Crux Mathematicorum

Volume/tome 47, issue/numéro 3
March/mars 2021

Canadian Mathematical Society
Société mathématique du Canada

Crux Mathematicorum is a problem-solving journal at the secondary and university undergraduate levels, published online by the Canadian Mathematical Society. Its aim is primarily educational; it is not a research journal. Online submission:

`https://publications.cms.math.ca/cruxbox/`

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire publiée par la Société mathématique du Canada. Principalement de nature éducative, le Crux n'est pas une revue scientifique. Soumission en ligne:

`https://publications.cms.math.ca/cruxbox/`

ISSN 1496-4309 (Online)

ISSN 1496-4309 (électronique)

Supported by / Soutenu par :

- Intact Financial Corporation
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Editorial Board

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MathemAttic

No. 23

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **May 15, 2021**.

MA111. Peter and Steve had a 50-metre race. When Peter crosses the finish line, Steve was 5 metres behind. A rematch is set up, with Peter handicapped by being 5 metres behind the starting line. Assuming that the boys run at the same speed as before, who wins the race (or is it a tie) and by how much?

MA112. Circle C_1 has radius 13 and is tangent to line ℓ at Y. Circle C_2 has radius 23, is tangent to ℓ at Z and is externally tangent to C_1 at X. The line through X and Y intersects C_2 at P. Determine the length of PZ.

MA113. For any positive number t, $\lfloor t \rfloor$ denotes the integer part of t and $\{t\}$ denotes the "decimal" part of t. If $x + \{y\} = 7.32$ and $y + \lfloor x \rfloor = 8.74$, then determine $\{x\}$.

MA114. Let p, q, and r be positive constants. Prove that at least one of the following equations has real roots.

$$px^2 + 2qx + r = 0$$
$$rx^2 + 2px + q = 0$$
$$qx^2 + 2rx + p = 0$$

MA115. I met a person the other day that told me they will turn x years old in the year x^2. What year were they born?

. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA111. Lors d'une première course de 50 mètres, Pierre a devancé Stéphane par 5 mètres. Lors de la deuxième, Pierre a donc commencé 5 mètres derrière la ligne usuelle de départ. Supposant que les vitesses des deux coureurs demeurent toujours les mêmes, qui terminera premier dans cette deuxième course et par combien, ou est-ce un match nul?

MA112. Un cercle C_1 de rayon 13 est tangent à la ligne ℓ en Y. Le cercle C_2 de rayon 23 est tangent à la ligne ℓ en Z et est extérieurement tangent à C_1 en X. Enfin, la ligne passant par X et Y intersecte C_2 en P. Déterminer la longueur de PZ.

MA113. Pour un nombre réel positif t, $\lfloor t \rfloor$ dénote sa partie entière et $\{t\}$ dénote sa partie fractionnaire. Si $x + \{y\} = 7.32$ et $y + \lfloor x \rfloor = 8.74$, déterminer $\{x\}$.

MA114. Soient p, q et r des réels positifs. Démontrer qu'au moins une des équations suivantes a des racines réelles:

$$px^2 + 2qx + r = 0$$
$$rx^2 + 2px + q = 0$$
$$qx^2 + 2rx + p = 0$$

MA115. L'autre jour, j'ai rencontré quelqu'un m'affirmant qu'il va atteindre l'âge de x ans dans l'an x^2. En quelle année cette personne aurait-elle été née ?

MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(8), p. 349–351.

MA86. On a $2 \times n$ board, you start from the square at the bottom left corner. You are allowed to move from square to adjacent square, with no diagonal moves, and each square must be visited at most once. Moreover, two squares visited on the path may not share a common edge unless you move directly from one of them to the other. We consider two types of paths, those ending on the square at the top right corner and those ending on the square at the bottom right corner. The diagram below shows that there are 4 paths of each type when $n = 4$. Prove that the numbers of these two types of paths are the same for $n = 2014$.

Originally problem 5 of the 2014 Alberta High School Mathematics Competition.

We received 3 submissions, all of which were correct and complete. We present the solution by the Sigma Problem Solving Group.

Let x_n be the number of paths ending at the top right corner, and y_n be the number of paths ending at the bottom right corner, for $n \geq 4$.

Label the squares of the grid as (a, b), where $a = 1, 2, ..., n$ from right to left, and $b = 1, 2$ from bottom to top. That is, the bottom right corner is $(1, 1)$ and the top left corner is $(n, 2)$.

Firstly, we have the initial values, by direct counting, $x_4 = 4, x_5 = 6$ and $y_4 = 4, y_5 = 7$.

Consider now, a $2 \times (n + 1)$ square grid labelled as above. Beginning at $(n + 1, 1)$ there are two choices, to go right, to $(n, 1)$, or to go up, to $(n + 1, 2)$. If we move right, then we have x_n paths to $(1, 2)$ and y_n paths to $(1, 1)$.

If we move up from $(n + 1, 1)$, then we have to move right at least twice and reach $(n - 1, 2)$. Reaching the top right corner from $(n - 1, 2)$ is equivalent to reaching the bottom right corner from $(n - 1, 1)$, as can be seen by flipping the board vertically. Hence we have y_{n-1} paths to reach the top right corner from $(n - 1, 2)$ and similarly, x_{n-1} paths to reach the bottom right corner from there. Thus, we get the coupled recursions, for $n \geq 5$,

$$x_{n+1} = x_n + y_{n-1}$$
$$y_{n+1} = y_n + x_{n-1}$$

We define a new sequence, $z_n = x_{n+4} + y_{n+4}, n \geq 0$. Then subtracting the above equations, we have,

$$\begin{aligned} & z_{n-3} = z_{n-4} - z_{n-5},\ n \geq 5 \\ \implies & z_{n+2} = z_{n+1} - z_n, \quad n \geq 0 \end{aligned} \tag{1}$$

The first 7 values of z_n, beginning from $n = 0$ are: $0, -1, -1, 0, 1, 1, 0$. We have, from (1),

$$\begin{aligned} & z_{k+3} = z_{k+2} - z_{k+1} \\ \implies & z_{k+3} = z_{k+1} - z_k - z_{k+1} \\ \implies & z_{k+3} = -z_k \\ \implies & z_{k+6} = -z_{k+3} = z_k \end{aligned}$$

Thus, for $k \geq 0$, $z_{k+6} = z_k$ and as $z_0 = 0$, we get that $z_{6k} = 0$.

Hence, we have that $x_{6k+4} = y_{6k+4}$, for $k \geq 0$. Since $2014 = 6 \cdot 335 + 4$, there are an equal number of both paths on a 2×2014 board.

MA87. One or more pieces of clothing are hanging on a clothesline. Each piece of clothing is held up by either 1, 2 or 3 clothespins. Let a_1 denote the number of clothespins holding up the first piece of clothing, a_2 the number of clothespins holding up the second piece of clothing, and so forth. You want to remove all the clothing from the line, obeying the following rules:

(i) you must remove the clothing in the order that they are hanging on the line;

(ii) you must remove either 2, 3 or 4 clothespins at a time, no more, no less;

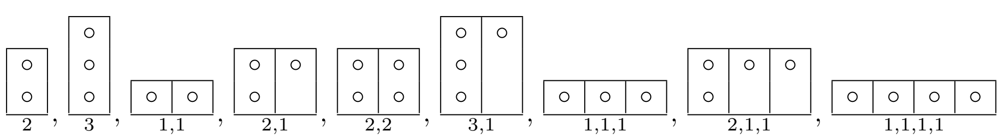
(iii) all the pins holding up a piece of clothing must be removed at the same time.

Find all sequences $a_1, a_2, \ldots, a_n$ of any length for which all the clothing can be removed from the line.

Originally problem 3 of the 2017 Alberta High School Mathematics Competition.

We received 2 submissions of which 1 was correct and complete. We present the solution by Zachary Cormack.

What combinations of clothespins can be removed? There are nine possibilities, up to isomorphic permutation:

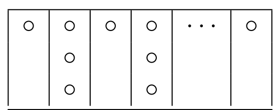


2, 1 is the same as 1, 2 in terms of being a group of 3 being removed, or 2, 1, 1 being the same as 1, 2, 1 for a group of 4 being removed, for example.

All arrangements made of these groups work. So, what sequences don't?

Divide the line of clothes into groups taken from the nine groups illustrated above, but not using the group 3,1 with possibly some isolated 1's left over. A single 1 cannot be removed by itself; let's call this an island. The removable groups (everything that's not an island) will be called seas. Call an island "sunk" if it can be grouped together with a sea on one side and the combined set removed (possibly in two moves, for instance if an island were grouped with a $\{2, 2\}$ sea). The clotheslines that work are those in which all islands may be sunk.

A sea of a single 3 can only sink an island on one side. All other seas can sink 1 or 2 islands. The number of islands can be no more than the number of seas plus one. It follows that, if any of the seas can sink two islands, then clearly, all islands may be sunk. And so, the islands cannot be sunk if and only if all seas consist of a 3, and even then, there must be more islands than seas. The sequences which don't work must clearly only be of this form:



$1, 3, 1, 3, \ldots, 1$

Therefore, any clothesline in which the number of pins does not alternate between one and three, and begin and end with one, may have all clothes removed from it.

Note. Proposal 4601 in *Crux* Volume 47, issue 1 is an extension of this problem.

MA88. *Proposed by Konstantin Knop.*

a) Sort the numbers from 1 to 100 in increasing order of their digit-sums; in case of a tie, sort in increasing order of the numbers themselves. Consider the resulting sequence

$$a(n): \ a(1) = 1, a(2) = 10, a(3) = 100, \ldots$$

Find at least one number $n > 1$ such that $a(n) = n$.

b) Consider the same problem but for numbers from 1 to 100 000 000.

We received 3 submissions, of which 1 was correct and complete. We present the solution by Richard Hess, with help by Josh Jordan.

a) By ordering and counting we find that for the first 100 numbers we get $a(24) = 24$ and $a(97) = 97$.

b) For the case of the first 100 million numbers I got lucky early to find $a(11) = 11$. My friend Josh Jordan wrote a program and found the following:

$$\begin{aligned} &a(11) = 11, \\ &a(13212) = 13212, \\ &a(12951621) = 12951621, \\ &a(91686881) = 91686881. \end{aligned}$$

MA89. *Proposed by Bill Sands.*

Two robots R2 and D2 are at the origin O on the x, y plane. R2 can move twice as fast as D2. There are two treasures located on the plane, and whichever robot gets to each treasure first gets to keep it (in case of a tie, neither robot gets the treasure). One treasure is located at the point $P = (-3, 0)$, and the other treasure is located at a point $X = (x, y)$. Find all $X \neq O$ so that D2 can prevent R2 from getting both treasures, no matter what R2 does. Which such X has the largest value of y?

Note: D2 does not care if R2 gets one of the treasures, only that R2 shouldn't get both treasures. D2 also doesn't care if it gets either treasure itself, it only wants to prevent R2 from getting both treasures.

We received one partial solution to the problem, by Richard Hess. The solution below is based on his submission.

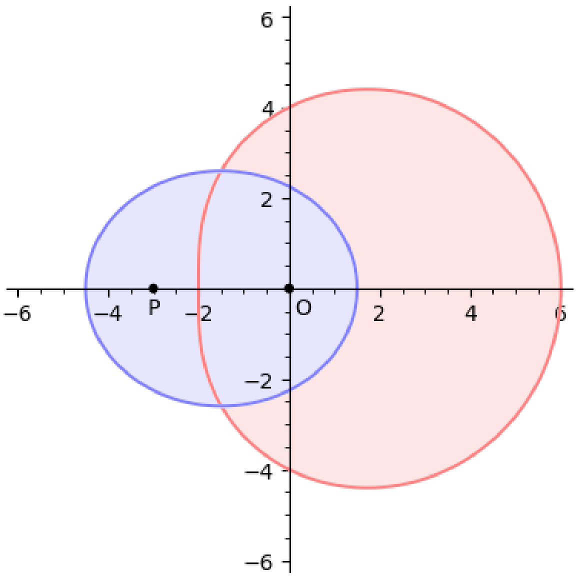


Figure 1: D2 can win if X is within the red shaded region or on its boundary.

If $|OX| + |XP| < 2|OP|$ then R2 can move to X and then to P before D2 can reach P. Therefore R2 wins if

$$\sqrt{x^2 + y^2} + \sqrt{(x+3)^2 + y^2} < 6. \tag{1}$$

This inequality describes the inside of the ellipse with foci O and P and major axis length 6 (marked in blue in Figure 1).

Similarly, R2 can win if $|OP| + |PX| < 2|OX|$ or

$$3 + \sqrt{(x+3)^2 + y^2} < 2\sqrt{x^2 + y^2}. \tag{2}$$

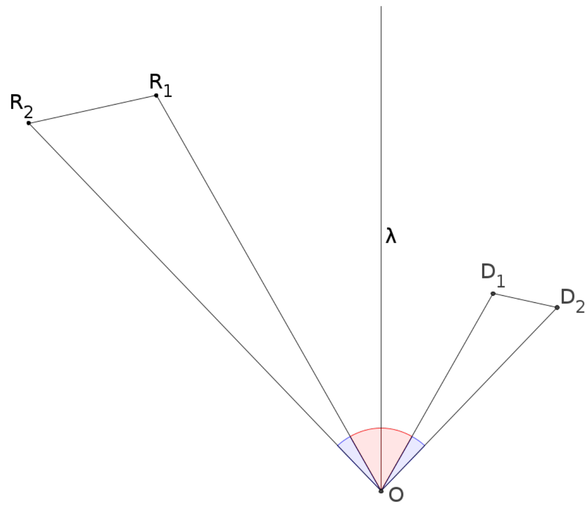


Figure 2: Strategy for D2.

The region described by this inequality is the region on the outside of the red curve in Figure 1. We will show that D2 can win if X satisfies neither (1) nor (2), i.e. is in the red shaded region in Figure 1 or on its boundary.

Let λ be the angle bisector of $\angle POX$. Until R2 picks up the first treasure, D2 will move as follows. If R2 moves to a point R, then D2 will move to a point D such that $|OR| = 2|OD|$ and such that the angle between OD and λ is equal to the angle between λ and OR (D2 and R2 will be on opposite sides of λ). To see that this is always possible suppose R2 and D2 have already moved to points R_1 and D_1 respectively (see Figure 2), thus $|OR_1| = 2|OD_1|$. Suppose R2 now moves to a point R_2 and let D_2 be the corresponding point for D2 to move to. Then $|OR_2| = 2|OD_2|$ and $\angle R_1OR_2 = \angle D_1OD_2$ which implies that OR_1R_2 and OD_1D_2 are similar and thus $|R_1R_2| = 2|D_1D_2|$. So D2 can reach D_2 at the same time that R2 reaches R_2.

Now suppose R2 picks up a treasure, say P. By the strategy outlined, D2 will be at a point D that lies on OX with property $|OP| = 2|OD|$. By assumption $|OP| + |PX| \geq 2|OX|$ and thus

$$|PX| = |OP| + |PX| - |OP| \geq 2|OX| - 2|OD| = 2(|DX|).$$

Therefore R2 cannot reach X before D2.

The largest value of y among all winning points $X = (x, y)$ clearly occurs on the curve defined by $3 + \sqrt{(x+3)^2 + y^2} = 2\sqrt{x^2 + y^2}$. Solving for y^2 we obtain (...insert long and tedious algebra...)

$$y^2 = -x^2 + 2x + 8 + 4\sqrt{2x + 4}.$$

With a bit of calculus, we find that this function has a maximum at $x = \sqrt{3}$, for which

$$y^2 = -3 + 2\sqrt{3} + 8 + 4\sqrt{2\sqrt{3}+4} = 5 + 2\sqrt{3} + 4(1+\sqrt{3}) = 9 + 6\sqrt{3}$$

and thus the largest value of y is obtained at $X = (\sqrt{3}, \sqrt{9+6\sqrt{3}})$.

MA90. *Proposed by Michel Bataille.*

Two positive integers are called co-prime if they share no common divisors other than 1. Find all pairs of co-prime x, y such that $\dfrac{y(x+y)}{x-y}$ is a positive integer.

We received 7 correct solutions, some not very elegant and one running to twelve pages. We present the solution by Basu Aaratrick, and Richard Hess (done independently).

Note that the requirement that $\dfrac{y(x+y)}{x-y}$ is a positive integer implies that $x > y$.

$$\frac{y(x+y)}{x-y} = y + \frac{2y^2}{x-y}.$$

Since y and $x-y$ are coprime, $x-y$ must divide 2, and so is equal to 1 or 2. Therefore,

$$(x, y) = (u+1, u), (v+2, v),$$

where u is any positive integer and v is any positive odd integer.

Editor's comment. Note that $y(x+y)/(x-y) \in \mathbb{Z} \Leftrightarrow x(x+y)/(x-y) \in \mathbb{Z}$.

PROBLEM SOLVING VIGNETTES

No. 15

Shawn Godin

Divisibility Rules

At the heart of number theory, as well as many problems dealing with numbers, lies the fundamental theorem of arithmetic. This theorem states that any integer greater than 1 is prime or can be decomposed uniquely (ignoring order) into a product of primes. Hence, many problems rely on being able to factor an integer into its prime factors. To help us in our exploration we will be using congruences that were introduced by Don Rideout [2019 : 45(3), p. 118-121] and have been used in other columns in this series.

Throughout our investigation, we will use the fact that a n-digit positive integer N can be represented as

$$N = a_{n-1} \times 10^{n-1} + a_{n-2} \times 10^{n-2} + \cdots + a_2 \times 10^2 + a_1 \times 10 + a_0$$

where $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $a_{n-1} \neq 0$. We will derive some divisibility rules based on properties of powers of ten modulo a power of a prime. This column is inspired by some explorations of grade 9 student Priya Janaky Aiyer that were shared with me. It is my hope that Priya will continue her mathematical explorations.

We will start our journey with one of the easiest and most well known divisibility rules, that for 2. In grade school, children learn that a number is divisible by 2 (is even) if its units digit is 0, 2, 4, 6 or 8. If we look, modulo 2, then a number N is divisible by 2 if it is congruent to 0, hence

$$N = a_{n-1} \times 10^{n-1} + a_{n-2} \times 10^{n-2} + \cdots + a_2 \times 10^2 + a_1 \times 10 + a_0 \equiv a_0 \equiv 0 \pmod 2$$

since $10 \equiv 0 \pmod 2$. As $a_0 \equiv 0 \pmod 2 \Rightarrow a_0 \in \{0, 2, 4, 6, 8\}$, we see why the divisibility rule works.

Since $2 \mid 10$ we must have $2^n \mid 10^k$ for any positive integers n and k with $k \geq n$. As such, we can extend our rule. For example, if we consider $4 = 2^2$, then $4 \mid 10^k$ with $k \geq 2$, and so

$$N = a_{n-1} \times 10^{n-1} + a_{n-2} \times 10^{n-2} + \cdots + a_2 \times 10^2 + a_1 \times 10 + a_0 \equiv a_1 \times 10 + a_0 \pmod 4$$

so we can conclude that any number N is divisible by 4 if and only if the two digit number formed by the tens and units digits of N is divisible by 4. Thus since $4 \mid 76$ we can conclude that $4 \mid 147\,976$, indeed *any* integer whose tens and units digits are 76 will be divisible by 4.

We can extend this idea so that any number where the three digit number formed by the last three digits is divisible by $8 = 2^3$ is itself divisible by 8; any number where the four digit number formed by the last four digits is divisible by $16 = 2^4$ is itself divisible by 16; and, in general, any number where the n digit number formed by the last n digits is divisible by 2^n is itself divisible by 2^n. This means that when dividing by 2^n, we only need to consider the last n digits in our division. In some cases this might be a significant reduction in our work-load, but in others the reduction might be minimal. We can actually use our knowledge of powers of two to reduce things further.

If we reexamine the divisibility by 4, we remove all digits corresponding to powers of 10 two and larger because $4 = 2^2 \mid 2^2 \times 5^2 = 100$, but $4 \mid 2^2 \times k$ for *any* integer k, hence $4 \mid 2^2 \times 5 = 20$. How does that help? When we remove the digits other than the tens and units, we have removed a multiple of 10^2, which is a multiple of 4. We can do the same for 20. So, if we reexamine 147 976, we get

$$147\ 976 \equiv 76 \equiv 76 - 20 = 56 \equiv 36 \equiv 16 \pmod 4 .$$

Another way to look at things, since we are removing multiples of 20, the units digit is unchanged. All we are looking at is the tens digit *modulo* 2! Hence, we can think of the divisibility rule for 2^n as a series of $n-1$ steps, which reduce our original number to something it is equivalent to, modulo 2^n. Then the final step is dividing a much smaller number by 2^n, indeed the final number should be less than 10×2^n. For example, since $4 = 2^2$, then $147\ 976 \equiv 76 \pmod 4$, and since $7 \equiv 1 \pmod 2$, then $76 \equiv 16 \pmod 4$. This process can be extended for other powers of 2.

For example to check the divisibility of N by $8 = 2^3$ we do the following process:

- consider $N_1 = a_2 \times 10^2 + a_1 \times 10 + a_0$, the number formed by the last three digits, since $8 \mid 10^3 = 2^3 \times 5^3$,
- since $8 \mid 200 = 2^3 \times 5^2$, reduce a_2 modulo 2 (to either 0 or 1), let this be a_2' and we are now considering the number $N_2 = a_2' \times 10^2 + a_1 \times 10 + a_0$,
- since $8 \mid 40 = 2^3 \times 5$, reduce $a_2' \times 10 + a_1$ modulo 4 (to either 0, 1, 2 or 3), let this be a_1' and we are now considering the number $N_3 = a_1' \times 10 + a_0$.
- $8 \mid N$ if and only if $8 \mid N_3$.

So for our example we have $N = 147\ 976$, so $N_1 = 976$ (we have just subtracted 147×10^3 from N). Since $9 \equiv 1 \pmod 2$, $N_2 = 176$ (we have just subtracted 4×200 from N_1). Next, $17 \equiv 1 \pmod 4$ so $N_3 = 16$. Finally, $8 \mid 16$ and therefore $8 \mid 147\ 976$.

Next we will move on to divisibility by 3 and 9. This was talked about by Don Rideout in the aforementioned article, but we will reproduce the result here as the ideas will be expanded on later. If we consider a number N modulo 9, then, since $10 \equiv 1 \pmod 9$ we get

$$\begin{aligned} N &= a_{n-1} \times 10^{n-1} + a_{n-2} \times 10^{n-2} + \cdots + a_2 \times 10^2 + a_1 \times 10 + a_0 \\ &\equiv a_{n-1} \times 1^{n-1} + a_{n-2} \times 1^{n-2} + \cdots + a_2 \times 1^2 + a_1 \times 1 + a_0 \pmod 9 \\ &\equiv a_{n-1} + a_{n-2} + \cdots + a_2 + a_1 + a_0 \pmod 9 \end{aligned}$$

hence a number is congruent to the sum of its digits, modulo 9.

At this point we introduce the *digital root* of a number. To calculate the digital root of a number N, first calculate the sum of its digits. If this is a single digit number, you are done. If it is not, then continue calculating the digital sum of the results until you are left with a single digit. The final single digit is the digital root of N. We can conclude that a number is congruent to its digital root modulo 9, I will leave the details to the reader. Hence we get the following divisibility rules for a positive integer N:

If the digital root of N is 9, then N is divisible by 9.

If the digital root of N is 3, 6, or 9, then N is divisible by 3.

For example, the digital root of 853 640 928 is 9, since

$$8+5+3+6+4+0+9+2+8=45$$

and $4+5=9$, hence $9 \mid 853\ 640\ 928$, and so it is also a multiple of 3. Similarly, the digital root of 147 265 404 is 6, hence $9 \nmid 147\ 265\ 404$, but since $3 \mid 6$, we have $3 \mid 147\ 265\ 404$.

So far we have only talked about divisibility rules for primes and prime powers. It turns out that these types of rules are the only ones necessary. For example, to test the divisibility by $6 = 2 \times 3$ we just use the rules for 2 and 3. Hence 63 945 is not divisible by 6 since it fails the divisibility test for 2, having an odd units digit. Similarly 29 038 is not divisible by 6 since its digital root is 4.

Next we will look at divisibility by 11 as it closely resembles that of 9. Since $10 \equiv -1 \pmod{11}$ then, considering N modulo 11 we get

$$\begin{aligned}
N &= a_{n-1} \times 10^{n-1} + a_{n-2} \times 10^{n-2} + \cdots + a_2 \times 10^2 + a_1 \times 10 + a_0 \\
&\equiv a_{n-1} \times (-1)^{n-1} + a_{n-2} \times (-1)^{n-2} + \cdots + a_2 \times (-1)^2 + a_1 \times (-1) + a_0 \pmod{11} \\
&\equiv a_0 - a_1 + a_2 - \cdots + a_{n-1} \times (-1)^{n-1} \pmod{11}
\end{aligned}$$

So a number is divisible by 11 if its *alternating* digital sum is divisible by 11. Hence since

$$9-2+1-4+7-4+2-1+3=11,$$

we can conclude $11 \mid 312\ 474\ 129$. If we are only interested in divisibility, we may do the alternating sum from the *leftmost* digit as this will be equal to the alternating digital sum or its opposite. Hence since

$$8-4+2-8+4-0+3-6+0-3=-4,$$

we can conclude that $11 \nmid 8\ 428\ 403\ 603$, but $8\ 428\ 403\ 603 \equiv 4 \not\equiv -4 \pmod{11}$ since the leftmost digit corresponds to an *odd* power of 10.

An interesting extension of this idea can combine several divisibility tests into one. We start by rewriting any number in terms of powers of 1000 instead of 10. That is, we can consider $312\ 474\ 129 = 312 \times 1000^2 + 474 \times 1000 + 129$. Then, since $1000 \equiv -1 \pmod{1001}$, looking at 312 474 129 modulo 1001 we get

$$\begin{aligned} 312\ 474\ 129 &= 312 \times 1000^2 + 474 \times 1000 + 129 \\ &\equiv 312 \times (-1)^2 + 474 \times (-1) + 129 \pmod{1001} \\ &\equiv 312 - 474 + 129 \pmod{1001} \\ &\equiv -33 \pmod{1001} \end{aligned}$$

Therefore, since $1001 = 7 \times 11 \times 13$, then we can conclude that

$$\begin{aligned} 312\ 474\ 129 &\equiv -33 \equiv 2 \pmod{7} \\ 312\ 474\ 129 &\equiv -33 \equiv 0 \pmod{11} \\ 312\ 474\ 129 &\equiv -33 \equiv 6 \pmod{13} \end{aligned}$$

and hence $7 \nmid 312\ 474\ 129$, $11 \mid 312\ 474\ 129$, and $13 \nmid 312\ 474\ 129$. We have performed three divisibility tests at once! The method will always take any positive integer N and, by looking at blocks of three digits starting from the right (the last block may have 1, 2 or 3 digits) and performing the alternating sum we can check the divisibility by 7, 11 and 13 by checking, at largest, a 3-digit number.

We will finish by looking at another divisibility test for 7, also explored in Rideout's article, and show a second version of it. Let's consider the following known divisibility test for 7:

> *A number $N > 10$ is divisible by 7 if and only if the number formed by subtracting twice the units digit from the number formed by the remaining digits is also divisible by 7.*

For example, $23 \times 7 = 161$ is divisible by 7 as well as $16 - 2 \times 1 = 14$. For larger multiples of 7 we can repeat the process as long as we want. For example, $94\ 654 \times 7 = 662\ 578$ is a multiple of 7. Iterating the rule we get

$$\begin{aligned} 662\ 578 &\to 66\ 257 - 2 \times 8 = 66\ 241 \\ &\to 6\ 624 - 2 \times 1 = 6\ 622 \\ &\to 662 - 2 \times 2 = 658 \\ &\to 65 - 2 \times 8 = 49 \end{aligned}$$

and since $49 = 7 \times 7$, it shows our original is also a multiple of 7 (or we could have gone one step further and gotten $4 - 2 \times 9 = -14$ which is also a multiple of 7).

So why does this work? If we write $N = 10a + b$ then

$$\begin{aligned} 7 \mid N \Leftrightarrow && N &\equiv 0 \pmod{7} \\ \Leftrightarrow && 10a + b &\equiv 0 \pmod{7} \\ \Leftrightarrow && 3a - 6b &\equiv 0 \pmod{7} \\ \Leftrightarrow && a - 2b &\equiv 0 \pmod{7} \;\square \end{aligned}$$

Can we get a similar result using the last two digits of N? That is if we write $N = 100a + b$ then

$$\begin{aligned} 7 \mid N \Leftrightarrow && N &\equiv 0 \pmod 7 \\ \Leftrightarrow && 100a + b &\equiv 0 \pmod 7 \\ \Leftrightarrow && 2a - 6b &\equiv 0 \pmod 7 \\ \Leftrightarrow && a - 3b &\equiv 0 \pmod 7 \end{aligned}$$

which means that we can do the same thing using the last two digits, which speeds things up a bit. However instead of subtracting twice the last digit we subtract 3 times the 2-digit number formed by the last two digits. Our example above then becomes

$$\begin{aligned} 662\,578 &\to 6\,625 - 3 \times 78 = 6\,391 \\ &\to 63 - 3 \times 91 = -210 \end{aligned}$$

which is a multiple of 7. The process needs fewer steps but the steps are a little more difficult. Notice if we went one step further and wrote $N = 1000a + b$ it leads to $7 \mid N \Leftrightarrow 7 \mid (a - b)$ (I will leave the details to you), which is related to the method we devised for 7, 11 and 13 developed earlier.

Hopefully these divisibility tests will aid in your problem solving efforts. You may amuse yourself by using these tests to factor integers into prime factors. Other divisibility tests can be developed using congruences. I leave the following explorations for your amusement.

1. Since $10 = 2 \times 5$, develop a method for testing divisibility by powers of 5 analogous to the one we developed for powers of 2.
2. In his TED talk (`youtu.be/M4vqr3_R0Ik`), Arthur Benjamin performs a "trick" where he gets volunteers take a number that had come up during his presentation and to multiply it by *any* three digit number. The volunteers then recite the digits of the result back to him *in any order*, leaving one of the digits out. At that point he tells them the digit that was left out. Determine how he did it. (This trick occurs at 5:45 to 7:45 on the video.)
3. Given $N = 10a + b$, prove that $11 \mid N \Leftrightarrow 11 \mid (a - b)$. Extend this to larger blocks of numbers as we did for the divisibility rule for 7.
4. Given $N = 10a + b$, prove that $13 \mid N \Leftrightarrow 13 \mid (a - 4b)$. Extend this to larger blocks of numbers as we did for the divisibility rule for 7.
5. Observing that $999 = 27 \times 37$, and $1000 \equiv 1 \pmod{999}$ develop a divisibility rule for 27 and 37 similar to the one developed for 7, 11, and 13. (Hint, it will involve *adding* blocks of three digits.)
6. Generalize the result of the last problem by observing that

$$10^n - 1 = \overbrace{999\ldots 99}^{n\text{ 9s}} = 9 \times \overbrace{111\ldots 11}^{n\text{ 1s}}$$

so, if we can factor the *rep-unit number* $\overbrace{111\ldots11}^{n\ 1\text{s}}$ (see Don Rideout's article as well as #6 of this column [2019: 45(6), p. 313-317]), we can develop a simultaneous test for all its factors (and 9) by observing that

$$10^n \equiv 1 \left(\mod \overbrace{111\ldots11}^{n\ 1\text{s}}\right)$$

and working with blocks of n digits.

7. Similar to the previous problem if we can factor $10^n + 1$ we can develop simultaneous divisibility tests for its factors.

I hope you enjoy developing some of these tests. We will see further applications of congruences and divisibility in future issues.

OLYMPIAD CORNER

No. 391

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **May 15, 2021**.

OC521. In the plane there are two identical circles with radius 1, which are tangent externally. Consider a rectangle containing both circles, each side of which touches at least one of them. Determine the largest and the smallest possible area of such a rectangle.

OC522. Find the largest natural number n such that the sum

$$\lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \cdots + \lfloor\sqrt{n}\rfloor$$

is a prime number.

OC523. Let $(a_n)_{n\geq 1}$ be a sequence such that $a_n > 1$ and $a_{n+1}^2 \geq a_n a_{n+2}$ for all $n \geq 1$. Prove that the sequence $(x_n)_{n\geq 1}$ defined by $x_n = \log_{a_n} a_{n+1}$ for $n \geq 1$ is convergent and find its limit.

OC524. Let $p \geq 2$ be a natural number and let $(M, \cdot)$ be a finite monoid such that $a^p \neq a$ for all $a \in M \setminus \{e\}$, where e is the identity element of M. Prove that $(M, \cdot)$ is a group.

OC525. Consider the sequence $(a_1, a_2, \ldots, a_n)$ with terms from the set $\{0, 1, 2\}$. We will call a *block* a subsequence of the form $(a_i, a_{i+1}, \ldots, a_j)$, where $1 \leq i \leq j \leq n$, and $a_i = a_{i+1} = \cdots = a_j$. A block is called *maximal* if it is not contained in any longer block. For example, in the sequence $(1, 0, 0, 0, 2, 1, 1)$ the maximal blocks are (1), $(0, 0, 0)$, (2) and $(1, 1)$. Let K_n be the number of such sequences of length n with terms from the set $\{0, 1, 2\}$ in which all maximal blocks have odd lengths. Moreover, let L_n be the number of all sequences of length n with terms from the set $\{0, 1, 2\}$ in which the numbers 0 and 2 do not appear in adjacent positions. Prove that $L_n = K_n + \dfrac{1}{3}K_{n-1}$ for all $n > 1$.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC521. Dans le plan se trouvent deux cercles de rayon 1, tangents extérieurement. Considérer alors des rectangles entourant les deux cercles et dont chaque côté touche au moins un d'eux. Déterminer la plus petite et la plus grande surface possibles pour de tels rectangles.

OC522. Déterminer le plus gros nombre naturel n tel que la somme

$$\lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \cdots + \lfloor\sqrt{n}\rfloor$$

est un nombre premier.

OC523. Soit $(a_n)_{n\geq 1}$ une suite telle que $a_n > 1$ et $a_{n+1}^2 \geq a_n a_{n+2}$ pour tout $n \geq 1$. Démontrer que la suite $(x_n)_{n\geq 1}$ définie par $x_n = \log_{a_n} a_{n+1}$ pour $n \geq 1$ est convergente et détermininer sa limite.

OC524. Soit $p \geq 2$ un nombre naturel et soit $(M, \cdot)$ une monoïde finie telle que $a^p \neq a$ pour tout $a \in M \setminus \{e\}$, où e est l'élément neutre de M. Démontre que $(M, \cdot)$ est effectivement un groupe.

OC525. Soit une suit $(a_1, \ldots, a_n)$ dont les éléments appartiennent à l'ensemble $\{0, 1, 2\}$. Nous dénoterons *bloc* toute sous suite de la forme $(a_i, a_{i+1}, \ldots, a_j)$ où $1 \leq i \leq j \leq n$ et $a_i = a_{i+1} = \cdots = a_j$. Un bloc est dit *maximal* s'il n'est pas partie d'un bloc de plus grande longueur. Par exemple, dans la suite $(1, 0, 0, 0, 2, 1, 1)$, les blocs maximaux sont (1), $(0, 0, 0)$, (2) et $(1, 1)$. Posons alors K_n le nombre de suites de longueur n tirées de l'ensemble $\{0, 1, 2\}$ dont tous les blocs maximaux sont de longueur impaire. Enfin, posons L_n le nombre de suites de longueur n tirées de l'ensemble $\{0, 1, 2\}$, dans lesquelles les nombres 0 et 2 ne se trouvent jamais en positions adjacentes. Démontrer que $L_n = K_n + \dfrac{1}{3}K_{n-1}$ pour tout $n > 1$.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(8), p. 364–365.

OC496. The six digits 1, 2, 3, 4, 5, and 6 are used to construct a one-digit number, a two-digit number and a three-digit number. Each digit must be used only once and all six digits must be used. The sum of the one-digit number and the two-digit number is 47 and the sum of the two-digit number and the three-digit number is 358. Find the sum of all three numbers.

Originally from 2015 International Mathematics Competition, EMIC, Individual Contest, Changchun, China.

We received 21 correct and complete submissions. We present a typical solution.

The one-digit number can be $1, 2, 3, 4, 5$, and 6. Using the facts that the sum of the one-digit and two-digit numbers is 47 and the sum of the two-digit and three-digit numbers is 358, once the one-digit number is known we can find the other numbers. Hence, we can construct the following table:

One-digit number	Two-digit number	Three-digit number
1	46	312
2	45	313
3	44	314
4	43	315
5	42	316
6	41	317

From this table we can see that the fifth row is the only admissible solution, a solution where the digits $1, 2, 3, 4, 5$, and 6 appear exactly once. Hence, 5 is the one-digit number, 42 is the two-digit number, and 316 is the three-digit number. The sum of all three numbers is $5 + 42 + 316 = 363$.

OC497. Does there exist a positive integer that is divisible by 2020 and has equal number of digits 0,1,2, ..., 9?

Originally from 41st International Mathematical Tournament of Towns (Spring 2020), Junior A-level, proposed by Mikhail Evdokimov.

We received 9 correct and complete submissions. We present the solution by Corneliu Manescu-Avram.

Yes, the number exists. Choose c, an even nonzero digit, and form the number $\overline{c0c0}$ that is divisible by 20. Split the remaining eight nonzero digits into pairs and form numbers of the form $\overline{abab}$ that are divisible by 101. Append the numbers of form $\overline{abab}$ and add at the end $\overline{c0c0}$. Obviously, every digit is repeated twice in the obtained number. The obtained number is divisible by $2020 = 4 \times 5 \times 101$. Indeed the last two digits $\overline{c0}$ form a number divisible by 20. Moreover, starting from left to right, each group of four digits is a number divisible by 101. Such an example is

$$12123434565679798080 = 2020 \times 6001700280039504.$$

Editor's comments: Other examples were found. Aditya Gupta proposed an example where each digit is repeated four times:

$$9999888877776666555544443333222211110000.$$

UCLan Cyprus Problem Solving group proposed an example where each digit appears only one time: 1237548960 and showed that this is the smallest number with the requested properties. The largest number with each digit appearing only once was found by Richard Hess: 9876521340.

OC498. A collection of 8 cubes consists of one cube with edge-length k for each integer k, $1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

(a) Any cube may be the bottom cube in the tower.

(b) The cube immediately on top of a cube with edge-length k must have edge-length at most $k+2$.

Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?

Originally from 2006 American Invitational Mathematics Examination (AIME).

We received 6 correct and complete submissions. We present a typical solution.

We proceed recursively.

Suppose we can build T_m towers using cubes with edge-lengths $1, 2, \ldots, m$. Next we count the towers that we can build using cubes of sizes $1, 2, \ldots, (m+1)$. If we remove the cube of size $(m+1)$ from such a tower (keeping all other blocks in order), we get a valid tower that includes only cubes of sizes $1, 2, \ldots, m$. Given a valid tower that includes only cubes of sizes $1, 2, \ldots, m$, where $m \geq 2$, we can insert the cube of size $m+1$ in exactly 3 places: at the beginning, immediately following the cube of size $(m-1)$, or immediately following the cube of size m in order to obtain a valid tower of cubes of sizes $1, 2, \ldots, (m+1)$. Thus, there are 3 times as many towers of cubes of sizes $1, 2, \ldots, (m+1)$ as there are towers that include only the cubes of sizes $1, 2, \ldots, m$, equivalently $T_{(m+1)} = 3T_m$. There are 2 towers of cubes of sizes 1, 2. So there are $T_8 = 2 \times 3^6 = 1458$ towers that we can build using cubes of sizes $1, 2, \ldots, 8$. So the remainder when T_8 is divided by 1000 is 458.

OC499. A self-avoiding rook walk on a chessboard (a rectangular grid of unit squares) is a path traced by a sequence of moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, i.e., the rook's path is non-self-intersecting.

Let $R(m,n)$ be the number of self-avoiding rook walks on an $m \times n$ (m rows, n columns) chess board which begin at the lower-left corner and end at the upper-left corner. For example, $R(m,1) = 1$ for all natural numbers m; $R(2,2) = 2$; $R(3,2) = 4$; $R(3,3) = 11$. Find a formula for $R(3,n)$ for each natural number n.

Originally from 40th Canadian Mathematical Olympiad (2008).

We received 5 correct submissions. We present the solution by UCLan Cyprus Problem Solving Group.

We consider the rook walks on a $3 \times n$ board that intersect the last column, the column n, and split them into three mutually disjoint sets. We denote by A_n, B_n, and C_n the counts of walks in the three sets. Specifically, A_n is the count of walks that enter the last column at the bottom and exit it in the middle, B_n is the count of walks that enter the last column at the bottom and exit it at the top, and C_n is the count of walks that enter the last column at the middle and exit it at the top.

Note that there is no other way to use the last column as part of a self-avoiding rook walk. If a walk enters at the top and exits at the bottom, then the top-left path is now blocked. Intuitively, the argument looks obvious but needs topological tools such as Brouwer's fixed-point theorem to be established rigorously.

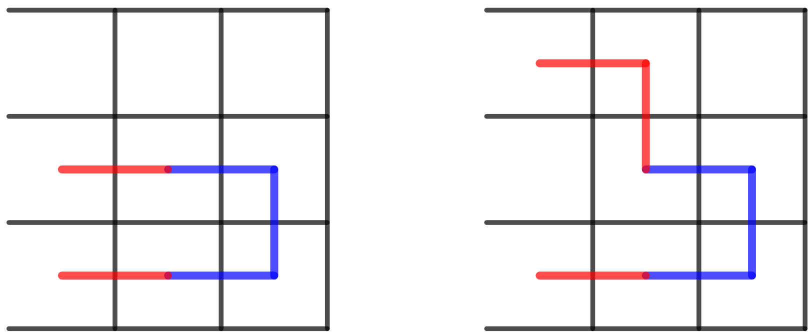
We now claim that for every $n \geqslant 2$

$$A_{n+1} = A_n + B_n, \quad B_{n+1} = A_n + B_n + C_n, \quad \text{and} \quad C_{n+1} = B_n + C_n. \qquad (1)$$

The following figure shows the first of these equalities. The others are proved similarly. The blue lines show the path that needs to be followed while crossing from the column n to $n+1$ in order to have a walk counted by A_{n+1}. The red lines show the only two ways in which the blue path can be extended. The total numbers of ways to complete the rook walk in the left part is A_n while in the right part is B_n.

It is easy to check that $A_2 = B_2 = C_2 = 1$. If we define $A_1 = C_1 = 0$ and $B_1 = 1$ then (1) holds for every $n \geqslant 1$. It is now easy to see by induction that $A_n = C_n$ for every $n \in \mathbb{N}$. Hence, for every $n \geqslant 1$ we get

$$\begin{aligned}
A_{n+1} &= A_n + B_n, \\
B_{n+1} &= 2A_n + B_n, \\
B_{n+2} &= 2A_{n+1} + B_{n+1} \\
&= 2(A_n + B_n) + B_{n+1} = (B_{n+1} - B_n) + 2B_n + B_{n+1} = 2B_{n+1} + B_n.
\end{aligned}$$



Note that $B_{n+2}-2B_{n+1}-B_n=0$ is a homogeneous linear second order difference equation with characteristic equation $x^2-2x-1=0$. Its roots are $1\pm\sqrt{2}$ and therefore, there are constants a and b such that

$$B_n = a(1+\sqrt{2})^n + b(1-\sqrt{2})^n$$

for every $n\geqslant 1$. Using the initial conditions $B_1=B_2=1$, or for convenience using B_0 defined as $B_0=B_2-2B_1=-1$, we find that for every $n\geqslant 1$

$$B_n = \frac{(\sqrt{2}-1)(1+\sqrt{2})^n-(\sqrt{2}+1)(1-\sqrt{2})^n}{2}.$$

Note that $B_{n+1}=A_n+B_n+C_n$ is the total number of self-avoiding rook walks on a $3\times n$ board that intersect the last column, the column n. Therefore

$$\begin{aligned}
R(3,n) &= B_2+B_3+\cdots+B_{n+1}\\
&= B_0+B_1+B_2+B_3+\cdots+B_{n+1}\\
&= \frac{1}{2}\left[(\sqrt{2}-1)\frac{(1+\sqrt{2})^{n+2}-1}{(1+\sqrt{2})-1}-(\sqrt{2}+1)\frac{(1-\sqrt{2})^{n+2}-1}{(1-\sqrt{2})-1}\right]\\
&= \frac{(\sqrt{2}-1)(1+\sqrt{2})^{n+2}+(\sqrt{2}+1)(1-\sqrt{2})^{n+2}}{2\sqrt{2}}-1\\
&= \frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{2\sqrt{2}}-1\,.
\end{aligned}$$

We point out that $\{R(3,n), n\geq 1\}$ is the sequence A005409 recorded by The On-line Encyclopedia of Integer Sequences (OEIS), `https://oeis.org/A005409`

We mention the first few terms of the sequence $\{R(3,n), n\geq 1\}$ that were submitted by Richard Hess: 1, 4, 11, 28, 69, 168, 407, 984, 2377, 5744, 13859, 33460.

OC500. An $n \times m$ matrix is nice if it contains every integer from 1 to mn exactly once and 1 is the only entry which is the smallest both in its row and in its column. Prove that the number of $n \times m$ nice matrices is $\dfrac{(nm)!n!m!}{(n+m-1)!}$.

Originally from the 2019 Miklós Schweitzer Memorial Competition in Mathematics.

We received 3 correct submissions. We present the solution by Oliver Geupel.

Let s and t be integers such that $0 \le s < n$ and $0 \le t < m$. Break up an empty $n \times m$ table into an upper-left $s \times t$ block A, an upper right $s \times (m-t)$ block B, a lower-left $(n-s) \times t$ block C, and a lower-right $(n-s) \times (m-t)$ block D. Fill the block D as follows:

$$D = \begin{bmatrix} 1 & 2 & \dots & m-t \\ m-t+1 & m-t+2 & \dots & 2(m-t) \\ \dots & \dots & \dots & \dots \\ (n-s)(m-t)+1 & (n-s-1)(m-t)+2 & \dots & (n-s)(m-t) \end{bmatrix},$$

which is a nice $(n-s)\times(m-t)$ matrix. Given this situation at start, we randomly fill the remaining cells of the table with all the integers from $(n-s)(m-t)+1$ to mn to obtain an $n \times m$ matrix M.

Let $P(s,t)$ denote the probability that the outcome M is nice. If M is nice then the number $(n-s)(m-t)+1$ is either in block B or in C. The probability that it is in B is

$$\frac{s(m-t)}{mn-(n-s)(m-t)} = \frac{s(m-t)}{ms+nt-st}.$$

If it is in B, then its actual position in B as well as the remaining $m-t-1$ entries in the same row have no impact on the fact, whether M is nice or not. Moreover, the precise order of entries in D is not important, as long as D remains nice. Hence, the conditional probability that M is nice, under the condition that the element $(n-s)(m-t)+1$ is in block B, is $P(s-1,t)$.

With similar observations on block C, we obtain

$$P(s,t) = \frac{s(m-t)}{ms+nt-st}\cdot P(s-1,t) + \frac{(n-s)t}{ms+nt-st}\cdot P(s,t-1).$$

We prove by induction on $\min\{s,t\}$ that

$$P(s,t) = \frac{(m+n-1-s)!(m+n-1-t)!}{(m+n-1)!(m+n-1-s-t)!}. \tag{1}$$

The base case $P(s,0) = P(0,t) = 1$ is immediate. By induction, we have

$$\frac{s(m-t)}{ms+nt-st}\cdot P(s-1,t) = \frac{(m+n-1-s)!(m+n-1-t)!}{(m+n-1)!(m+n-s-t)!}\cdot\frac{s(m-t)(m+n-s)}{ms+nt-st}$$

and

$$\frac{(n-s)t}{ms+nt-st}\cdot P(s,t-1)=\frac{(m+n-1-s)!(m+n-1-t)!}{(m+n-1)!(m+n-s-t)!}\cdot\frac{(n-s)t(m+n-t)}{ms+nt-st}.$$

Since

$$s(m-t)(m+n-s)+(n-s)t(m+n-t)=(ms+nt-st)(m+n-s-t),$$

the formula (1) follows, which completes the induction.

Now, $P(n-1,m-1)$ is the probability that M is nice under the condition that the lower-right entry is 1. We can drop the condition that 1 is placed in the lower-right entry, since the property of M to be nice is invariant under permutation of rows and columns. Consequently, the total number of nice $n\times m$ matrices is

$$(mn)!P(n-1,m-1)=\frac{(mn)!m!n!}{(m+n-1)!},$$

which completes the proof.

Lagrange Interpolation

Sushanth Malipati

Lagrange Interpolation is an advanced trick for Math Olympiad problems. It is very useful and aids in solving function problems.

1 Lagrange Interpolating Polynomial

Definition 1 *The Lagrange Interpolating polynomial $P(x)$ of degree $\leq (n-1)$ that passes through n points $(x_1, y_1 = f(x_1)), (x_2, y_2 = f(x_2)), ..., (x_n, y_n = f(x_n))$ is given by*

$$P(x) = \sum_{j=1}^{n} P_j(x)$$

where

$$P_j(x) = y_j \prod_{k=1;k\neq j}^{n} \frac{x - x_k}{x_j - x_k}.$$

The definition (taken from `Wolfram.com`) written explicitly is:

$$P(x) = \frac{(x-x_2)(x-x_3)...(x-x_n)}{(x_1-x_2)(x_1-x_3)...(x_1-x_n)}y_1 + \frac{(x-x_1)(x-x_3)...(x-x_n)}{(x_2-x_1)(x_2-x_3)...(x_2-x_n)}y_2$$
$$+ \cdots + \frac{(x-x_1)(x-x_2)...(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)...(x_n-x_{n-1})}y_n$$

This works because this polynomial is an nth degree polynomial approximation to $f(x)$ and the nth degree polynomial passing through $(n+1)$ points is unique.

2 Examples

Before doing a problem, here is an exercise to get a handle on writing the polynomials.

Exercise 1 Write the Lagrange Polynomial for $n = 3$ points.

Let the points be $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. Then we get $P_1(x)$ to be

$$y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)},$$

we get $P_2(x)$ to be

$$y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)},$$

and lastly we get $P_3(x)$ to be

$$y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}.$$

Adding these three up we get $P(x)$:

$$y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

Example 2 (`Brilliant.org`) If $P(x)$ is a cubic polynomial with $P(1) = 1$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, find $P(6)$.

Using the formula for generating the polynomial we get:

$$\begin{aligned} P(x) &= 1\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + 2\frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} \\ &+ 3\frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + 5\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} \\ &= \frac{-1}{6}(x-2)(x-3)(x-4) + (x-1)(x-3)(x-4) \\ &+ \frac{-3}{2}(x-1)(x-2)(x-4) + \frac{5}{6}(x-1)(x-2)(x-3) \end{aligned}$$

We calculate each $P_j(x)$ and add them to get the final $P(x)$. Now plugging in $x = 6$ we get $\boxed{16}$.You can also use Finite Differences to solve this problem. Here is the finite difference table method way of solving this problem:

$$\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & \\ 3 & 3 & 2 & & \\ 4 & 5 & & & \end{array}$$

From here we get

$$\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 & 1 \\ 4 & 5 & 4 & 3 & 1 \\ 5 & 9 & 7 & 4 & 1 \\ 6 & \boxed{16} & 11 & 5 & 1 \end{array}$$

Example 3 (HMMT) Let $P(x)$ be the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, ..., 2020$. Compute $P(2021^2)$.

Applying the formula, we get:

$$P(x) = \sum_{k=0}^{2020} k \prod_{0\le j\le 2020 j\ne k} \frac{x-j^2}{k^2-j^2} = \sum_{k=0}^{2020} \frac{2(-1)^k k}{(2020-k)!(2020+k)!(x-k^2)} \prod_{0\le j\le 2020; j\ne k} x - j^2.$$

What we do above is factor out the denominator because it's a constant and seems like a denominator of a binomial coefficient. Substituting for $x = 2021^2$, we see the binomial coefficient arise, and using Pascal's identity multiple times:

$$\begin{aligned}
P(2021^2) &= \sum_{k=0}^{2020} \frac{4042!(-1)^k k}{(2021-k)!(2021+k)!} = \sum_{k=0}^{2020} \binom{4042}{2021-k}(-1)^k k \\
&= 2021 - \left(\sum_{k=0}^{2020}(-1)^{k+1} k \left(\binom{4041}{2021-k} - \binom{4041}{2020-k}\right)\right) \\
&= 2021 - \left(\sum_{k=0}^{2020}(-1)^{k+1} \binom{4041}{2021-k}\right) \\
&= 2021 - \left(\sum_{k=0}^{2020}(-1)^{k+1} \left(\binom{4040}{2021-k} - \binom{4040}{2020-k}\right)\right) \\
&= 2021 - \binom{4040}{2020}
\end{aligned}$$

3 Problems

Problem 1 (`Brilliant.org`) Let $f(x)$ be a quintic polynomial such that $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, $f(4) = 3$, $f(5) = 5$, $f(6) = 8$. Find $f(7)$

Problem 2 (HMMT) Given that P is a real polynomial of degree at most 2012 such that $P(n) = 2^n$ for $n = 1, 2, ..., 2012$, what choice(s) of $P(0)$ produces the minimal possible value of $P(0)^2 + P(2013)^2$?

Problem 3 (CHMMC) Suppose that $P(x)$ is a monic polynomial with 20 roots, each distinct and of the form $\frac{1}{3^k}$ for $k = 0, 1, 2, \dots, 19$. Find the coefficient of x^{18} in $P(x)$.

Problem 4 (USAMO) Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

FOCUS ON...

No. 45

Michel Bataille

Quadratics (II)

Introduction

In this number, we examine various links between quadratic polynomials and polynomials of degree 3 or 4. In particular, we will illustrate Cardano's and Ferrari's methods sometimes used for solving cubic and quartic equations, respectively.

Connections with the cubic polynomial

Our first example is problem **CC48**, [2012 : 403 ; 2013 : 438], a nice problem about cubic polynomials, which shows how their quadratic derivatives can lead to a solution.

> Determine whether there exist two real numbers a and b such that both $(x-a)^3+(x-b)^2+x$ and $(x-b)^3+(x-a)^2+x$ contain only real roots.

Let $P_1(x)=(x-a)^3+(x-b)^2+x$ and $P_2(x)=(x-b)^3+(x-a)^2+x$. We show that it is not possible to choose real numbers a and b such that all the roots of the polynomials P_1 and P_2 are real numbers. For the purpose of a contradiction, we assume that all the roots of P_1 and P_2 are real numbers.

Writing $P_1(x)=(x-a)^3+((x-a)+(a-b))^2+(x-a)+a$, we readily see that

$$P_1(x)=(x-a)^3+(x-a)^2+(1+2a-2b)(x-a)+(a-b)^2+a.$$

Similarly, we have

$$P_2(x)=(x-b)^3+(x-b)^2+(1+2b-2a)(x-b)+(a-b)^2+b$$

and from assumption, we deduce that all the roots of

$$Q_1(x)=x^3+x^2+(1+2a-2b)x+(a-b)^2+a$$

and

$$Q_2(x)=x^3+x^2+(1+2b-2a)x+(a-b)^2+b$$

are also real numbers. It follows that the functions $x\mapsto Q_1(x)$ and $x\mapsto Q_2(x)$ cannot be strictly monotone. Thus, the discriminants of the quadratics

$$Q_1'(x)=3x^2+2x+(1+2a-2b) \quad \text{and} \quad Q_2'(x)=3x^2+2x+(1+2b-2a)$$

cannot be negative. In other words, we must have $1-3(1+2a-2b)\geq 0$ and $1-3(1+2b-2a)\geq 0$ that is, $b-a\geq \frac{1}{3}$ and $b-a\leq -\frac{1}{3}$, which is clearly impossible. This contradiction completes the proof.

We point out to the interested reader that the featured solution in [2013: 438] does not use derivatives.

The following solution to a 1988 U.S.A. Olympiad problem also rests upon properties of the quadratic derivative:

> The cubic equation $x^3+ax^2+bx+c=0$ has three real roots. Show that $a^2-3b \geq 0$, and that $\sqrt{a^2-3b}$ is less than or equal to the difference between the largest and smallest roots.

Let $f(x)=x^3+ax^2+bx+c$. As in the previous example, the quadratic derivative $f'(x)=3x^2+2ax+b$ cannot be strictly positive for all real x, hence its discriminant $\Delta = 4(a^2-3b)$ cannot be negative.

Now, suppose that x_0, x_1, x_2 are the roots of $f(x)=0$ with $x_0 < x_1 < x_2$.

Because $x_0+x_1+x_2=-a$ and $x_0x_1+x_1x_2+x_2x_0=b$, we have

$$\begin{aligned}(x_2-x_0)^2 &= (x_0+x_2)^2-4x_0x_2\\ &= (-a-x_1)^2-4(b-x_1(x_0+x_2))\\ &= (-a-x_1)^2-4(b-x_1(-a-x_1)) = -3x_1^2-2ax_1+a^2-4b.\end{aligned}$$

The inequality $x_2-x_0 \geq \sqrt{a^2-3b}$, which is equivalent to

$$-3x_1^2-2ax_1+a^2-4b \geq a^2-3b,$$

that is, to $f'(x_1) \leq 0$, does hold because x_1 is between the roots of $f'(x)$ (observe that if α, β are the roots of $f'(x)$ with $\alpha < \beta$, then we must have $x_0 < \alpha < x_1 < \beta < x_2$). $\square$

Consider the cubic equation $x^3-3px+q=0$. To solve this equation, Cardano's method consists in looking for solutions of the form $z+\frac{p}{z}$. This leads to the equation $z^6+qz^3+p^3=0$, which can be solved by the quadratic $X^2+qX+p^3=0$ (Lagrange's resolvent). We illustrate the method through a problem posed in the July 2009 issue of *The Mathematical Gazette*:

> Solve the equation of the 6th order $(a+x^3)(1-bx)^3=x^3$ where a and $b(\neq 0)$ are real parameters.

We will content ourselves with reducing the problem to solving quadratic equations.

Since $\frac{1}{b}$ is not a solution, the given equation (E) is equivalent to

$$ab^3+(bx)^3+\left(\frac{bx}{bx-1}\right)^3=0.$$

Remarking that $bx+\dfrac{bx}{bx-1}=bx\cdot\dfrac{bx}{bx-1}$ and recalling that

$$u^3+v^3=(u+v)^3-3uv(u+v),$$

we readily deduce that (E) is also equivalent to

$$p\left(\frac{b^2x^2}{bx-1}\right)=0,$$

where $p(x)=x^3-3x^2+ab^3=(x-1)^3-3(x-1)+ab^3-2$.

Let us apply Cardano's method to the cubic equation $x^3-3x+ab^3-2=0$. Lagrange's resolvent is readily obtained: $X^2-(2-ab^3)X+1=0$. If α^3 is a solution to this quadratic and $\omega=e^{2\pi i/3}$, it quickly follows that

$$p(x)=(x-s_1)(x-s_2)(x-s_3),$$

where

$$s_1=1+\alpha+\frac{1}{\alpha},\quad s_2=1+\omega\alpha+\frac{1}{\omega\alpha},\quad s_3=1+\omega^2\alpha+\frac{1}{\omega^2\alpha}.$$

Thus, the equation (E) rewrites as

$$(b^2x^2-s_1bx+s_1)(b^2x^2-s_2bx+s_2)(b^2x^2-s_3bx+s_3)=0,$$

and the six solutions (counting multiplicity) can now be found by solving the three quadratics $b^2x^2-s_ibx+s_i=0,\quad(i=1,2,3)$.

For example, the numerical example $a=\frac{1}{2},\ b=2$ yields the six solutions

$$\frac{1+i}{2}\text{ (double)},\ \frac{1-i}{2}\text{ (double)},\ \frac{-1+\sqrt{5}}{4},\ \frac{-1-\sqrt{5}}{4}$$

(details are left to the reader).

Connections with the quartic polynomial

In part I, we have already met biquadratic equations, of the form $ux^4+vx^2+w=0$. These quartic equations are of course immediately solved *via* the quadratic equation $uX^2+vX+w=0$. Other quartic equations can easily be solved, namely reciprocal equations. Consider for example the following exercise:

Find $t\in[0,\frac{\pi}{2}]$ given that

$$\cos 2t=\frac{4\sin^4 t\cos^4 t}{6\sin^4 t+\cos^4 t}.$$

Let $x=\cos 2t$. From $2\cos^2 t=1+\cos 2t$ and $2\sin^2 t=1-\cos 2t$, the given condition writes as $x(7-10x+7x^2)=1-2x^2+x^4$, that is,

$$x^4-7x^3+8x^2-7x+1=0.$$

Such a reciprocal equation is classically solved *via* quadratic equations: equivalently, the equation writes as $x^2-7x+8-\frac{7}{x}+\frac{1}{x^2}=0$ or

$$y^2-7y+6=0$$

if we set $y = x + \frac{1}{x}$ (since $x^2 + \frac{1}{x^2} = y^2 - 2$). This gives $y = 1$ or $y = 6$. If $1 = y = x + \frac{1}{x}$ leads to $x^2 - x + 1 = 0$ with no real solutions, the equation $6 = x + \frac{1}{x}$ gives $x = 3 + 2\sqrt{2}$ or $x = 3 - 2\sqrt{2}$. Since $|x| = |\cos 2t| \leq 1$, we must have $x = \cos 2t = 3 - 2\sqrt{2}$ and so $\cos^2 t = \frac{1+\cos 2t}{2} = 2 - \sqrt{2}$. We conclude that $t = \arccos(\sqrt{2-\sqrt{2}})$. $\square$

To solve a quartic equation, quite rarely one has to resort to the general method invented by the Italian mathematician Ferrari (who, like Cardano, lived in the sixteenth century). The method is rather simple in theory, but can lead to very complicated calculations in practice. Its principle is to seek λ such that

$$x^4 + ax^3 + bx^2 + cx + d = (x^2 + (a/2)x + \lambda)^2 - (q(x))^2$$

where the degree of $q(x)$ is not greater than 1. This leads first to a cubic equation for λ (the resolvent) and then to a factorization of the quartic as the product of two quadratic polynomials, hence to the roots. Here is an example, providing another solution to problem **4317** [2018 : 71 ; 2019 : 98]:

Solve the following system of equations over reals:

$$\begin{cases} a+b+c+d & = & 4 \\ abc+abd+acd+bcd & = & 2 \\ abcd & = & -\frac{1}{4} \end{cases}$$

Let

$$u = \frac{1+\sqrt{3}}{2},\ u' = \frac{1-\sqrt{3}}{2},\ v = \frac{5-3\sqrt{3}}{2},\ v' = \frac{5+3\sqrt{3}}{2}.$$

It is easily checked that $(u, u, u, v), (u', u', u', v')$ (and all their permutations) are solutions for (a, b, c, d). We show that there are no other solutions. To this end, we consider real numbers a, b, c, d satisfying the system and set

$$m = ab + ac + ad + bc + bd + cd.$$

Then, (a, b, c, d) is a list of the roots of the polynomial

$$p(x) = x^4 - 4x^3 + mx^2 - 2x - \frac{1}{4}.$$

To solve the equation $p(x) = 0$, we use Ferrari's method and look for λ ensuring that

$$p(x) = (x^2 - 2x + \lambda)^2 - (q(x))^2,$$

as indicated above. The calculation of $(x^2 - 2x + \lambda)^2 - p(x)$ gives

$$(q(x))^2 = (2\lambda + 4 - m)x^2 - 2(2\lambda - 1)x + \lambda^2 + \frac{1}{4}.$$

The discriminant $\Delta(\lambda)$ of this quadratic polynomial must satisfy $\Delta(\lambda) = 0$ (the cubic resolvent). Being of degree 3 and with real coefficients,

$$\Delta(\lambda) = 4\left((2\lambda - 1)^2 - \left(\lambda^2 + \frac{1}{4}\right)(2\lambda + 4 - m)\right) = -8\lambda^3 + 4m\lambda^2 + m - 18\lambda$$

has at least one real root for λ. From now on, we assume that λ has been so chosen. Then $m = \frac{8\lambda^3+18\lambda}{1+4\lambda^2}$ and after some algebra, we finally obtain

$$p(x) = (x^2 - 2x + \lambda)^2 - \left(\frac{2(2\lambda - 1)x}{\sqrt{4\lambda^2 + 1}} - \frac{\sqrt{4\lambda^2 + 1}}{2} \right)^2.$$

Factoring this difference of two squares and setting $2\lambda = \sinh t$, the equation $p(x) = 0$ finally writes as

$$\left(x^2 - 2x \cdot \frac{e^t - 1}{\cosh t} + \frac{e^t}{2} \right) \left(x^2 - 2x \cdot \frac{e^{-t} + 1}{\cosh t} - \frac{e^{-t}}{2} \right) = 0.$$

The discriminants of the first and second factors respectively are

$$\Delta_1 = \frac{-2e^t(\cosh t - 2)^2}{\cosh^2 t} \leq 0 \qquad \text{and} \qquad \Delta_2 = \frac{2e^{-t}(\cosh t + 2)^2}{\cosh^2 t} > 0$$

and since the four roots of p are real, we must have $\cosh t = 2$, hence $\sinh t = \sqrt{3}$ or $-\sqrt{3}$.

If $\sinh t = \sqrt{3}$, then $e^t = \cosh t + \sinh t = 2 + \sqrt{3}$, $e^{-t} = \cosh t - \sinh t = 2 - \sqrt{3}$ and we have to solve

$$\left(x - \frac{1 + \sqrt{3}}{2} \right)^2 \left(x^2 - x(3 - \sqrt{3}) + \frac{\sqrt{3}}{2} - 1 \right) = 0.$$

The (easily found) solutions are u (triple) and v. Similarly, we find u' (triple) and v' in the case $\sinh t = -\sqrt{3}$. This completes the proof.

Exercises

1. Let p be a real number such that $\frac{13}{32} < p < 8$. Prove that the equation

$$x^3 - 6x^2 + 2(p - 2)x - p = 0$$

has only one real solution.

2. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ where a, b, c, d are complex numbers. Prove that the sum of two of its roots is equal to the sum of the two remaining roots if and only if P' and P''' have a common root. Application: find the roots of the polynomial $x^4 + 2x^3 + 2x^2 + x + \frac{1}{16}$.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **May 15, 2021**.

4621. *Proposed by Michel Bataille.*

In the plane, circles $\mathcal{C}$ and $\mathcal{C}'$ are externally tangent at T. Points P, Q of $\mathcal{C}'$ are such that $P \neq T$ and $\angle PQO = 90^\circ$ where O is the centre of $\mathcal{C}$. Points A, B of $\mathcal{C}$ are such that PA and QB are tangent to $\mathcal{C}$. If the line PT intersects $\mathcal{C}$ again at S, prove that $PS \cdot QT = PA \cdot QB$.

4622. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle and AA', BB' and CC' be its medians. Let Γ be the circumcircle of $\triangle\ A'B'C'$ and let $\Gamma \cap AA' = \{A''\}$, $\Gamma \cap BB' = \{B''\}$, $\Gamma \cap CC' = \{C''\}$. Show that if $\overrightarrow{AA''} + \overrightarrow{BB''} + \overrightarrow{CC''} = \overrightarrow{0}$, then ABC is an equilateral triangle.

4623. *Proposed by Nguyen Viet Hung.*

Let $p(x) = x^2 + 2ax - b - 1$ and $q(x) = x^2 + 2bx - a - 4$ be two polynomials with integer coefficients. Determine all pairs (a, b) of non-negative integers such that these two polynomials simultaneously have integer solutions.

4624. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers with $a + b + c = 3$. Prove that

$$\sqrt{\frac{ab}{2a+b+c}} + \sqrt{\frac{bc}{2b+c+a}} + \sqrt{\frac{ca}{2c+a+b}} \leq \frac{3}{2}.$$

4625. *Proposed by Corneliu Manescu-Avram.*

Let a, m and n be positive integers greater than 1. Prove that $a^2 + a + 1$ divides $(a+1)^m + a^n$ if and only if m is odd and 3 divides $m + n$.

4626. *Proposed by Alpaslan Ceran.*

Let $ABCDE$ and $ABFKM$ be regular pentagons and suppose that AC intersects EB at a point L. Show that

$$\frac{|LK|}{|DL|} = \frac{\sqrt{5}+1}{2}.$$

4627. *Proposed by Dong Luu.*

Let P be a point not on the circumcircle (O) of a given triangle ABC, nor on the extensions of any of its sides. Define U, V, W to be the projections of P on the lines BC, CA, AB, and X, Y, Z to be the vertices of the triangle formed by the perpendicular bisectors of the segments PA, PB, PC. Suppose that the circumcircle of ΔXYZ intersects (O) in two points, E and F. Prove that the foot of the perpendicular from P to the line EF is the circumcenter of triangle UVW.

4628. *Proposed by Russ Gordon and George Stoica.*

Let A be a nonempty set of positive integers that is closed under addition and such that $\mathbb{N} \setminus A$ contains infinitely many elements. Prove that there exists a positive integer $d \geq 2$ such that $A \subseteq \{nd : n \in \mathbb{N}\}$.

4629. *Proposed by Minh Nguyen.*

Determine the smallest natural number n such that in any n-element subset of $\{1, 2, \ldots, 2020\}$ there exist four different a,b,c,d satisfying $a + 2b + 3c + 4d \leq 2020$.

4630. *Proposed by George Stoica.*

We consider an equilateral triangle ABC with the circumradius $R = 1$ and a point D on or inside its circumcircle. Prove that $3 \leq AD + BD + CD \leq 4$.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2021**.

La rédaction souhaite remercier Frédéric Morneau-Guérin, professeur à l'Université TÉLUQ, d'avoir traduit les problèmes.

4621. *Proposée par Michel Bataille.*

Dans le plan, les cercles $\mathcal{C}$ et $\mathcal{C}'$ sont tangents extérieurement en T. Les points P et Q de $\mathcal{C}'$ sont tels que $P \neq T$ et $\angle PQO = 90°$, où O désigne le centre de $\mathcal{C}$. Les points A et B de $\mathcal{C}$ sont tels que PA et QB sont tangents à $\mathcal{C}$. Si la droite PT rencontre $\mathcal{C}$ en S, montrez que $PS \cdot QT = PA \cdot QB$.

4622. *Proposée par Mihaela Berindeanu.*

Soit ABC un triangle acutangle. Soient AA', BB' et CC' ses médianes. Désignons par Γ le cercle circonscrit à $\triangle\ A'B'C'$ et $\Gamma \cap AA' = \{A''\}$, $\Gamma \cap BB' = \{B''\}$, $\Gamma \cap CC' = \{C''\}$. Montrez que si $\overrightarrow{AA''} + \overrightarrow{BB''} + \overrightarrow{CC''} = \overrightarrow{0}$, alors ABC est un triangle équilatéral.

4623. *Proposée par Nguyen Viet Hung.*

Soient $p(x) = x^2 + 2ax - b - 1$ et $q(x) = x^2 + 2bx - a - 4$ deux polynômes à coefficients entiers. Identifiez toutes les paires (a, b) d'entiers non négatifs pour lesquels ces deux polynômes admettent simultanénement des solutions entières.

4624. *Proposée par George Apostolopoulos.*

Soient a, b, c des nombres réels positifs satisfaisant $a + b + c = 3$. Montrez que

$$\sqrt{\frac{ab}{2a+b+c}} + \sqrt{\frac{bc}{2b+c+a}} + \sqrt{\frac{ca}{2c+a+b}} \leq \frac{3}{2}.$$

4625. *Proposée par Corneliu Manescu-Avram.*

Soient a, m et n des entiers positifs supérieurs à 1. Montrez que $a^2 + a + 1$ divise $(a+1)^m + a^n$ si et seulement si m est impair et 3 divise $m + n$.

4626. *Proposée par Alpaslan Ceran.*

Soient $ABCDE$ et $ABFKM$ des pentagones réguliers. Supposons que AC rencontre EB au point L. Montrez que

$$\frac{|LK|}{|DL|} = \frac{\sqrt{5}+1}{2}.$$

4627. *Proposée par Dong Luu.*

Soit P un point n'appartenant pas au cercle circonscrit (O) d'un triangle donné ABC et n'appartenant pas non plus à l'extension de l'un de ces côtés. Soient U, V et W les projections de P sur les droites BC, CA, et AB. Soient encore X, Y et Z les sommets du triangle formé par les médiatrices relatives aux segments PA, PB, PC, respectivement. Supposons que le cercle inscrit de ΔXYZ rencontre (O) en deux points notés respectivement E et F. Montrez que le pied de la perpendiculaire de P à la droite EF est le centre du cercle circonscrit au triangle UVW.

4628. *Proposée par Russ Gordon et George Stoica.*

Soit A un ensemble non vide d'entiers positifs qui est stable sous l'addition et tel que $\mathbb{N} \setminus A$ contient une infinité d'éléments. Montrez qu'il existe un entier positif $d \geq 2$ tel que $A \subseteq \{nd : n \in \mathbb{N}\}$.

4629. *Proposée par Minh Nguyen.*

Déterminez quel est le nombre naturel minimal n pour lequel tout sous-ensemble à n éléments de $\{1, 2, \ldots, 2020\}$ admet quatre éléments distincts a,b,c,d satisfaisant $a + 2b + 3c + 4d \leq 2020$.

4630. *Proposée par George Stoica.*

Considérons un triangle équilatéral ABC dont le cercle circonscrit est de rayon $R = 1$. Soit D un point situé sur ou à l'intérieur du cercle circonscrit. Montrez que $3 \leq AD + BD + CD \leq 4$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2020: 46(8), p. 415–419.

4571. *Proposed by Ed Barbeau. Dedicated in memoriam to Richard K. Guy.*

What is the smallest square integer expressible as the product of three distinct nonzero integers in arithmetic progression?

We received 20 submissions, out of which 13 were complete and correct. We present the solution by Charles Burnette, lightly edited.

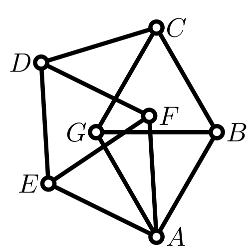
The smallest such integer is 36. Observe that $36 = (-9)(-4)(1)$ where -9, -4, and 1 are in arithmetic progression. To see that no smaller perfect square satisfies this property, we invoke the following facts:

- 1 only has two distinct divisors.
- The square of a prime number cannot be the product of three distinct integers in arithmetic progression. The only triples of distinct integers whose product is the square of a prime p are $(-p^2, -1, 1)$ and $(-p, -1, p)$, neither of which are in arithmetic progression.
- The number 16 cannot be the product of three distinct integers in arithmetic progression either. The only triples of distinct integers whose product is 16 are $(-16, -1, 1)$, $(-8, -2, 1)$, $(-8, -1, 2)$, $(-4, -2, 2)$, $(-4, -1, 4)$, $(-2, -1, 8)$, and $(1, 2, 8)$, none of which are in arithmetic progression.

Editor's comment. The next squares that can be written as the product of three distinct integers in arithmetic progression are $8^2 = (-8)(-2)(4)$, $9^2 = (-9)(-3)(3)$, and $35^2 = (1)(25)(49)$, which is the smallest square that is the product of three positive integers in arithmetic progression, as some readers pointed out.

4572. *Proposed by Veselin Jungić. Dedicated in memoriam to Richard K. Guy.*

In 1961, Canadian mathematicians Leo and William Moser introduced a geometric object consisting of seven vertices and eleven line segments of the unit length. This object is now known as the Moser spindle: see p. 390-396 of this issue for more details.



In the Moser spindle, find the measure of the angle $\angle GAF$.

We received 20 solutions, all correct. Presented is the one by the Eagle Problem Solvers.

Quadrilaterals $ABCG$ and $AFDE$ each consist of two equilateral triangles sharing an edge, so they are congruent. Let α be the angle of rotation about A that takes $ABCG$ to $AFDE$; then $\angle BAF = \angle CAD = \alpha$. Triangle CAD is isosceles with sides of length $\sqrt{3}$, $\sqrt{3}$, and 1, so by the Law of Cosines

$$\alpha = \cos^{-1}\left(\frac{3+3-1}{2\cdot 3}\right) = \cos^{-1}\left(\frac{5}{6}\right).$$

Thus the measure $\angle GAF$ is

$$\frac{\pi}{3} - \alpha = \frac{\pi}{3} - \cos^{-1}\left(\frac{5}{6}\right) = \cos^{-1}\left(\frac{5+\sqrt{33}}{12}\right).$$

Editor's comments. The last equation can be obtained from the known equation $\cos^{-1}(5/6) = \pi/3 - \angle GAF$ by taking the cosine on both sides, using the formula for the sum of angles, applying $\sin = \sqrt{1-\cos^2}$ and solving for $\cos(\angle GAF)$.

4573. *Proposed by J. Chris Fisher.*

For any triangle ABC let γ be the circle through A and B that surrounds the incircle α and is tangent to it, while β is a circle inside the triangle that is tangent to the sides AC and BC. Then β is externally tangent to γ if and only if it is also tangent to the line parallel to (but not equal to) AB that is tangent to the incircle.

This result was conjectured following the solution of Honsberger problem H4 [2018: 143-144], which related H4 to Problem 2.6.4 in H. Fukagawa and D. Pedoe, Japanese Temple Geometry Problems: San Gaku*, The Charles Babbage Research Centre (1989) page 37.*

We received three submissions, all correct, and feature the solution by Sergey Sadov.

Let $A'B'C$ be the triangle homothetic to $\triangle ABC$ (with homothety centre C) such that β is the incircle of $\triangle A'B'C$. If line $A'B'$ moves (staying always parallel to AB) from AB towards C, there is a unique position where the circle γ happens to be externally tangent to β. We need to prove that in this position $A'B'$ is tangent to α.

Because of the uniqueness of such a position, we can rearrange the setup: Given the incircle α of ΔABC, define the line $A'B'$ parallel to AB (with $A' \in AC$, $B' \in BC$) by the condition that it is tangent to α (on the side opposite to AB), and define β to be the incircle of $\Delta A'B'C$. We have to prove that there exists a circle containing the points A, B and tangent to α and β.

The proof below is based on Casey's theorem and is analogous to a short proof of Feuerbach's Tangency Theorem (Shailesh Shirali, On the Generalized Ptolemy Theorem. *Crux Mathematicorum* 22:2 (1996), 49–53).

Casey's theorem, or the Generalized Ptolemy Theorem, states an existence criterion for a circle to be tangent to four given circles in terms of their tangential distances. In our case we need a criterion for the existence of a circle that passes through two points (circles of radius zero) being tangent externally to one given (proper) circle and internally to another.

Consider first the abstract situation. Let the circles ω_i, $i = 1, 2, 3, 4$ be such that no one of them lies inside another. The suitable form of Casey's criterion says that there exists a circle tangent to ω_1 internally and to $\omega_{2,3,4}$ externally if and only if

$$t_{12}^4 + t_{13}^4 + t_{14}^4 + t_{23}^4 + t_{24}^4 + t_{34}^4 - 2(t_{12}^2 t_{34}^2 + t_{13}^2 t_{24}^2 + t_{14}^2 t_{23}^2) = 0,$$

where t_{ij} is the length of the common tangent (bitangent) to the circles ω_i and ω_j: external if $i, j \in \{2, 3, 4\}$ and internal if $i = 1$.

It is convenient to introduce the notation $P_{ij,kl} = t_{ik}t_{jl} + t_{il}t_{jk} - t_{ij}t_{kl}$. The polynomial in Casey's criterion factors out as

$$-(t_{12}t_{34} + t_{13}t_{24} + t_{14}t_{23})P_{12,34}P_{13,24}P_{14,23}.$$

We will apply Casey's criterion to the 4 circles: $\omega_1 = \alpha$, $\omega_2 = \beta$, $\omega_3 = A$ and $\omega_4 = B$ (of which the last two are point-circles).

Denote the relevant tangency points:

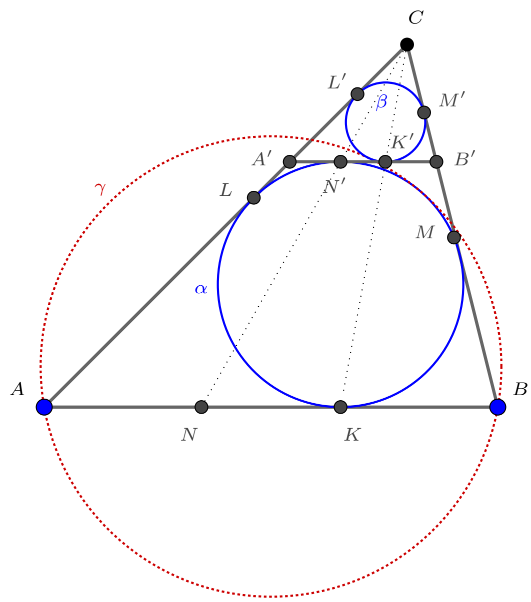
$$\begin{array}{ll} K = AB \cap \alpha, & K' = A'B' \cap \beta, \\ L = AC \cap \alpha, & L' = AC \cap \beta, \\ M = BC \cap \alpha, & M' = BC \cap \beta, \\ & N' = A'B' \cap \alpha. \end{array}$$

Let N be the (auxiliary) point of tangency of the line AB with corresponding excircle.

Instead of t_{ij} we will write $t_{\alpha A}$, $t_{\alpha\beta}$, etc. When at least one of the circles is a point, there is no difference between external and internal bitangents. However, we must consider the *internal* bitangent for the pair (α, β). We have

$$\begin{array}{ll} t_{\alpha\beta} = K'N', & t_{AB} = AB, \\ t_{\alpha A} = AK, & t_{\alpha B} = BK, \\ t_{\beta A} = AL', & t_{\beta B} = BM'. \end{array}$$

It remains to express these lengths in terms of the sides $a = BC$, $b = AC$, $c = AB$ of $\triangle ABC$ and to check that one of the P-factors in Casey's criterion equals zero. We will assume that $b \geq a$ as in the following figure.



Introduce also the following notation:

$$s = \frac{a+b+c}{2}, \quad u = s-a, \quad v = s-b, \quad w = s-c.$$

Let k be the homothety coefficient for the pair $\triangle A'B'C$, $\triangle ABC$, so that

$$k = \frac{CA'}{CA} = \frac{CB'}{CB} = \frac{A'L'}{AL}, \text{ etc.}$$

As in Honsberger problem H4, we find an equation for k: since $AN = BK = v = s - b$ and $k \cdot AN = A'N' = A'L = w - kb$, it follows that

$$k = \frac{w}{s}.$$

Therefore

$$\begin{aligned} &t_{AB} = c, \qquad t_{\alpha A} = u, \qquad t_{\alpha B} = v,\\ &t_{\alpha\beta} = k \cdot NK = k(u-v) = k(b-a),\\ &t_{\beta A} = b - k \cdot CL = b - kw,\\ &t_{\beta B} = a - k \cdot CM = a - kw. \end{aligned}$$

Now,

$$\begin{aligned} t_{\alpha A}t_{\beta B} &= u(a-kw),\\ t_{\alpha B}t_{\beta A} &= v(b-kw),\\ t_{\alpha\beta}t_{AB} &= kc(b-a). \end{aligned}$$

Next,

$$ua - vb = a(s-a) - b(s-b) = (b-a)(b+a-s) = (b-a)w.$$

Therefore

$$t_{\alpha A}t_{\beta B} - t_{\alpha B}t_{\beta A} = (b-a)w(1-k) = (b-a)w\frac{c}{s} = (b-a)ck = t_{\alpha\beta}t_{AB}$$

and we get the vanishing Casey's factor

$$P_{\alpha A,\beta B} = 0.$$

The verification of Casey's criterion is complete and the existence of the circle γ through A and B tangent to both α and β is proved.

Editor's comment. Let K' be the point where CK (in the notation of the featured solution) intersects $A'B'$, and let T be the point where the second tangent from K' touches α. The proposer proved that T is the point where α is tangent to γ, so that γ is the circle through A, B, and T.

4574. *Proposed by George Apostolopoulos.*

Let $x_1, \ldots, x_n$ be positive real numbers with $x_i < 64$ such that $\sum_{i=1}^n x_i = 16n$. Prove that

$$\sum_{i=1}^n \frac{1}{8-\sqrt{x_i}} \geq \frac{n}{4}.$$

We received 20 submissions of which 16 were correct and complete. We present 2 solutions.

Solution 1, by Michel Bataille.

Let $t_i = \dfrac{\sqrt{x_i}}{8}$ $(i = 1, \ldots, n)$. The problem then becomes to show that

$$\sum_{i=1}^n \frac{1}{1-t_i} \geq 2n$$

under the constraints $0 < t_i < 1$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^n t_i^2 = \frac{n}{4}$.

Now, for $t \in (0,1)$, we have

$$\frac{1}{1-t} = 1 + t^2 + t + \frac{t^3}{1-t}$$

and

$$t + \frac{t^3}{1-t} \geq 3t^2$$

(an inequality equivalent to $t(2t-1)^2 \geq 0$). We deduce that

$$\sum_{i=1}^n \frac{1}{1-t_i} = \sum_{i=1}^n (1+t_i^2) + \sum_{i=1}^n \left(t_i + \frac{t_i^3}{1-t_i}\right) \geq n + \frac{n}{4} + \sum_{i=1}^n 3t_i^2 = \frac{5n}{4} + 3\cdot\frac{n}{4} = 2n.$$

Solution 2, by Paul Bracken, Titu Zvonaru and the UCLan Cyprus Problem Solving Group (done independently).

For $x \in (0, 64)$ the following inequalities are equivalent to each other:

$$\begin{aligned} &\frac{1}{8-\sqrt{x}} \geqslant \frac{x+16}{128} \\ \iff\ & x\sqrt{x} - 8x + 16\sqrt{x} \geqslant 0 \\ \iff\ & \sqrt{x}(\sqrt{x}-4)^2 \geqslant 0. \end{aligned}$$

Therefore

$$\sum_{i=1}^{n} \frac{1}{8-\sqrt{x_i}} \geqslant \sum_{i=1}^{n} \frac{x_i+16}{128} = \frac{16n+16n}{128} = \frac{n}{4}\,.$$

Equality holds if and only if $x_i = 16$ for every i.

4575. *Proposed by Nguyen Viet Hung.*

Determine the coefficient of x in the following polynomial

$$\left(1+\binom{n}{0}x\right)\left(1+\binom{n}{1}x\right)^2\left(1+\binom{n}{2}x\right)^3\cdots\left(1+\binom{n}{n}x\right)^{n+1}.$$

We received 27 submissions and 26 of them were all correct and complete.

It is easy to check that the coefficient of x is

$$\binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} + \cdots + (n+1)\binom{n}{n}. \tag{1}$$

To get a closed form of equation (1), we present the following three solutions, two of which were common approaches.

Solution 1.

We regard equation (1) as the derivative of a function evaluated at $x = 1$. Using the binomial theorem, equation (1) can be written as

$$\begin{aligned} &\frac{d}{dx}\left[\binom{n}{0}x + \binom{n}{1}x^2 + \cdots + \binom{n}{n}x^{n+1}\right]\Big|_{x=1} \\ &= \frac{d}{dx}[x(1+x)^n]\big|_{x=1} \\ &= 2^n + n2^{n-1} \\ &= (n+2)2^{n-1}. \end{aligned}$$

Solution 2.

We evaluate equation (1) directly using the binomial identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}, \quad 1 \le k \le n.$$

Using the binomial theorem, equation (1) can be written as

$$\sum_{k=1}^{n} k\binom{n}{k} + 2^n = \sum_{k=1}^{n} n\binom{n-1}{k-1} + 2^n = n2^{n-1} + 2^n = (n+2)2^{n-1}.$$

Solution 3, by UCLan Cyprus Problem Solving Group.

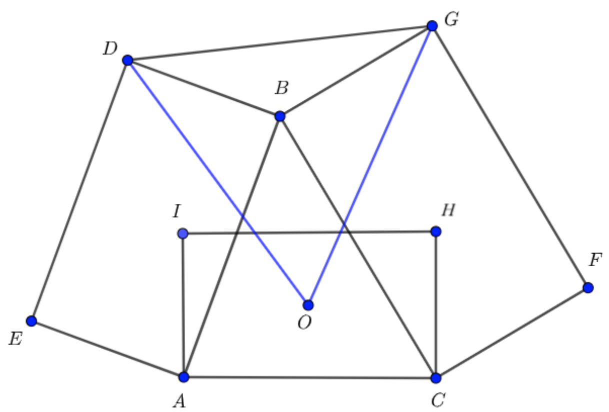
The $n+1$ residents of a city are going to elect a committee. This committee has to include the mayor of the city but there are no other restrictions. Then the committee will elect a chair which may be the mayor or not.

If the committee contains k other members apart from the mayor, this can be done in $(k+1)\binom{n}{k}$ ways. So in total, this can be done in $\sum_{k=0}^{n}(k+1)\binom{n}{k}$ ways.

We can also count this as follows: If the mayor also chairs the committee, then this can be done in 2^n ways. If he does not chair the committee, then this can be done in $n \cdot 2^{n-1}$ ways. (We choose first the chair and then the other committee members out of the $n-1$ citizens left.)

4576. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Let $ABDE$, $BCFG$ and $ACHI$ be three similar rectangles as given in the figure. Suppose $\frac{AB}{AE}$ is constant and let O be the center of $ACHI$. Show that $OD = OG$ and $\angle GOD$ is constant when A and C are fixed but B can move.



We received 12 submissions, of which one was incomplete. Most of the 11 correct solutions used complex numbers; we shall sample two of them.

Solution 1, by Oliver Geupel.

We consider the problem in the complex plane where point O is zero and the midpoint of segment CH is 1. Capital letters represent a point as well as its corresponding complex number. Let $\angle COH = 2\varphi$ and $OA = r$, so that $\cos\varphi = \frac{1}{r}$. We have $A = -re^{i\varphi}$ and $C = re^{-i\varphi}$. By similarity of rectangles, we have

$$D = A + (B - A)re^{i\varphi} = r\left(-e^{i\varphi} + re^{2i\varphi} + Be^{i\varphi}\right),$$

$$G = C + (B - C)re^{-i\varphi} = r\left(e^{-i\varphi} - re^{-2i\varphi} + Be^{-i\varphi}\right).$$

Also,

$$\frac{2e^{i\varphi}}{r} = 2\cos\varphi(\cos\varphi + i\sin\varphi) = 1 + \cos 2\varphi + i\sin 2\varphi = 1 + e^{2i\varphi},$$

or

$$e^{i\varphi} - r = -e^{i\varphi} + re^{2i\varphi}.$$

Hence,

$$\left(e^{-i\varphi} - re^{-2i\varphi}\right)e^{2i\varphi} = e^{i\varphi} - r = -e^{i\varphi} + re^{2i\varphi}.$$

Thus, $D = Ge^{2i\varphi}$. Consequently, $OD = OG$ and $\angle GOD = 2\varphi = \angle COH$.

Solution 2, by the UCLan Cyprus Problem Solving Group.

Assume that $AB = kAE$. We use complex numbers with B at the origin: for each point X we write z_X for the complex number representing X. Using the fact that multiplication by i represents an anticlockwise rotation through 90° about the point B, we find that $z_A = kz_Di$ and $z_C = -kz_Gi$. Then

$$z_I = z_A + (z_C - z_A)\frac{i}{k} = kz_Di + (-kz_Gi - kz_Di)\frac{i}{k} = kz_Di + z_G + z_D\,.$$

It follows that

$$z_O = \frac{z_C + z_I}{2} = \frac{kz_Di - kz_Gi + z_D + z_G}{2}$$

Now let M be the midpoint of DG; that is, $z_M = (z_D + z_G)/2$. We observe that

$$z_O - z_M = \frac{ki(z_D - z_G)}{2},$$

from which it follows that OM is perpendicular to DG with $OM = \frac{k}{2}DG$; whence,

$$k = \frac{MO}{MG} = \frac{AC}{AI} = \cot\angle ICA$$

so that the triangles GMO and IAC are similar.

We can now conclude that $OD = OG$ with $\angle GOD = 2\angle ICA$, which is constant.

Editor's comment. Sergey Sadov generalized the problem by replacing the three given similar rectangles by three similar parallelograms; his conclusion is that the triangles ODG and OAI are similar.

4577. *Proposed by Nikolai Osipov.*

For any integer k, solve the equation

$$xy^2 + (kx^2 + 1)y + x^4 + 1 = 0$$

in integers x, y.

We received 6 submissions, of which 2 solutions were incomplete. We present the solution by Sergey Sadov, slightly modified by the editor.

For convenience, we use (x, y, k) to denote any solution, where k represents H.

1. Note first that $y = 0$ yields no solution and $(0, -1, k)$ is a solution for all k. Hence we may assume that $xy \neq 0$.

2. If $x = \pm 1$, then y must divide 2, so $y \in \{\pm 1, \pm 2\}$. It turns out that the 4 possibilities correspond to integer values of k to yield 8 triples (x, y, k) below:

$(1, 1, -4), (1, -1, 2), (1, 2, -4), (1, -2, 2), (-1, 1, -2), (-1, -1, 0), (-1, 2, 0), (-1, -2, -2).$

In the sequel we assume that $|x| \geq 2$.

3. Let $z = \dfrac{1 + x^4}{y}$, which is an integer since by the given equations we have

$$z = -xy - kx^2 - 1 \tag{1}$$

and

$$yz = x^4 + 1. \tag{2}$$

Note that $y \equiv -1 \pmod{x}$ and $z \equiv -1 \pmod{x}$. Hence we may write $y = ux - 1, z = vx - 1$ for some $u, v \in \mathbb{Z}$. It follows that $xy \equiv -x \left(\bmod\ x^2\right)$, hence by (1) we have $z+1 \equiv -xy \equiv x \left(\bmod\ x^2\right)$. Since $z+1 = vx$, we have $vx \equiv x \left(\bmod\ x^2\right)$. Therefore $v = qx + 1$ for some integer q.

Next $yz = uvx^2 - (u + v)x + 1$. Then by (2), we have $yz \equiv -1 \left(\bmod\ x^2\right)$, so we get $(u + v)x \equiv 0 \left(\bmod\ x^2\right)$. Since $v \equiv -1 \pmod{x}$, it follows that $u = px - 1$ for some integer p. Therefore

$$y = px^2 - x - 1, \qquad z = qx^2 + x - 1. \tag{3}$$

4. Using (2) and (3), we can exclude the possibility that $|p| > 1$ and $|q| > 1$ simultaneously as follows. Since $|x| \geq 2$, then $x^2 \geq 2|x|$ and $x^2 \geq \frac{1}{4}$, we deduce that if $|t| \geq 2$, then $|tx^2 \pm x - 1| \geq x^2(2 - 1/2 - 1/4) = (5/4)x^2$.Suppose $|p| > 1$ and $|q| > 1$. Substituting $t = p$ and $t = q$ respectively into the above inequalities, we get $|yz| \geq (5/4)^2 x^4 > x^4 + 1$, contradicting (2). Hence we conclude that at least one of the inequalities $|p| \leq 1$ and $|q| \leq 1$ is true.

Thus there are 6 cases to consider: $p \in \{0, \pm 1\}$ and (separately) $q \in \{0, \pm 1\}$.

5. In all cases, we exploit (2), specifically the fact that $x^4 + 1$ is divisible by the integer expressed as a quadratic polynomial from (3), of the form $\epsilon x^2 \pm x - 1$,

where $\epsilon \in \{0, \pm 1\}$. This divisibility condition turns out to be sufficiently strong to imply that only a finite number (if any) of suitable integer values of x exist.

In essence, we use Euclid's algorithm to find the GCD of the polynomial x^4+1 and the given polynomial of degree 1 or 2, avoiding however the divisions by integers greater than 1. Then we analyse the result to see what integer values of x are admissible. It will be useful to express k in terms of x and y or in terms of x and z:

$$k = -\frac{xy^2 + y + x^4 + 1}{x} = -\frac{x^5 + x + z^2 + z}{x^2 z}.$$

In Cases 1–3 below, for given value of admissible p, we use $y = px^2 - x - 1$ as a divisor-to-be of $x^4 + 1$, and in Cases 4–6, for given value of admissible q, we use $z = qx^2 + x - 1$ as such a candidate divisor.

Case 1. $p = -1$, so $y = -(x^2 + x + 1)$. We have

$$\begin{aligned} &x^4 + 1 = (x^2 + x + 1)(x^2 - x) + (x + 1), \\ &x^2 + x + 1 = (x + 1)x + 1. \end{aligned}$$

Conclusion: $\gcd(x^4 + 1, x^2 + x + 1) = 1$ for any integer x. In order for y to be a divisor of x^4+1, we must have $|y| = 1$. The value $y = 1$ yields $x^2+x+2 = 0$, which produces no solution. The value $y = -1$ yields $x^2 + x = 0$, so $(x, y) = (0, -1)$ or $(-1, -1)$, which have already been produced.

Case 2. $p = 0$, so $y = -(x + 1)$. We have

$$x^4 + 1 = (x^3 - x^2 + x - 1)(x + 1) + 2.$$

Hence $\gcd(yz, y) = \gcd(y, 2)$. It follows that $|y| = 1$ or $|y| = 2$. Considering the 4 possible values of y, we find the following new solutions: $(x, y, k) = (-2, 1, -4)$ and $(-3, 2, -4)$. The value $y = -1$ yields $x = 0$ and a solution already listed in item 1 above. The value $y = -2$ yields $x = 1$ and a solution already listed above in item 2.

Case 3. $p = 1$, so $y = x^2 - x - 1$. We have

$$\begin{aligned} &x^4 + 1 = (x^2 - x - 1)(x^2 + x + 2) + 3(x + 1), \\ &x^2 - x - 1 = (x + 1)(x - 2) + 1. \end{aligned}$$

Hence $\gcd(x^4+1, x^2-x-1) = \gcd(x^2-x-1, 3(x+1)) = \gcd(x^2-x-1, x+1) = 1$ since 3 does not divide x^4+1. We conclude that x^4+1 and x^2-x-1 are coprime for any integer x. Therefore we have $|y| = 1$.

The value $y = 1$ yields $x^2 - x - 2 = 0$ and the new solution $(x, y, k) = (2, 1, -5)$. (The solution when $x = -1$ is not new). The value $y = -1$ yields $x^2 - x = 0$ and solutions already listed in items 1 and 2 above.

Case 4. $q = -1$, so $z = -(x^2 - x + 1)$. Here

$$\begin{aligned} &x^4 + 1 = (x^2 - x + 1)(x^2 + x) - (x - 1), \\ &x^2 - x + 1 = (x - 1)x + 1. \end{aligned}$$

Hence $\gcd(yz, z) = 1$ and we have $|z| = 1$. So either $x^2 - x = 0$ or $x^2 - x + 2 = 0$. No new solutions result from this case.

Case 5. $q = 0$, so $z = x - 1$. Here

$$x^4 + 1 = (x-1)(x^3 + x^2 + x + 1) + 2.$$

Therefore $|z| = 1$ or $|z| = 2$, yielding 2 new solutions:

$$\begin{aligned} z = 1 &\Rightarrow (x, y, k) = (2, 17, -9), \\ z = 2 &\Rightarrow (x, y, k) = (3, 41, -14). \end{aligned}$$

Case 6. $q = 1$, so $z = x^2 + x - 1$. Here

$$\begin{aligned} &x^4 + 1 = (x^2 + x - 1)(x^2 - x + 2) - 3(x-1), \\ &x^2 + x - 1 = (x-1)(x+2) + 1. \end{aligned}$$

Like in Case 3, we conclude that $|z| = 1$. The value $z = 1$ yields $x^2 + x - 2 = 0$ and a new solution $(x, y, k) = (-2, 17, 8)$.

In summary, there are 15 solutions (x, y, k) including infinite class $(0, -1, k), k \in \mathbb{Z}$ and the 14 solutions given by:

$$\begin{aligned} &(1,-2,2), (1,-1,2), (1,1,-4), (1,2,-4), (-1,-2,-2), (-1,-1,0), (-1,1,-2), \\ &(-1,2,0), (-2,1,-4), (-3,2,-4), (2,1,-5), (2,17,-9), (3,41,-14), (-2,17,8). \end{aligned}$$

4578. *Proposed by Ed Barbeau. Dedicated in memoriam to Richard K. Guy.*

Suppose that $\{a, b, c\}$ and $\{u, v, w\}$ are two distinct sets of three integers for which $a + b + c = u + v + w$ and $a^2 + b^2 + c^2 = u^2 + v^2 + w^2$. What is the minimum possible value assumed by $|abc - uvw|$?

We received 9 solutions, one of which was incorrect. We present the solution by UCLan Cyprus Problem Solving Group.

We observe that if (a, b, c, u, v, w) is a suitable sextuple, then so is

$$(a', b', c', u', v', w') = (a-1, b-1, c-1, u-1, v-1, w-1).$$

Furthermore, $abc - uvw = a'b'c' - uvw$. From the above, we may assume that $w = 0$. Thus we are trying to minimize $|abc|$.

There are two possible interpretations, one in which a, b, c (respectively u, v, w) have to be pairwise distinct and one in which we are allowing, for example, $a = b$.

We will show that in the first interpretation the minimum is 12 and in the second interpretation the minimum is 4. The examples are $(1, 2, 6)$ and $(0, 4, 5)$ in the first case, and $(1, 1, 4)$ and $(0, 3, 3)$ in the second case. It remains to show that these cannot be improved.

We start with the second interpretation which is simpler. Since $a^2+b^2+c^2 = u^2+v^2$ then a, b, c cannot all be odd as then $u^2 + v^2 \equiv 3 \bmod 4$ which is impossible. So it remains to check that we cannot have $|abc| = 2$.

Note that if $abc = 0$, say because $a = 0$, then $b^2 + c^2 = u^2 + v^2$ and $b + c = u + v$ which gives that $\{a, b, c\} = \{u, v, w\}$.

If $|abc| = 2$, then without loss of generality we have $a = \pm 2, b = \pm 1, c = \pm 1$. Then $u^2 + v^2 = a^2 + b^2 + c^2 = 6$ which is easily seen to be impossible.

In the first interpretation it remains to prove that $|abc| \notin \{4, 6, 8, 10\}$. Without loss of generality we need to consider the triples $(\pm 2n, \pm 1, \pm 1)$ and $(\pm n, \pm 2, \pm 1)$ for $n \in \{2, 3, 4, 5\}$. These give

$$u^2 + v^2 \in \{9, 18, 14, 38, 21, 66, 30, 102\}\,.$$

Since we additionally demand $u, v \neq 0$, they can all be checked by hand that they are impossible except in the case $u = 3, v = -3$. Then without loss of generality $a = \pm 4, b = \pm 1, c = \pm 1$. But then $u + v + w = 0$, while $a + b + c \neq 0$.

Note: In order to avoid laborious checking by hand of the impossibility of the above, let us recall that if p is a prime of the form $p \equiv 3 \bmod 4$ with $p|u^2+v^2$ then we also have that $p|u$ and $p|v$, which together imply that $p^2|u^2 + v^2$. Applying it with $p = 3, 7, 19$ immediately excludes all possibilities except $u^2 + v^2 \in \{9, 18\}$. For those two we can immediately see that they lead to $u = 0, v = 0$ or $u = v$.

Editor's comment: The proposer's intended interpretation was the one requiring uniqueness, and, indeed, the stipulation that $\{a, b, c\}$ and $\{u, v, w\}$ are sets, as opposed to (ordered) triples, would suggest that this is the correct interpretation. Nonetheless, both versions of the problem are of interest, and the two versions were equally represented in the submitted solutions.

4579. *Proposed by George Stoica.*

Let $a, b, c \in \mathbb{Z}^*$ such that $\dfrac{ab}{c} + \dfrac{bc}{a} + \dfrac{ca}{b} \in \mathbb{Z}$. Prove that $\dfrac{ab}{c}, \dfrac{bc}{a}, \dfrac{ca}{b} \in \mathbb{Z}$.

We received 16 submissions, all correct. We present a composite of the nearly identical but independent solutions by 8 different solvers.

By the given assumption, the function

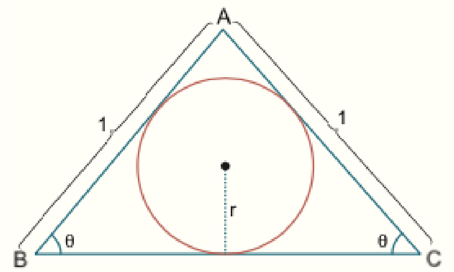
$$f(x) = (x - \frac{ab}{c})(x - \frac{bc}{a})(x - \frac{ca}{b}) = x^3 - (\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b})x^2 + (a^2 + b^2 + c^2)x - abc$$

is a monic polynomial with integer coefficients. Hence by the well-know Rational Roots theorem, every rational root of $f(x)$ must be an integer. It then follows that $\dfrac{ab}{c}, \dfrac{bc}{a}$, and $\dfrac{ca}{b}$ must all be integers.

Editor's comment: Walther Janous pointed out this proposal was problem #6 of the 2016 Austrian Mathematics Olympiad.

4580. *Proposed by Alpaslan Ceran.*

In an isosceles triangle ABC with $AB = AC = 1$, find the length of BC which maximizes the inradius.



We received 36 solutions, all of which were correct. We present the solution by Brian Bradie.

Let x denote the length of BC. Then the inradius is

$$r = \frac{x}{2}\sqrt{\frac{1-\frac{x}{2}}{1+\frac{x}{2}}} = \frac{x(2-x)}{2\sqrt{4-x^2}},$$

and

$$\frac{dr}{dx} = \frac{4-2x-x^2}{2(2+x)\sqrt{4-x^2}}.$$

For $0 < x < 2$, the only critical point of r is $x = \sqrt{5}-1$. Because

$$\frac{dr}{dx} > 0 \text{ for } 0 < x < \sqrt{5}-1 \quad \text{and} \quad \frac{dr}{dx} < 0 \text{ for } \sqrt{5}-1 < x < 2,$$

it follows that r achieves an absolute maximum when $x = \sqrt{5}-1$. Thus, in an isosceles triangle ABC with $AB = AC = 1$, the inradius is maximum when the length of BC is $\sqrt{5}-1$.