to LM from O are isogonal with respect to the sides OL and OM; that is, OP' is perpendicular to the side LM. But the line joining O' (the circumcenter and, therefore, the midpoint of OP) to S (the midpoint of PP') is parallel to OP', and therefore is the perpendicular bisector of the chord LM of the circumcircle. Consequently, S is equidistant from L and M, as claimed.



Proof of the main result. Define P' to be the isogonal conjugate of P with respect to ΔABC. (That is, the lines joining P and P' to any vertex are isogonal with respect to the angle at that vertex.) The lemma implies that the midpoint of PP' is equidistant from the points U, V, and W, and is thus the circumcenter of ΔUVW. The problem is thereby reduced to proving that EF is the perpendicular bisector of PP'. To that end we define a new triangle $X'Y'Z'$ whose vertices are the intersections of the perpendicular bisectors of $P'A, P'B$, and $P'C$. Because X is on the perpendicular bisectors of PB and PC, we deduce that X is equidistant from B, P, and C. Similarly, X' is equidistant from those same points. It follows that X and X' are both on the perpendicular bisector of BC, whence O, X, X'

are collinear.

We now use directed angles to prove that OB is tangent to the circle (BXX'):

$$\begin{aligned}
\angle XBO &= \angle BXO + \angle XOB = \frac{1}{2}(\angle BXC + \angle COB)\\
&= \angle BPC + CAB\\
&= \angle PBC + \angle BCP + \angle CBA + \angle ACB\\
&= (\angle CBA + \angle PBC) + (\angle ACB + \angle BCP)\\
&= \angle PBA + \angle ACP\\
&= \angle CBP' + \angle P'CB\\
&= \angle CP'B = \frac{1}{2}\angle CX'B\\
&= \angle OX'B = \angle XX'B,
\end{aligned}$$

which implies that OB is tangent to (BXX'), as claimed. As a consequence, this circle is perpendicular to the circumcircle (O) of the given ΔABC, whence inversion in (O) interchanges X with X'. Similarly, this inversion interchanges Y with Y' and Z with Z' while it fixes the points E and F (which are the points common to the circles $(O), (XYZ)$, and therefore also to $(X'Y'Z')$. In other words, the line EF contains the points whose power with respect to (XYZ) equals their power with respect to $(X'Y'Z')$. Furthermore, X, X', Y, Y' are concyclic points (because any circle through a point and its inverse is perpendicular to the circle of inversion, and is therefore fixed by the inversion; that is, the line OY must meet the circle $(XX'Y)$ again at Y'). This implies that the lines XY and $X'Y'$ must intersect at a point, call it K on the line EF (because the powers $KX \cdot KX' = KY \cdot KY'$, so that K must lie on the line of equal powers with respect to the circles (XYZ) and $(X'Y'Z')$. Recall that XY is the perpendicular bisector of PC, while $X'Y'$ is the perpendicular bisector of $P'C'$; consequently, K is equidistant from P, C, and P'. Similarly, XZ intersects $X'Z'$ at a point of EF (distinct from K) that is equidistant from P and P'. We conclude that EF is the perpendicular bisector of PP'; since S is (by definition) the foot of the perpendicular from P to EF, it must be the midpoint of PP', which (by the lemma) completes the proof.

4628. *Proposed by Russ Gordon and George Stoica.*

Let A be a nonempty set of positive integers that is closed under addition and such that $\mathbb{N} \setminus A$ contains infinitely many elements. Prove that there exists a positive integer $d \geq 2$ such that $A \subseteq \{nd : n \in \mathbb{N}\}$.

We received 7 submissions and they were all complete and correct. We present two solutions.

Solution 1, by Oliver Geupel, Roy Barbara and UCLan Cyprus Problem Solving Group (done independently), slightly modified by the editor.

Let $a \in A$. Note that if $a = 1$, then $A = \mathbb{N}$, contradicting to our assumption. Thus, $a > 1$. Let $p_1, \ldots, p_k$ be the prime divisors of A. If for some $1 \leqslant i \leqslant k$ we have $A \subseteq \{np_i : n \in \mathbb{N}\}$ then we are done.

So we may assume that for each $1 \leqslant i \leqslant k$ there is an $a_i \in A$ with $p_i \nmid a_i$. Then we have $\gcd(a, a_1, \ldots, a_k) = 1$. This would imply that each positive integer $n > F(a, a_1, \ldots, a_k)$ can be written as an integer linear combination of a,a_1,. . .,a_k with non-negative coefficients, where $F(a, a_1, \ldots, a_k)$ is the Frobenius number associated to a,a_1,. . ., a_k.

Note that $\gcd(a, a_1, \ldots, a_k) = 1$ implies that $F(a, a_1, \ldots, a_k) < \infty$. In other words, $\mathbb{N} \setminus A$ is finite, a contradiction.

Solution 2, by the proposer, slightly modified by the editor.

Fix some $k \in A$. We have that $k \geq 2$, otherwise $A = \mathbb{N}$, a contradiction.

In the following discussion, we would identify $i \in \mathbb{Z}_k$ with its smallest positive residue modulo k. For each $i \in \mathbb{Z}_k$, we denote

$$[i] := \{i, i+k, i+2k, i+3k, \ldots\}, \quad A_i = [i] \cap A.$$

By the additivity assumption, if $A_i \neq \emptyset$, then $[i] \setminus A_i$ is finite.

Let $G = \{i \in \mathbb{Z}_k : A_i \neq \emptyset\}$. We claim that G is a subgroup of $\mathbb{Z}_k$. Indeed, if $i, j \in G$, then $i + j \in G$ and $-i = (k-1)i \in G$ since A is closed under addition.

Note that

$$\mathbb{N} \setminus A = \cup_{i \in \mathbb{Z}_k}([i] \setminus A_i)$$

is an infinite set, so there must exist some $j \in \mathbb{Z}_k$ such that $[j] \setminus A_j$ is an infinite set, i.e., $A_j = \emptyset$. This implies that G is a proper subgroup of $\mathbb{Z}_k$, so $G = d\mathbb{Z}_k$ for some $d > 1$ and $d \mid k$.

We conclude that $A = \cup_{i \in G} A_i \subseteq \{nd : n \in \mathbb{N}\}$.

4629. *Proposed by Minh Nguyen.*

Determine the smallest natural number n such that in any n-element subset of $\{1, 2, \ldots, 2020\}$ there exist four different a,b,c,d satisfying $a+2b+3c+4d \leq 2020$.

We received 9 submissions and 8 of them were complete and correct. We present the solution by the majority of the solvers.

Using the rearrangement inequality, the minimum value of $S := a + 2b + 3c + 4d$ is achieved when $a > b > c > d$. Because the n-element subset could also include the greatest n numbers from the set, the minimum value of S is achieved when d is the smallest number in the subset and $(a, b, c, d) = (a, a-1, a-2, a-3)$. Therefore $S_{\min} = 10a - 20$ and the inequality $S \leq 2020$ leads to $a = 204$ (and thus $d = 201$) being the smallest value of a in the subset of the greatest n numbers from the set for which the given inequality is satisfied. Thus, the smallest such n is $2020 - 200 = 1820$.

4630. *Proposed by George Stoica.*

We consider an equilateral triangle ABC with the circumradius 1 and a point D on or inside its circumcircle. Prove that $3 \leq AD + BD + CD \leq 4$.

There were 8 correct and 4 incomplete solutions submitted. We present two solutions.

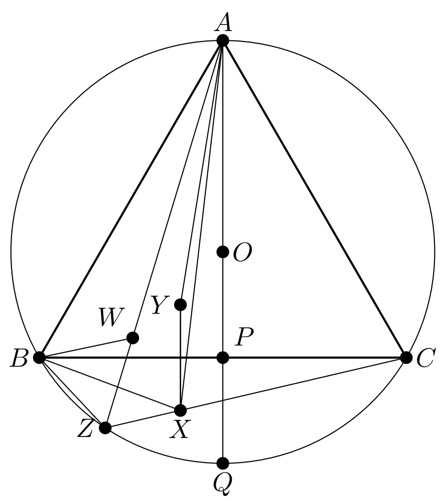
Solution 1, using the ingredients of several submissions.

Let O be the centre of the circumcircle of triangle ABC, P be the midpoint of BC, and AQ be the diameter of the circle through P. By symmetry, it suffices to consider points in the sector OBC of the circumcircle.

Observe that R, the sector of the circle bounded by the chord and short arc joining B and C is reflected with axis BC to a region S that contains triangle OBC. Let $X \in R$ and $Y \in S$ be corresponding points with respect to the reflection. Since $\angle AYX$ exceeds $90°$, $AY \leq AX$ whence

$$AY + BY + CY \leq AX + BX + CX.$$

Therefore, to find the maximizing position of D, we need only look in R while to find the minimizing position, in S.



For the maximizing problem, suppose that D is at position X in R. Let CX produced meet the arc BC at Z. Since $\angle AZX = \angle AZC = 60°$ and $\angle ZAX \leq 60°$, $\angle AXZ \geq 60°$ and so $AZ \geq AX$. Hence

$$\begin{aligned} AX + BX + CX &\leq AX + BZ + ZX + CX \\ &\leq AZ + BZ + CZ \\ &= 2AZ \leq 2AQ = 4, \end{aligned}$$

with equality if and only if $X = Z = Q$.

The last inequality follows since AQ is a diameter of the circumcircle. The fact that $BZ + CZ = AZ$ needs more argument. Let W be located on AZ so that $WZ = BZ$. Since $\angle AZB = \angle ACB = 60^\circ$, triangle BWZ is equilateral and $WB = BZ$. Since $\angle BAW = \angle BCZ$ and

$$\angle BWA = 180^\circ - \angle BWZ = 120^\circ = \angle BZC,$$

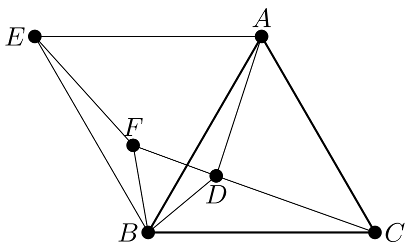
it follows that $\angle ABW = \angle CBZ$. Because $BA = BC$ and $BW = BZ$, triangles ABW and CBZ are conguent (SAS) and so

$$BZ + CZ = ZW + WA = AZ.$$

For the minimizing problem, suppose that D lies within the triangle ABC. A 60° rotation with centre B takes C to A, A to E and D to F, so that $AEBC$ is a $60^\circ - 120^\circ$ rhombus. Since $BF = BD$, triangle BDF is equilateral and $BD = FD$. Also $AD = EF$. Hence

$$AD + BD + CD = EF + FD + DC \geq EC.$$

But CE right bisects AB, and so CE is equal to twice the altitude of triangle ABC, namely 3. Equality occurs if D coincides with the circumcenter O.



Solution 2, by UCLan Cyprus Problem Solving Group.

We need consider only points in the sector OBC. Consider first the maximization problem. If D lies within the triangle OBC, then an argument similar to that in Solution 1 establishes that

$$BD + CD \leq BO + CO = 2,$$

Since AD is less than a diameter of the circumcircle,

$$AD + BD + CD \leq 2 + 2 = 4.$$

Now let D lie in the segment bounded by the chord and arc BC, and let AD intersect the circle again at G. By Ptolemy's theorem

$$AB \cdot CG + AC \cdot BG = BC \cdot AG \quad \text{whence} \quad CG + BG = AG.$$

Therefore

$$AD + BD + CD \leq AG + BG + CG = 2AG \leq 4,$$

with equality if and only if $D = G$ and AG is a diameter.

For the minimization problem, let $\mathbf{x} = \overrightarrow{OX}$ for $X = A, B, C, D$ and corresponding lower cases. Then

$$(\mathbf{a} - \mathbf{d}) \cdot \mathbf{a} \leq \|\mathbf{a} - \mathbf{d}\| \|\mathbf{a}\| = \|\mathbf{a} - \mathbf{d}\| = AD$$

with similar assertions for $\mathbf{b}$ and $\mathbf{c}$. Then

$$\begin{aligned} AD + BD + CD &\geq (\mathbf{a} - \mathbf{d}) \cdot \mathbf{a} + (\mathbf{b} - \mathbf{d}) \cdot \mathbf{b} + (\mathbf{c} - \mathbf{d}) \cdot \mathbf{c} \\ &= 3 - \mathbf{d} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) \\ &= 3 - \mathbf{d} \cdot \mathbf{0} \\ &= \mathbf{3}, \end{aligned}$$

since ABC is an equilateral triangle.