

# Canadian Mathematical Olympiad 2021

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**Note:** Each problem starts on a new page.

### Problem No. 1.

Let  $ABCD$  be a trapezoid with  $AB$  parallel to  $CD$ ,  $|AB| > |CD|$ , and equal edges  $|AD| = |BC|$ . Let  $I$  be the center of the circle tangent to lines  $AB$ ,  $AC$  and  $BD$ , where  $A$  and  $I$  are on opposite sides of  $BD$ . Let  $J$  be the center of the circle tangent to lines  $CD$ ,  $AC$  and  $BD$ , where  $D$  and  $J$  are on opposite sides of  $AC$ . Prove that  $|IC| = |JB|$ .

**Solution.** Let  $\{P\} = AC \cap BD$  and let  $\angle APB = 180 - 2a$ . Since  $ABCD$  is an isosceles trapezoid,  $APB$  is an isosceles triangle. Therefore  $\angle PBA = a$ , which implies that  $\angle PBI = 90^\circ - a/2$  since  $I$  lies on the external bisector of  $\angle PBA$ . Since  $I$  lies on the bisector of  $\angle CPB$ , it follows that  $\angle BPI = a$  and hence that  $IPB$  is isosceles with  $|IP| = |PB|$ . Similarly  $JPC$  is isosceles with  $|JP| = |PC|$ . So, in the triangles  $CPI$  and  $BPJ$  we have  $PI \equiv PB$  and  $PJ \equiv CP$ . Since  $I$  and  $J$  both lie on the internal bisector of  $\angle BPC$ , it follows that triangles  $CPI$  and  $BPJ$  are congruent. Therefore  $|IC| = |JB|$ .  $\square$

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## Problem No. 2.

Let  $n \geq 2$  be some fixed positive integer and suppose that  $a_1, a_2, \dots, a_n$  are positive real numbers satisfying  $a_1 + a_2 + \dots + a_n = 2^n - 1$ .

Find the minimum possible value of

$$\frac{a_1}{1} + \frac{a_2}{1 + a_1} + \frac{a_3}{1 + a_1 + a_2} + \dots + \frac{a_n}{1 + a_1 + a_2 + \dots + a_{n-1}}.$$

**Solution.** We claim the the minimum possible value of this expression is  $n$ . Observe that by AM-GM, we have that

$$\begin{aligned} & \frac{a_1}{1} + \frac{a_2}{1 + a_1} + \dots + \frac{a_n}{1 + a_1 + a_2 + \dots + a_{n-1}} \\ &= \frac{1 + a_1}{1} + \frac{1 + a_1 + a_2}{1 + a_1} + \dots + \frac{1 + a_1 + a_2 + \dots + a_n}{1 + a_1 + a_2 + \dots + a_{n-1}} - n \\ &\geq n \cdot \sqrt[n]{\frac{1 + a_1}{1} \cdot \frac{1 + a_1 + a_2}{1 + a_1} \dots \frac{1 + a_1 + a_2 + \dots + a_n}{1 + a_1 + a_2 + \dots + a_{n-1}}} - n \\ &= n \cdot \sqrt[n]{1 + a_1 + a_2 + \dots + a_n} - n \\ &= 2n - n = n. \end{aligned}$$

Furthermore, equality is achieved when  $a_k = 2^{k-1}$  for each  $1 \leq k \leq n$ . □

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## Problem No. 3.

At a dinner party there are  $N$  hosts and  $N$  guests, seated around a circular table, where  $N \geq 4$ . A pair of two guests will chat with one another if either there is at most one person seated between them or if there are exactly two people between them, at least one of whom is a host. Prove that no matter how the  $2N$  people are seated at the dinner party, at least  $N$  pairs of guests will chat with one another.

**Solution.** Let a *run* refer to a maximal group of consecutive dinner party guests all of whom are the same type (host or guest). Suppose that there are exactly  $k$  runs of hosts and  $k$  runs of guests. Let  $G_i$  and  $H_i$  denote the number of runs of guests and hosts, respectively, of length exactly  $i$ . Furthermore, let  $X$  denote the number of hosts surrounded by two runs of guests, both of length exactly 1. We claim that the number of pairs of guests who chat is at least

$$2N - 3k + G_1 + 2H_1 + H_2 - X.$$

The number of pairs of guests who chat with no host between them is at least the sum of  $\max\{2\ell - 3, 0\}$  over all guest run lengths  $\ell$ . This sum is at least  $2N - 3k + G_1$ . The number of pairs of guests who chat with exactly two hosts between them is  $H_2$ . Furthermore, the number of pairs of guests who chat with exactly one host between them is at least  $2H_1 - X$ . This is because any host surrounded by two runs of guests causes at least two pairs of guests to chat unless these runs are both of length exactly 1. This proves the claim. Now note that

$$2H_1 + H_2 + N \geq 3k$$

because each run of hosts contributes at least three to the left hand side. Furthermore, pairing each run counted in  $X$  with the guest run of length 1 immediately following it in clockwise order shows that  $G_1 \geq X$ . Combining these inequalities yields that  $2N - 3k + G_1 + 2H_1 + H_2 - X \geq N$ , completing the proof of the desired result.  $\square$

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## Problem No. 4.

A function  $f$  from the positive integers to the positive integers is called *Canadian* if it satisfies

$$\gcd(f(f(x)), f(x+y)) = \gcd(x, y)$$

for all pairs of positive integers  $x$  and  $y$ .

Find all positive integers  $m$  such that  $f(m) = m$  for all Canadian functions  $f$ .

**Solution.** Define an  $m \in \mathbb{N}$  to be *good* if  $f(m) = m$  for all such  $f$ . It will be shown that  $m$  is good if and only if  $m$  has two or more distinct prime divisors. Let  $P(x, y)$  denote the assertion

$$\gcd(f(f(x)), f(x+y)) = \gcd(x, y)$$

for a pair  $x, y \in \mathbb{N}$ . Let  $x$  be a positive integer with two or more distinct prime divisors and let  $p^k$  be largest power of one of these prime divisors such that  $p^k \mid x$ . If  $x = p^k \cdot q$ , then  $p^k$  and  $q$  are relatively prime and  $x > p^k, q > 1$ . By  $P(q, x - q)$ ,

$$\gcd(f(f(q)), f(x - q + q)) = \gcd(f(f(q)), f(x)) = \gcd(q, x - q) = q$$

which implies that  $q \mid f(x)$ . By  $P(p^k, x - p^k)$ ,

$$\gcd(f(f(p^k)), f(x - p^k + p^k)) = \gcd(f(f(p^k)), f(x)) = \gcd(p^k, x - p^k) = p^k$$

which implies that  $p^k \mid f(x)$ . Since  $p^k$  and  $q$  are relatively prime,  $x = p^k \cdot q$  divides  $f(x)$ , which implies that  $f(x) \geq x$ . Now assume for contradiction that  $f(x) > x$ . Let  $y = f(x) - x > 0$  and note that, by  $P(x, y)$ , it follows that

$$f(f(x)) = \gcd(f(f(x)), f(x + f(x) - x)) = \gcd(x, f(x) - x) = \gcd(x, f(x)).$$

Therefore  $f(f(x)) \mid x$  and  $f(f(x)) \mid f(x)$ . By  $P(x, x)$ , it follows that

$$\gcd(f(f(x)), f(2x)) = \gcd(x, x) = x.$$

This implies that  $x \mid f(f(x))$ , which when combined with the above result, yields that  $f(f(x)) = x$ . Since  $x \mid f(x)$  and  $x$  is divisible by at least two distinct prime numbers,  $f(x)$  is also divisible by at least two distinct prime numbers. As shown previously, this implies that  $f(x) \mid f(f(x)) = x$ , which is a contradiction since  $f(x) > x$ . Therefore  $f(x) = x$  for all positive integers  $x$  with two or more distinct prime divisors.

Now it will be shown that all  $m \in \mathbb{N}$  such that either  $m$  has one prime divisor or  $m = 1$  are not good. In either case, let  $m = p^k$  where  $k \geq 0$  and  $p$  is a prime number and consider the function satisfying that  $f(p^k) = p^{k+1}$ ,  $f(p^{k+1}) = p^k$  and  $f(x) = x$  for all  $x \neq p^k, p^{k+1}$ . Note that this function also satisfies that  $f(f(x)) = x$  for all positive integers  $x$ . If  $x + y \neq p^k, p^{k+1}$ , then  $P(x, y)$  holds by the Euclidean

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algorithm since  $f(f(x)) = x$  and  $f(x + y) = x + y$ . If  $x + y = p^{k+1}$ , then  $P(x, y)$  is equivalent to  $\gcd(x, p^k) = \gcd(x, p^{k+1} - x) = \gcd(x, p^{k+1})$  for all  $x < p^{k+1}$  which holds since the greatest power of  $p$  that can divide  $x$  is  $p^k$ . If  $x + y = p^k$ , then  $P(x, y)$  is equivalent to  $\gcd(x, p^{k+1}) = \gcd(x, p^k - x) = \gcd(x, p^k)$  for all  $x < p^k$  which holds as shown above. Note that if  $m = 1$  then this case cannot occur. Since this function satisfies  $P(x, y)$ ,  $m$  is good if and only if  $m$  has two or more distinct prime divisors.  $\square$

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## Problem No. 5.

Nina and Tadashi play the following game. Initially, a triple  $(a, b, c)$  of nonnegative integers with  $a + b + c = 2021$  is written on a blackboard. Nina and Tadashi then take moves in turn, with Nina first. A player making a move chooses a positive integer  $k$  and one of the three entries on the board; then the player increases the chosen entry by  $k$  and decreases the other two entries by  $k$ . A player loses if, on their turn, some entry on the board becomes negative.

Find the number of initial triples  $(a, b, c)$  for which Tadashi has a winning strategy.

**Solution.** The answer is  $3^{\text{number of 1's in binary expansion of } 2021} = 3^8 = 6561$ .

Throughout this solution, we say two nonnegative integers *overlap in the  $2^\ell$  position* if their binary representations both have a 1 in that position. We say that two nonnegative integers *overlap* if they overlap in some position. Our central claim is the following.

**Claim 1.** *A triple  $(x, y, z)$  is losing if and only if no two of  $x, y, z$  overlap.*

Let  $d_\ell(a)$  denote the bit in the  $2^\ell$  position of the binary representation of  $a$ . Let  $\&$  denote the bitwise and operation:  $x \& y$  is the number satisfying  $d_\ell(x \& y) = d_\ell(x)d_\ell(y)$  for all  $\ell$ .

**Lemma 1.** *Let  $x, y, z$  be nonnegative integers, at least one pair of which overlaps. Define  $x' = (x+y)\&(x+z)$  and  $y', z'$  cyclically. At least one of the inequalities  $x < x', y < y', z < z'$  holds.*

*Proof.* Let  $\ell$  be maximal such that two of  $x, y, z$  overlap in the  $2^\ell$  position. We case on how many of the additions  $x + y, x + z, y + z$  involve a carry from the  $2^\ell$  position to the  $2^{\ell+1}$  position, and on the values of  $d_{\ell+1}(x), d_{\ell+1}(y), d_{\ell+1}(z)$ . Because at least two of  $d_\ell(x), d_\ell(y), d_\ell(z)$  equal 1, at least one of the additions  $x + y, x + z, y + z$  involves a carry from the  $2^\ell$  position.

**Case 1.** *One carry.*

WLOG let  $x + y$  be the addition with the carry. Then,  $d_\ell(x) = d_\ell(y) = 1$  and  $d_\ell(z) = 0$ . Since  $x, y, z$  do not overlap in any position left of the  $2^\ell$  position, the binary representations of  $z, z'$  agree left of the  $2^\ell$  position. As the additions  $x + z$  and  $y + z$  do not involve a carry from the  $2^\ell$  position, we have  $d_\ell(x + z) = d_\ell(y + z) = 1$ , and thus  $d_\ell(z') = 1$ . Thus  $z' > z$ , as desired.

**Case 2.** *At least two carries:  $x + y$  and  $x + z$  carry and  $d_{\ell+1}(y) = d_{\ell+1}(z) = 0$ , or cyclic equivalent. ( $y + z$  may or may not carry.)*

Let  $i$  be maximal such that  $d_{\ell+1}(x) = \dots = d_{\ell+i}(x) = 1$  (possibly  $i = 0$ ). By maximality of  $\ell$ ,  $d_{\ell+1}(y) = \dots = d_{\ell+i}(y) = d_{\ell+1}(z) = \dots = d_{\ell+i}(z) = 0$ . By maximality of  $i$ ,  $d_{\ell+i+1}(x) = 0$ .

If  $d_{\ell+i+1}(y) = d_{\ell+i+1}(z) = 0$ , then  $d_{\ell+i+1}(x + y) = d_{\ell+i+1}(x + z) = 1$ , so  $d_{\ell+i+1}(x') = 1$ . The binary representations of  $x$  and  $x'$  agree to the left of the  $2^{\ell+i+1}$  position, so  $x' > x$ .

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Otherwise, WLOG  $d_{\ell+i+1}(y) = 1$  and  $d_{\ell+i+1}(z) = 0$ . (Note that, here we in fact have  $i \geq 1$ .) Then  $d_{\ell+i+1}(y+z) = d_{\ell+i+1}(x+z) = 1$ , so  $d_{\ell+i+1}(z') = 1$ . The binary representations of  $z$  and  $z'$  agree to the left of the  $2^{\ell+i+1}$  position, so  $z' > z$ .

**Case 3.** *At least two carries, and the condition in Case 2 does not occur.*

WLOG let  $x+y, x+z$  involve carries. Since the condition in Case 2 does not occur,  $d_{\ell+1}(y) = 1$  or  $d_{\ell+1}(z) = 1$ . In either case,  $d_{\ell+1}(x) = 0$ . WLOG  $d_{\ell+1}(y) = 1$  and  $d_{\ell+1}(z) = 0$ .

Since the condition in Case 2 does not occur,  $y+z$  does not involve a carry from the  $2^\ell$  position. (Otherwise,  $x+y$  and  $y+z$  carry and  $d_{\ell+1}(x) = d_{\ell+1}(z) = 0$ .) Then  $d_{\ell+1}(x+z) = d_{\ell+1}(y+z) = 1$ , so  $d_{\ell+1}(z') = 1$ . The binary representations of  $z$  and  $z'$  agree to the left of the  $2^{\ell+1}$  position, so  $z' > z$ .  $\square$

*Proof of Claim 1.* Proceed by strong induction on  $x+y+z$ . There is no base case.

Suppose by induction the claim holds for all  $(x, y, z)$  with sum less than  $N$ . Consider a triple  $(x, y, z)$  with  $x+y+z = N$ .

Suppose no two of  $x, y, z$  overlap. If all moves from this position lead to positions with a negative coordinate,  $(x, y, z)$  is a losing position, as claimed. Otherwise, the player increases or decreases all coordinates by  $k$ . Consider the smallest  $m$  such that  $d_m(k) = 1$ . The player's move will toggle each of  $d_m(x), d_m(y), d_m(z)$ . Since at most one of the original  $d_m(x), d_m(y), d_m(z)$  is 1, at least two of the new  $d_m(x), d_m(y), d_m(z)$  will be 1. So, two of the new  $x, y, z$  overlap. By induction, the new  $(x, y, z)$  is winning. Thus the original  $(x, y, z)$  is losing, as claimed.

Conversely, suppose at least one pair of  $x, y, z$  overlap. By Lemma 1, at least one of  $x < x', y < y', z < z'$  holds. WLOG  $x < x'$ . Let the player to move choose  $k = x' - x$ , decrease  $y, z$  by  $k$ , and increase  $x$  by  $k$ . The new coordinates are nonnegative, as

$$y - k = x + y - x' \geq 0$$

because  $x' \leq x + y$ , and similarly for the  $z$  coordinate. Moreover, the binary representation of the new  $x$  consists of the 1's in the binary representations of both  $x + y$  and  $x + z$ ; the binary representation of the new  $y$  consists of the 1's in that of  $x + y$  but not  $x + z$ ; and the binary representation of the new  $z$  consists of the 1's in that of  $x + z$  but not  $x + y$ . So, no two of the new  $x, y, z$  overlap. By induction, the new  $(x, y, z)$  is losing. Thus the original  $(x, y, z)$  is winning, as claimed.  $\square$

We use Claim 1 to count the losing positions  $(x, y, z)$  with

$$x + y + z = 2021 = 11111100101_2.$$

In each position where  $d_\ell(2021) = 0$ , losing positions must have  $d_\ell(x) = d_\ell(y) = d_\ell(z) = 0$ . In each position where  $d_\ell(2021) = 1$ , the bit triplet  $(d_i(x), d_i(y), d_i(z))$  is one of  $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ . This gives a count of  $3^8 = 6561$ .

$\square$