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`https://publications.cms.math.ca/cruxbox/`

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IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

Math in the Time of Coronavirus. While we find ourselves adjusting many of our habits, changing our everyday activities due to various necessary restrictions, for me, engaging in mathematics offered a kind of repose and feeling of normalcy. I now teach via videoconferencing with a toddler in the background and hold research meetings over the phone while walking my dog. However, my ***Crux*** work has changed very little, and I am indeed grateful for that. So if you too are looking for a mathematical distraction from the pandemic, look no further.

This issue has a couple of non-standard features. First, Chris Fisher has put together a comprehensive index for Michel Bataille's *Focus On* ... column. There, you will find columns arranged according to topics (Algebra, Geometry, Inequalities, Calculus, Combinatorics, Trigonometry) with citations and short content descriptions. Secondly, in this issue we have 25 Bonus Problems. While we have high standards for problem acceptance, we simply receive too many good problems. As we try to balance each issue's problem offerings between authors and topics, we inevitably acquire a backlog. To ensure that no problem stays in the waiting-to-be-published stage for too long, we will be occasionally publishing a Bonus Problems list. This gives authors a chance to cite their problems, while providing our readers with more materials. Please note that this material is truly bonus: we will not be considering solutions to these problems.

Stay healthy and safe.

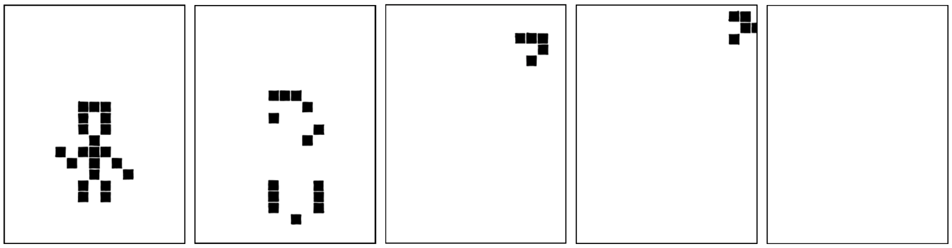
Kseniya Garaschuk

IN MEMORIAM

Fans of recreational mathematics will mourn the recent death of John H. Conway. While many of his important discoveries in geometry and group theory can't be explained without many hours of lead-in, (even the grand antiprism is fairly mindboggling) and even the many of our readers will have experimented with Conway's "Game of Life." Maybe you have read "On Numbers and Games", or the more popular "Winning Ways" which he coauthored with Elwyn Berlekamp and Richard Guy (by sad coincidence, all three authors of this tour de force have died within a little over a year of each other.) Maybe at some point you learned his "Doomsday Rule" for finding the day of the week of any day in history, or how to win at Nim or Hackenbush. However it happened, whatever it was: so many of us are the richer for John's time among us, and if you aren't yet - it's not too late!

Randall Munroe's XKCD webcomic gave John the rare tribute of a memorial cartoon. Here are some snapshots by Mr. Munroe's generous permission. For the full animation, see `https://imgs.xkcd.com/comics/rip_john_conway.gif`

Robert Dawson



MathemAttic

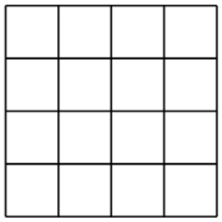
No. 14

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

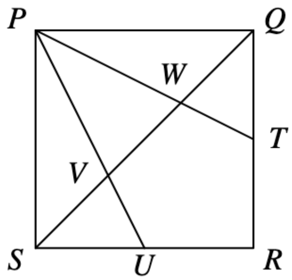
To facilitate their consideration, solutions should be received by **June 15, 2020**.

MA66. The 16 small squares shown in the diagram each have a side length of 1 unit. How many pairs of vertices (intersections of lines) are there in the diagram whose distance apart is an integer number of units?

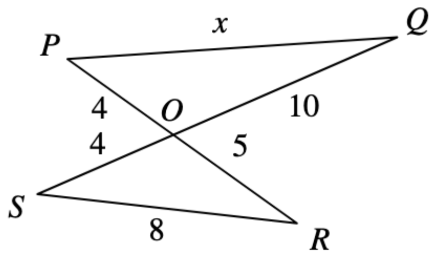


MA67. Consider numbers of the form $10n + 1$, where n is a positive integer. We shall call such a number *grime* if it cannot be expressed as the product of two smaller numbers, possibly equal, both of which are of the form $10k + 1$, where k is a positive integer. How many grime numbers are there in the sequence $11, 21, 31, 41, \ldots, 981, 991$?

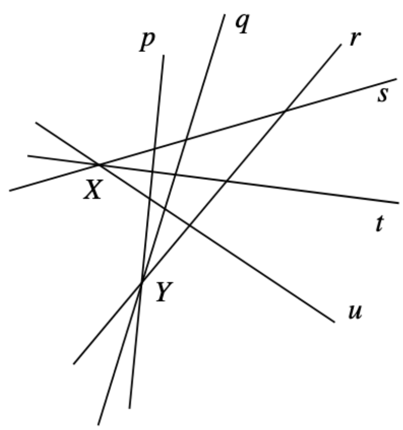
MA68. $PQRS$ is a square. The points T and U are the midpoints of QR and RS respectively. The line QS cuts PT and PU at W and V respectively. What fraction of the area of the square $PQRS$ is the area of the pentagon $RTWVU$?



MA69. The diagram shows two straight lines PR and QS crossing at O. What is the value of x?



MA70. Challengeborough's underground train network consists of six lines, p, q, r, s, t, u, as shown. Wherever two lines meet, there is a station which enables passengers to change lines. On each line, each train stops at every station. Jessica wants to travel from station X to station Y. She does not want to use any line more than once, nor return to station X after leaving it, nor leave station Y having reached it. How many different routes, satisfying these conditions, can she choose?



. . . .

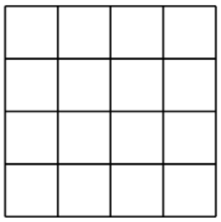
Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2020**.

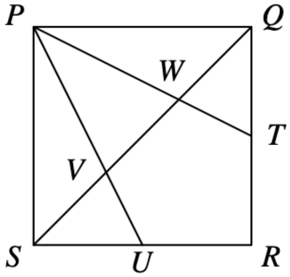
La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA66. Les 16 petits carrés illustrés ci-bas sont tous de côtés 1 unité. Combien de paires de sommets se trouvent à une distance entière d'unités?

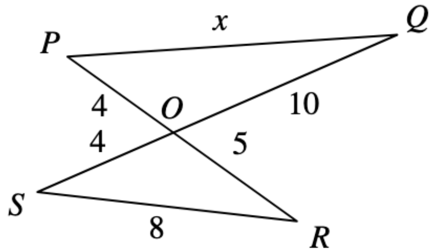


MA67. Soient les entiers de la forme $10n + 1$, où n est entier positif. Un tel nombre est dit *remier* s'il n'est pas possible de le représenter comme produit de deux plus petits entiers possiblement égaux, toujours de la forme $10k + 1$ où k serait entier positif. Combien de nombres remiers y a-t-il parmi $11, 21, 31, 41, \ldots, 981, 991$?

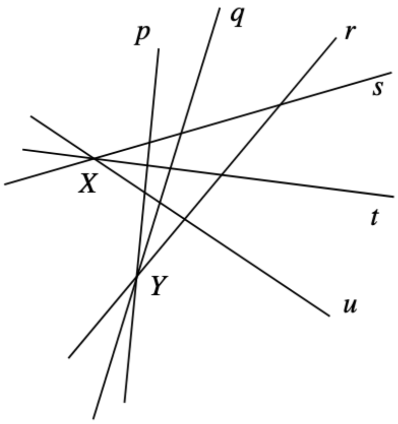
MA68. $PQRS$ est un carré. Les points T et U sont les mi points de QR et RS respectivement. La ligne QS intersecte PT et PU en W et V respectivement. Quelle fraction de la surface du carré $PQRS$ est occupée par le pentagone $RTWVU$?



MA69. Le diagramme ci-bas montre deux lignes PR et QS intersectant en O. Quelle est la valeur de x?



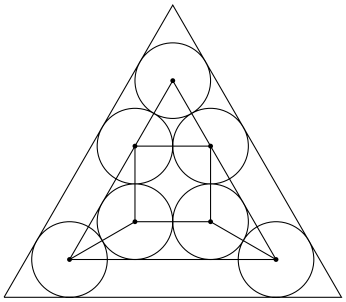
MA70. Le métro de Winnibourg consiste de six lignes, p, q, r, s, t, u, telles qu'indiquées ci-bas. Lorsque deux lignes se rencontrent, on y retrouve une station permettant de changer de ligne. De plus, le métro s'arrête à toute station sur sa ligne. Jéhane désire voyager de la station X à la station Y. Mais elle refuse d'utiliser une quelconque ligne plus qu'une fois, en plus de ne jamais revenir une deuxième fois à la station X, ni de quitter la station Y après y être arrivée. Déterminer le nombre de telles routes différentes.



MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(9), p. 495–496.

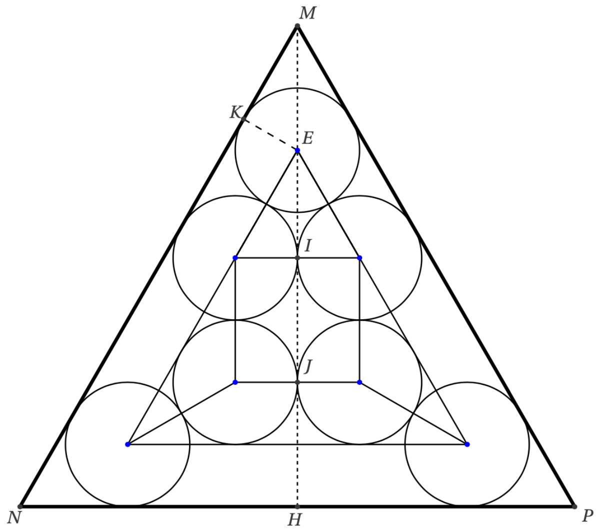
MA41. The diagram shows the densest packing of seven circles in an equilateral triangle.



Determine the exact fraction of the area of the triangle that is covered by the circles.

Originally from "Shaking Hands in Corner Brook and Other Math Problems" by Peter Booth, Bruce Shawyer and John Grant McLoughlin.

We received 7 submissions, of which 6 were correct and complete. We present the solution by Dominique Mouchet, modified by the editor.



On prend 1 comme longueur du côté du grand triangle équilatéral. On notera R le rayon des cercles. Exprimons la hauteur MH en fonction de R:

- le triangle MKE a pour angles $90-60-30$ et $EK = R$. Donc $ME = 2R$.
- EI est la hauteur d'un triangle équilatéral de côté $2R$. Donc
$$EI = 2R \cdot \frac{\sqrt{3}}{2} = R\sqrt{3}.$$
- $IJ = JH = 2R$.

Donc
$$MH = ME + EI + IJ + IH = 2R + R\sqrt{3} + 2R + 2R = R\big(6+\sqrt{3}\big).$$

Comme $MH = \dfrac{\sqrt{3}}{2}$, on obtient
$$R = \frac{\sqrt{3}}{2\big(6+\sqrt{3}\big)} = \frac{\sqrt{3}\big(6-\sqrt{3}\big)}{2\cdot 33} = \frac{2\sqrt{3}-1}{22}.$$

La fraction p de la surface du triangle couverte par les 7 triangles est donc:
$$p = \frac{7\pi R^2}{\frac{\sqrt{3}}{4}} = \frac{28\pi}{\sqrt{3}}\left(\frac{13-4\sqrt{3}}{484}\right) = \frac{7\pi}{363}\left(13\sqrt{3}-12\right) \approx 0.6371.$$

MA42. Find all functions of the form $f(x) = \dfrac{a+bx}{b+x}$ where a and b are constants such that $f(2) = 2f(5)$ and $f(0) + 3f(-2) = 0$.

Originally Question 2 of 1980 J.I.R. McKnight Mathematics Scholarship Paper.

We received 8 submissions, all of which were correct and complete. We present the solution by José Luis Díaz-Barrero, modified by the editor.

The condition gives us that
$$\frac{a+2b}{b+2} = 2\left(\frac{a+5b}{b+5}\right) \quad \text{and} \quad \frac{a}{b} + 3\left(\frac{a-2b}{b-2}\right) = 0.$$

The above results in the nonlinear system of equations:
$$\begin{aligned} 8b^2 + (10+a)b - a &= 0, \\ 6b^2 - 4ab + 2a &= 0. \end{aligned} \tag{1}$$

The resultant of the above system is
$$-4a(a+1)(19a-300).$$

Substituting the zeros of a in the above we see that $(a, b) = (0, 0)$, $(a, b) = (-1, -1)$ and $(a, b) = (300/19, 10/19)$ solve (1). Thus

$$f(x) = 0, \quad f(x) = \frac{1+x}{1-x}, \quad \text{and} \quad f(x) = \frac{300+10x}{10+19x},$$

are the only functions which satisfy the stated condition.

MA43. If n is not divisible by 4, prove that $1^n + 2^n + 3^n + 4^n$ is divisible by 5 for any positive integer n.

Adapted from Problem 2 of the 1901 Competition in Hungarian Problem Book 1 (1963).

We received 13 submissions, all of which were correct and complete. We present the generalized solution by the Problem Solving Group from Missouri State University, modified by the editor.

We will show, more generally, that if p is any prime number and n is a positive integer then

$$1^n + 2^n + 3^n + \ldots + (p-1)^n$$

is a multiple of p if and only if n is not divisible by $p - 1$.

It is well known that since p is prime, there is an element $\alpha \in \mathbb{Z}_p$ (a primitive root) such that for all $i \in \mathbb{Z}_p, i = \alpha^k$ for some integer k, with $0 \le k \le p - 2$. Therefore

$$\sum_{i=1}^{p-1} i^n \equiv \sum_{k=0}^{p-2} \left(\alpha^k\right)^n \equiv \sum_{k=0}^{p-2} (\alpha^n)^k .$$

Note that $\alpha^n \equiv 1 \bmod p$ if and only if n is a multiple of $p - 1$.

We prove both directions:

$\Rightarrow$ (Contrapositive) If n is a multiple of $p - 1$, $\alpha^n \equiv 1 \bmod p$ and

$$\sum_{i=1}^{p-1} i^n \equiv \sum_{k=0}^{p-2} (\alpha^n)^k \equiv p - 1 \not\equiv 0 \bmod p.$$

Thus our sum is not a multiple of p.

$\Leftarrow$ If n is not a multiple of $p - 1$, $\alpha^n - 1 \not\equiv 0 \bmod p$, so $\alpha^n - 1 \in \mathbb{Z}_p$. Using the formula for finite geometric series (with $b = \alpha^n$), we have

$$\sum_{i=1}^{p-1} i^n \bmod p \equiv \sum_{k=0}^{p-2} (\alpha^n)^k \bmod p \equiv \left(b^{p-1} - 1\right)(b-1)^{-1} \bmod p \equiv 0 \bmod p.$$

Thus our sum is a multiple of p.

We observe the original problem considers the case when $p = 5$.

MA44. Find the largest positive integer which divides all expressions of the form $n^5 - n^3$ where n is a positive integer. Justify your answer.

Proposed by John McLoughlin.

We received 11 submissions, all of which were correct and complete. We present the joint solution by the Problem Solving Group from Missouri State University and Tianqi Jiang (solved independently), modified by the editor.

First note that when $n = 2$, we have that $2^5 - 2^3 = 24$. Thus the number we seek must be a factor of 24.

We show $3 \mid n^5 - n^3$. As $n^5 - n^3 = n^3(n+1)(n-1)$ is divisible by three consecutive integers, it follows one of these numbers is a multiple of 3. Thus $3 \mid n^5 - n^3$.

We show $8 \mid n^5 - n^3$ by considering cases. If $n = 2k$, then

$$n^5 - n^3 = (2k)^3(2k+1)(2k-1) = 8k^3(2k+1)(2k-1).$$

Thus $8 \mid n^5 - n^3$. If $n = 2k+1$, then

$$n^5 - n^3 = (2k+1)^3(2k+2)(2k) = (2k+1)^3 \cdot 2^2 \cdot k \cdot (k+1).$$

As one of k or $k+1$ is even, it follows $8 \mid n^5 - n^3$.

Since 3 and 8 are relatively prime, $24 \mid n^5 - n^3$. As we established 24 as an upper bound, our proof is complete.

MA45. A sequence $s_1, s_2, \ldots, s_n$ is harmonic if the reciprocals of the terms are in arithmetic sequence. Suppose $s_1, s_2, \ldots, s_{10}$ are in harmonic sequence. Given $s_1 = 1.2$ and $s_{10} = 3.68$, find $s_1 + s_2 + \cdots + s_{10}$.

Originally Question 11 of 1988 Illinois CTM, State Finals AA.

We received 2 submissions, both correct and complete. We present the solution by Doddy Kastanya.

The arithmetic sequence of interest is $a_1, a_2, \ldots, a_{10}$ where

$$s_1 = \frac{1}{a_1},\ s_2 = \frac{1}{a_2}, \ldots, s_{10} = \frac{1}{a_{10}}.$$

From the problem statement, we know that $a_1 = \frac{1}{1.2} = \frac{5}{6}$ and $a_{10} = \frac{1}{3.68} = \frac{100}{368}$.

For the arithmetic sequence, there are eight items in between a_1 and a_{10} with equal spacing. Turning the denominator for these two values to 9936, we get $a_1 = \frac{8280}{9936}$ and $a_{10} = \frac{2700}{9936}$. The spacing between two numbers is $\frac{620}{9936}$. With this knowledge, the other items can be determined: $a_2 = \frac{7660}{9936}$, $a_3 = \frac{7040}{9936}$, up to $a_9 = \frac{3320}{9936}$.

The corresponding values of s_1 through s_{10} can be easily calculated. Finally, the sum of s_1 through s_{10} is calculated as 20.46.

PROBLEM SOLVING VIGNETTES

No. 11

Shawn Godin

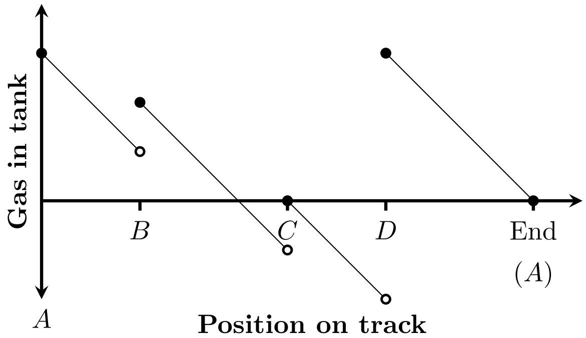
Picking Representations

In many cases, a problem's solution is aided by thinking about the problem in a different way than it was originally presented. This may be by looking at a different, but related problem whose solution leads back to the original. We can also think about a problem differently by choosing some other way to represent it. Analytic geometry is an example, where we can think of geometric problems algebraically or algebraic problems geometrically.

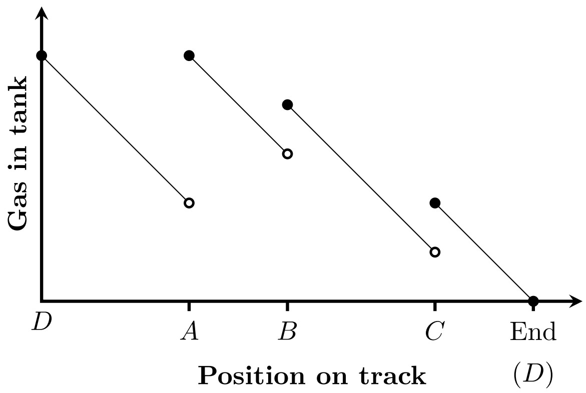
When I thought about this topic the following problem came to mind:

> *A gas powered go-cart is empty and on a track. Around the track are a number of gas cans. The total amount of gas in all the cans is equal to the amount of gas needed to go around the track once. Show that, no matter how the gas and cans are distributed, you can find a place to start so that you can make it all the way around the track.*

I was introduced to this problem by *Crux* Editorial Board member Ed Barbeau at a workshop he did for teachers over 20 years ago. It's one of those problems you can convince yourself must work, but coming up with an airtight argument that convinces others is another thing. The key to the insightful solution that was given by Ed was to imagine that we are allowed to have a "negative" amount of gas in our tank. Then the graph of the gas in our tank versus the distance driven will be a piecewise linear function where all the pieces of the graph will have equal, negative slopes; there will be a step discontinuity at the location of each gas can; and when we have finished one trip around the track our tank will, again, be empty.



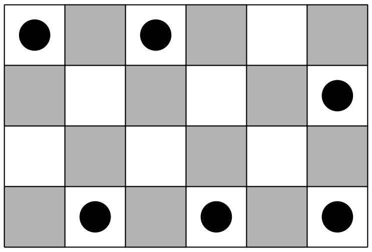
Thus if we draw the graph and find the lowest point, this will be the place that we should start.



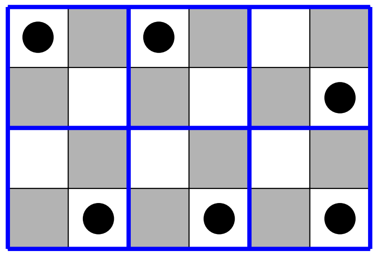
For example, in the graphs above, we are assuming that there are four gas cans A, B, C, and D. If we start at can A, we get the first graph above on the previous page. Thus, we see that we should have started at can D, which would have given us the second graph above.

Choosing the graphical representation not only helped make our argument clearer, it also gave way to the solution. Now, let's consider Problem 3 from the 2019 Canadian Mathematical Olympiad:

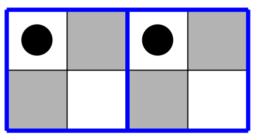
> *Let m and n be positive integers. A $2m \times 2n$ grid of squares is coloured in the usual chessboard fashion. Find the number of ways of placing mn counters on the white squares, at most one counter per square, so that no two counters are on white squares that are diagonally adjacent. An example of a way to place the counters when $m = 2$ and $n = 3$ is shown below.*



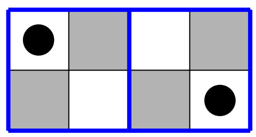
Since counters cannot be on diagonally adjacent squares, any 2×2 square drawn on the grid can only contain at most one counter. This suggests partitioning the grid into 2×2 squares, as in the diagram below. Since there are $\frac{(2m)(2n)}{(2)(2)} = mn$ 2×2 squares on the grid, each of these squares will have exactly one counter.



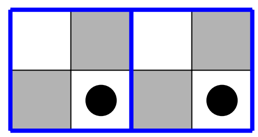
Notice that there are two configurations that the 2×2 squares can be in: either the counter can be in the upper corner (U), or the lower corner (L). When we look at two 2×2 squares beside each other we see that UU, UL and LL are all valid configurations.



UU

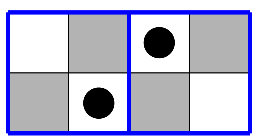


UL



LL

On the other hand the configuration LU is not allowed.



LU

Similarly, going from top to bottom we can have U followed by U or L, but L can only be followed by another L. Hence, our original $2m \times 2n$ grid was replaced with an $m \times n$ grid of 2×2 squares, which in turn can be replaced by an $m \times n$ grid filled with the symbols U and L.

The original example can now be replaced with

U	U	L
L	L	L

Notice that in any row or column we have a number of Us (possibly none) followed by a number of Ls. Once an L appears in a row or column, all entries to the right and below it are also Ls. Thus starting from the top row and proceeding downward, each new row has at least as many Ls as the row above it, and possibly more.

Considering the setup from the example in the problem statement, in which a 4×6 starting grid reduces to a 2×3 grid of Us and Ls, we can readily construct the 10 ways that the grid can be filled and count the number of Ls in each row (the case in bold blue is the example from the problem statement).

$$\begin{array}{|c|c|c|}\hline U&U&U\\\hline U&U&U\\\hline\end{array}\ \begin{array}{c}0\\0\end{array}\qquad \begin{array}{|c|c|c|}\hline U&U&U\\\hline U&U&L\\\hline\end{array}\ \begin{array}{c}0\\1\end{array}\qquad \begin{array}{|c|c|c|}\hline U&U&U\\\hline U&L&L\\\hline\end{array}\ \begin{array}{c}0\\2\end{array}\qquad \begin{array}{|c|c|c|}\hline U&U&U\\\hline L&L&L\\\hline\end{array}\ \begin{array}{c}0\\3\end{array}$$

$$\begin{array}{|c|c|c|}\hline U&U&L\\\hline U&U&L\\\hline\end{array}\ \begin{array}{c}1\\1\end{array}\qquad \begin{array}{|c|c|c|}\hline U&U&L\\\hline U&L&L\\\hline\end{array}\ \begin{array}{c}1\\2\end{array}\qquad \begin{array}{|c|c|c|}\hline U&U&L\\\hline L&L&L\\\hline\end{array}\ \begin{array}{c}1\\3\end{array}\qquad \begin{array}{|c|c|c|}\hline U&L&L\\\hline U&L&L\\\hline\end{array}\ \begin{array}{c}2\\2\end{array}$$

$$\begin{array}{|c|c|c|}\hline U&L&L\\\hline L&L&L\\\hline\end{array}\ \begin{array}{c}2\\3\end{array}\qquad \begin{array}{|c|c|c|}\hline L&L&L\\\hline L&L&L\\\hline\end{array}\ \begin{array}{c}3\\3\end{array}$$

Thus we can think of our problem in another way: a sequence of two non-negative integers representing the number of Ls in each row. For the configuration considered above these would be:

$$0,0;\quad 0,1;\quad 0,2;\quad 0,3;\quad 1,1;\quad 1,2;\quad 1,3;\quad 2,2;\quad 2,3;\quad 3,3.$$

Notice that, to satisfy the conditions of the problem, all of these sequences are non-decreasing, containing numbers less than or equal to 3 (the number of columns). Thus our problem is equivalent to finding the number of non-decreasing sequences of m terms chosen (possibly with repetition) from the set $\{0,1,2,\ldots,n\}$. In the example that would be the number of non-decreasing sequences of 2 terms from the set $\{0,1,2,3\}$.

We will choose another representation to attack the sequence problem. Since my numbers can be as large as 3, we will use 3 stars. Since we have 2 terms, we will use 2 bars. We will arrange these 5 symbols in some order, like

$$*|**|$$

This arrangement is interpreted by counting all the stars to the left of the leftmost bar, 1, and all the stars to the left of the rightmost bar, 3. Hence the arrangement above represents the non-decreasing sequence of 2 terms from the set $\{0,1,2,3\}$: $1,3$. Each sequence is represented by a unique arrangement of stars and bars, and each arrangement of stars and bars corresponds to a unique sequence. Hence counting the number of arrangements of 3 stars and 2 bars gives us the solution to the configuration in the problem statement.

We can count the number of ways to arrange the 3 stars and 2 bars in several ways to get $\dfrac{5!}{3!2!} = 10$. This can be interpreted as arranging 5 things, 3 of one type (stars) and 2 of another (bars). We can also interpret this as we have 5 positions to put our symbols and we must choose 3 of them to put the stars, leaving the rest of the places for bars. Alternatively, we could have picked the places for the bars first yielding $\binom{5}{2} = \binom{5}{3} = 10$.

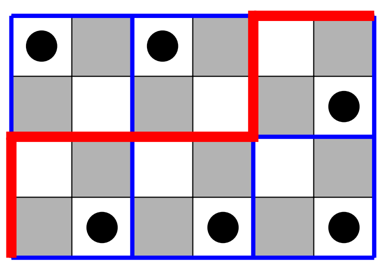
Returning to the problem, in the general case we have a $2m \times 2n$ grid of squares filled with counters. We are representing this by a non-decreasing sequence of m numbers, where each number is less than or equal to n, which represents the number of L's in our smaller grid. Converting this to m bars and n stars, we get the total number of ways our task can be completed is

$$\binom{m+n}{m} = \binom{m+n}{n} = \frac{(m+n)!}{m!n!}.$$

It is worthwhile to go back to the problem and convince yourself that you would get the same result with these slight variations on our technique:

- Counting the number of non-increasing sequences representing the number of Us in each row.
- Counting the number of non-increasing sequences representing the number of Us in each column.
- Counting the number of non-decreasing sequences representing the number of Ls in each column.

The official solution to the problem uses a slightly different approach. It also notices the differences between the Us and Ls and notes that the boundary separating these two types of 2×2 cells makes a path from the lower left corner of the big grid to the upper right corner, travelling either to the right or up. The original example with the boundary highlighted in red is in the diagram below. As an exercise, you may want to solve the problem using this representation.



Keep in mind that sometimes changing your point of view through a different representation of the problem may lead you to a solution. You may want to check out the seventh number of this column, *Counting Carefully* [2019: 386-389], where the stars and bars technique was used in a slightly different way. For your enjoyment, here are the rest of the problems from the 2019 Canadian Mathematical Olympiad.

The 2019 Canadian Mathematical Olympiad

1. Amy has drawn three points in a plane, A, B, and C, such that $AB = BC = CA = 6$. Amy is allowed to draw a new point if it is the circumcenter of a triangle whose vertices she has already drawn. For example, she can draw the circumcenter O of triangle ABC, and then afterwards she can draw the circumcenter of triangle ABO.

(a) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 7.

(b) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 2019.

(Recall that the circumcenter of a triangle is the center of the circle that passes through its three vertices.)

2. Let a and b be positive integers such that $a+b^3$ is divisible by $a^2+3ab+3b^2-1$. Prove that $a^2+3ab+3b^2-1$ is divisible by the cube of an integer greater than 1.

4. Let n be an integer greater than 1, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_1 = a_{n-1} = 0$. Prove that for any real number k,

$$|a_0| - |a_n| \leq \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}|.$$

5. David and Jacob are playing a game of connecting $n \geq 3$ points drawn in a plane. No three of the points are collinear. On each player's turn, he chooses two points to connect by a new line segment. The first player to complete a cycle consisting of an odd number of line segments loses the game. (Both endpoints of each line segment in the cycle must be among the n given points, not points which arise later as intersections of segments.) Assuming David goes first, determine all n for which he has a winning strategy.

The author would like to thank Ed Barbeau for reminding him about the details of the go-cart problem and providing valuable feedback that greatly improved the article.

OLYMPIAD CORNER

No. 382

The problems in this section appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **June 15, 2020**.

OC476. Let x be a real number such that both sums $S = \sin 64x + \sin 65x$ and $C = \cos 64x + \cos 65x$ are rational numbers. Prove that in one of these sums, both terms are rational.

OC477. Let $A = \{z \in \mathbb{C} \mid |z| = 1\}$.

(a) Prove that $(|z+1| - \sqrt{2})(|z-1| - \sqrt{2}) \leq 0 \ \forall z \in A$.

(b) Prove that for any $z_1, z_2, \ldots, z_{12} \in A$, there is a choice of signs "$\pm$" so that

$$\sum_{k=1}^{12} |z_k \pm 1| < 17.$$

OC478. Consider two noncommuting matrices $A, B \in \mathcal{M}_2(\mathbb{R})$.

(a) Knowing that $A^3 = B^3$, prove that A^n and B^n have the same trace for any nonzero natural number n.

(b) Give an example of two noncommuting matrices $A, B \in \mathcal{M}_2(\mathbb{R})$ such that for any nonzero $n \in \mathbb{N}$, $A^n \neq B^n$, and A^n and B^n have different trace.

OC479. We say that the function $f : \mathbb{Q}_+^* \to \mathbb{Q}$ has the property $\mathcal{P}$ if

$$f(xy) = f(x) + f(y) \qquad \forall x, y \in \mathbb{Q}_+^*.$$

(a) Prove that there do not exist injective functions with property $\mathcal{P}$.

(b) Do there exist surjective functions with property $\mathcal{P}$?

OC480. In the plane, there are points C and D on the same region with respect to the line defined by the segment AB so that the circumcircles of triangles ABC and ABD are the same. Let E be the incenter of triangle ABC, let F be the incenter of triangle ABD and let G be the midpoint of the arc AB not containing the points C and D. Prove that points A, B, E, F are on a circle with center G.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC476. Soit x un nombre réel tel que les deux sommes $S = \sin 64x + \sin 65x$ et $C = \cos 64x + \cos 65x$ sont rationnelles. Démontrer que dans une des sommes, les deux termes sont rationnels.

OC477. Soit $A = \{z \in \mathbb{C} \mid |z| = 1\}$.

(a) Démontrer que $(|z+1| - \sqrt{2})(|z-1| - \sqrt{2}) \leq 0 \ \forall z \in A$.

(b) Démontrer que pour tout $z_1, z_2, \ldots, z_{12} \in A$, il existe un choix de signes "$\pm$" tels que

$$\sum_{k=1}^{12} |z_k \pm 1| < 17.$$

OC478. Soient deux matrices qui ne commutent pas, $A, B \in \mathcal{M}_2(\mathbb{R})$.

(a) Si $A^3 = B^3$, démontrer que A^n et B^n ont la même trace $\forall n \in \mathbb{N}$, $n \neq 0$.

(b) Donner une exemple de deux matrices qui ne commutent pas, $A, B \in \mathcal{M}_2(\mathbb{R})$, telles que, pour tout non nul $n \in \mathbb{N}$, $A^n \neq B^n$, puis A^n et B^n sont de différentes traces.

OC479. La fonction $f : \mathbb{Q}_+^* \to \mathbb{Q}$ possède la propriété $\mathcal{P}$ si

$$f(xy) = f(x) + f(y) \qquad \forall x, y \in \mathbb{Q}_+^*.$$

(a) Démontrer qu'il n'existe aucune fonction injective possédant la propriété $\mathcal{P}$.

(b) Des fonctions surjectives avec la propriété $\mathcal{P}$ existent-elles?

OC480. Soient C et D deux points dans le même demi plan par rapport au segment AB, de façon à ce que les cercles circonscrits des triangles ABC et ABD soient les mêmes. Soit E le centre du cercle inscrit du triangle ABC et soit F le centre du cercle inscrit du triangle ABD ; soit aussi G le mi point de l'arc AB contenant ni C ni D. Démontrer que A, B, E, F se trouvent sur un cercle de centre G.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(10), p. 504–505.

OC451. Determine the least natural number a such that

$$a \geq \sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k)$$

for any nonzero natural number n and for any positive real numbers $a_1, a_2, \ldots, a_n$ whose sum is at most π.

We received 1 submission. We present the solution by Oliver Geupel.

We show that

$$\sup\left\{\sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k)\ :\ n \geq 1;\ a_1, \ldots, a_k > 0;\ \sum_{k=1}^{n} a_k \leq \pi\right\} = 1,$$

which implies that the least value of a is 1. For $1 \leq k \leq n$, let

$$x_k = \sum_{j=1}^{k} a_j.$$

Then, $0 = x_0 < x_1 < x_2 < \cdots < x_n \leq \pi$, and the sum

$$\sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k) = \sum_{k=1}^{n} (x_k - x_{k-1}) \cos x_k \tag{1}$$

is a right Riemann sum which underestimates the integral

$$I(x_n) = \int_0^{x_n} \cos x \,\mathrm{d}x$$

of the decreasing function $\cos x$ on the interval $[0, x_n]$. Since $I(x_n) = \sin x_n \leq 1$ for every $x_n \leq \pi$, we obtain that $a \leq 1$. However, if $a_k = \pi/(2n)$ and $x_k = k\pi/(2n)$ for any $1 \leq k \leq n$ then the sequence defined by the sums (1) converges towards $I(\pi/2) = 1$ as $n \to \infty$. Hence $a = 1$.

Editor's Comment. The restrictions on a_n's can be changed. For example, if

$$a_1 + \cdots + a_n \leq \pi/2,$$

then the value of the upper bound a is 1, as before. However, if

$$a_1 + \cdots + a_n = \pi,$$

then $a = 0$.

OC452. Let $ABCD$ be a square. Consider the points $E \in AB$, $N \in CD$ and $F, M \in BC$ such that triangles AMN and DEF are equilateral. Prove that $PQ = FM$, where $\{P\} = AN \cap DE$ and $\{Q\} = AM \cap EF$.

We received 11 *correct submissions. We present two solutions.*

Solution 1, by Miguel Amengual Covas.

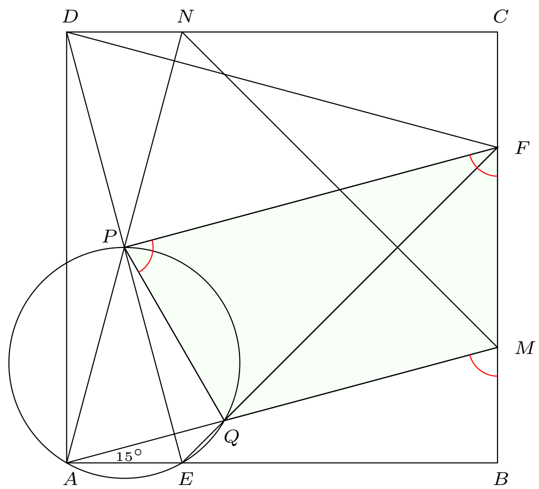
The right-angled triangles ABM and ADN have equal hypotenuses AM and AN, and the legs AB and AD are respectively equal. Thus $\triangle ABM$ and $\triangle ADN$ are congruent with $\angle MAB = \angle NAD$.

Now, $\angle DAB = \angle MAB + \angle NAM + \angle NAD$. Next, since $\angle DAB = 90^\circ$ and $\angle NAM = 60^\circ$, it follows that $\angle MAB$ and $\angle NAD$ are each 15°. Analogously, $\angle CDF = \angle ADE = 15^\circ$.

Clearly, then, $\triangle ABM$, $\triangle CDF$, $\triangle DAE$, $\triangle ADN$ are congruent (these are right-angled triangles which have equal legs AB, CD and DA and contain another pair of equal angles) with

$$AE = BM = FC = ND.$$

Consequently, $AEND$ is a rectangle, so that the segments AN and DE bisect each other. Thus P is the midpoint of segment DE. Then, we have in equilateral triangle DEF that $\angle FPE$ is a right angle and $\angle PFE = 30^\circ$.



Subtracting $AE = FC$ from both sides of $AB = BC$ gives $AB - AE = BC - FC$. This makes $EB = BF$ and consequently $\triangle EBF$ is an isosceles right-triangle with $\angle EFB = 45^\circ$. Therefore,

$$\begin{aligned}\angle PFM = \angle PFB = \angle PFE + \angle EFB &= 30^\circ + 45^\circ \\ &= 75^\circ \\ &= 90^\circ - 15^\circ \\ &= 90^\circ - \angle MAB \\ &= \angle AMB,\end{aligned}$$

implying that PF is parallel to AM, that is, $PF \parallel QM$.

Next, PQ subtends 60° angles at A and E, making $PAEQ$ cyclic and on chord EQ we have

$$\angle QPE = \angle QAE = \angle MAB = 15^\circ$$

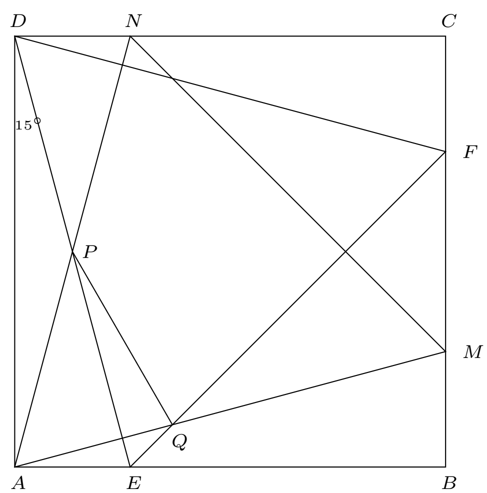
and

$$\angle FPQ = \angle FPE - \angle QPE = 90^\circ - 15^\circ = 75^\circ.$$

That is to say, the exterior angle QMB in quadrilateral $PQMF$ is equal to the interior and opposite angle P. Thus $PQMF$ is cyclic. Since $PF \parallel QM$, $PQMF$ is an isosceles trapezium. The conclusion follows.

Solution 2, by Miguel Amengual Covas.

As in Solution 1, we conclude that $\angle MAB = \angle NAD = \angle CDF = \angle ADE = 15^\circ$.



Suppose (wlog) the unity of measurement equal to the length of the side of the given square. Then

$$DN = AE = BM = CF = \tan 15^\circ. \tag{1}$$

Hence,

$$FM = BC - CF - MB = 1 - 2\tan 15^\circ. \tag{2}$$

Moreover, since $AE \parallel DN$, $AEND$ is a rectangle, so that segments AN and DE bisect each other. Therefore, P is the midpoint of DE and we have

$$PE = \frac{1}{2}DE = \frac{1}{2\cos 15^\circ}. \tag{3}$$

Observing that the equal segments FB and BE make EBF an isosceles right-angled triangle, the exterior angle theorem, applied to $\triangle AEQ$ at E, yields

$$\angle EQA = \angle QEB - \angle QAE = \angle FEB - \angle MAB = 45^\circ - 15^\circ = 30^\circ.$$

Now, the law of sines asserts that

$$\frac{QE}{\sin 15^\circ} = \frac{AE}{\sin 30^\circ},$$

and therefore

$$\frac{QE}{\sin 15^\circ} = 2 \cdot AE,$$

yielding (by (2))

$$QE = \frac{2\sin^2 15^\circ}{\cos 15^\circ}. \tag{4}$$

Applying the law of cosines to $\triangle PEQ$ we get

$$PQ^2 = PE^2 + QE^2 - 2 \cdot PE \cdot QE \cdot \cos 60^\circ.$$

Substituting for PE and QE from (4) and (5),

$$PQ^2 = \frac{1}{4\cos^2 15^\circ} + \frac{4\sin^4 15^\circ}{\cos^2 15^\circ} - \tan^2 15^\circ. \tag{5}$$

Now, we write the identity $2\sin 30^\circ = 1$ in the equivalent form $4\sin 15^\circ \cos 15^\circ = 1$, multiply it by $\tan 15^\circ$ and square, obtaining $16\sin^4 15^\circ = \tan^2 15^\circ$. Hence we can rewrite (6) as

$$PQ^2 = \frac{1+\tan^2 15^\circ}{4\cos^2 15^\circ} - \tan^2 15^\circ,$$

or, equivalently,

$$PQ^2 = \frac{1}{4\cos^4 15^\circ} - \tan^2 15^\circ.$$

This, in turn, is equivalent to

$$PQ^2 = \frac{1 - 4\sin^2 15^\circ \cos^2 15^\circ}{4\cos^4 15^\circ} = \frac{1-\sin^2 30^\circ}{4\cos^4 15^\circ} = \frac{3}{16\cos^4 15^\circ}.$$

Thus

$$PQ = \frac{\sqrt{3}}{4\cos^2 15^\circ} = \sqrt{3}\tan 15^\circ. \tag{6}$$

Taking into account that $\tan 15^\circ = 2-\sqrt{3}$, from (3) and (7) we conclude that

$$FM = 2\sqrt{3} - 3 = PQ.$$

OC453. Let $n \geq 2$ be an integer and let $A, B \in \mathcal{M}_n(\mathbb{C})$. If $(AB)^3 = O_n$, is it true that $(BA)^3 = O_n$? Justify your answer.

We received 4 correct submissions. We present two solutions.

Solution 1, by Oliver Geupel.

We prove that the deduction is correct if and only if $n \leq 3$.

First, let $n \leq 3$ and let λ be an eigenvalue of the matrix $C = BA$ with eigenvector v. Then, $\lambda v = Cv$ and

$$\lambda^4 v = C \cdot \lambda^3 v = C^2 \cdot \lambda^2 v = C^3 \cdot \lambda v = C^4 v = B(AB)^3 Av = BO_n Av = O_n.$$

Hence $\lambda = 0$. So 0 is the only eigenvalue of C. The characteristic polynomial of C is then λ^n. By the Cayley-Hamilton theorem, the matrix C satisfies its own characteristic equation, so that $C^n = O_n$ and therefore $(BA)^3 = C^3 = O_n$.

We now turn to the case where $n \geq 4$. Let

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consider the n-by-n block matrices

$$A = \begin{bmatrix} A_4 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix}, \qquad B = \begin{bmatrix} B_4 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix}.$$

Straightforward computations yield $(A_4B_4)^3 = O_4$ and

$$(B_4A_4)^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus,

$$(AB)^3 = \begin{bmatrix} (A_4B_4)^3 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix} = O_n$$

and

$$(BA)^3 = \begin{bmatrix} (B_4A_4)^3 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix} \neq O_n.$$

This completes the proof.

Solution 2, by Missouri State University Problem Solving Group.

We prove a more general result. Fix $k \geq 1$. Then

(1) If $n \leq k$ then $(AB)^k = 0$ implies $(BA)^k = 0$.

(2) If $n \geq k+1$ then there exist $n \times n$ matrices A and B such that $(AB)^k = 0$ but $(BA)^k \neq 0$.

Suppose $n \leq k$ and $(AB)^k = 0$. Every eigenvalue of AB is 0 and the characteristic polynomial of AB is x^n. But AB and BA have the same characteristic polynomial, hence $(BA)^k = 0$. This proves (1).

(2) Let $n = k+1$, let A be the $n \times n$ matrix whose (i,j) entry is 1 if and only if $i = j > 1$ and 0 otherwise, and let B be the $n \times n$ matrix whose (i,j) entry is 1 if and only if $j = i+1$ and 0 otherwise. Then $BA = B$ and $AB = C$ where the (i,j) entry of C is 1 if and only if $j = i+1 > 2$ and 0 otherwise.

A direct calculation yields $(AB)^{n-1} = 0$, but $(BA)^{n-1}$ is non-zero. By taking the direct sum of A and B with the zero matrix, we can get counterexamples for all larger n.

For the original problem with $k = 3$, we take $n = 4$ and get

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (AB)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

but

$$BA = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (BA)^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

OC454. Find all the functions $f : \mathbb{N} \to \mathbb{N}$ having the following property for each natural number m: if $d_1, d_2, \ldots, d_n$ are all the divisors of the number m, then

$$f(d_1)f(d_2) \cdot \ldots \cdot f(d_n) = m.$$

We received 8 submissions. We present the solution by Oliver Geupel.

It is readily checked that the following function is a solution to the problem:

$$f(m) = \begin{cases} p & \text{if } m = p^k \text{ where } p \text{ is a prime number and } k \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

We show that there are no other solutions. Suppose f is any solution.

Putting $m = 1$ in the given condition, we obtain $f(1) = 1$. Setting $m = p$ where p is a prime number in the given condition, we get $f(p) = f(1)f(p) = p$. A straightforward induction shows that $f(p^k) = p$ for $k \geq 1$.

Finally, let m have at least two distinct prime divisors, say $m = p_1^{k_1} p_2^{k_2} \cdots p_\ell^{k_\ell}$, where $p_1, \ldots, p_\ell$ are distinct prime divisors ($\ell \geq 2$) and $k_j \geq 1$ for $1 \leq j \leq \ell$. Let D denote the set of those divisors of m that have at least two distinct prime divisors. Then,

$$m = f(1) \left(\prod_{\substack{1 \leq j \leq \ell \\ 1 \leq i \leq k_j}} f(p_j^i) \right) \left(\prod_{d \in D} f(d) \right) = \left(\prod_{1 \leq j \leq \ell} p_j^{k_j} \right) \left(\prod_{d \in D} f(d) \right) = m \prod_{d \in D} f(d).$$

Hence,

$$\prod_{d \in D} f(d) = 1.$$

It follows $f(d) = 1$ for all $d \in D$. Since $m \in D$, we conclude that $f(m) = 1$.

OC455. Let D be a point on the base AB of an isosceles triangle ABC. Select a point E so that $ADEC$ is a parallelogram. On the line ED, take a point F such that $E \in DF$ and $EB = EF$. Prove that the length of the chord that the line BE cuts on the circumcircle of triangle ABF is twice the length of the segment AC.

We received 4 submissions. We present the solution by Ivko Dimitrić.

Without loss of generality, we may assume that the vertices are labeled counterclockwise and that $|DB| \leq |AD|$. Let ω be the circumcircle of $\triangle ABF$, O its center and K the point where the line BE meets ω again. Let M be the midpoint of BK, S the midpoint of BF and $N = \overleftrightarrow{CE} \cap \overleftrightarrow{BF}$. Further, set

$$\alpha = \angle CAB = \angle CBA = \angle ECB, \quad \theta = \angle FBE = \angle EFB \quad \text{and} \quad \varphi = \angle ECM.$$

To prove the claim, it suffices to show that $\triangle CBM$ is isosceles.

From parallelogram $ADEC$ we have $\angle CED = \alpha$ and from isosceles $\triangle BFE$ we get $\angle DEB = 2\theta$, so that $\angle CEB = \alpha + 2\theta$. The points O, E and S are collinear, because $BE = EF$ and E, O belong to the perpendicular bisector of BF at S. Since $|DB| \leq |AD|$ the foot of the perpendicular from B to CN belongs to the segment $\overline{CE}$ just as the foot of the perpendicular from C to AB belongs to $\overline{AD}$.

That implies that $\angle BEN \geq 90^\circ$ and $\angle MEO = \angle BES < 90^\circ$. Hence, S is between B and N and M is between E and K.

Since $OM \perp BK$ and O and C belong to the perpendicular bisector of AB we have $\angle ECO = \angle OME = 90^\circ$ and the quadrilateral $CEMO$ is cyclic so that

$$\angle ECM = \angle EOM = \varphi \quad \text{and} \quad \angle OEC = \angle OMC.$$

Since the triangles ESN and OEM are right-angled we have

$$\begin{aligned}\angle CNB = \angle ENS &= 90^\circ - \angle SEN \\ &= 90^\circ - \angle OEC \\ &= 90^\circ - \angle OMC \\ &= \angle CME = \angle CMB.\end{aligned}$$

Therefore, quadrilateral $BCMN$ is cyclic so that $\varphi = \angle NCM = \angle NBM = \theta$. Now, we have

$$\angle BCM = \angle BCE + \angle ECM = \alpha + \varphi$$

and from $\triangle CME$ we get

$$\angle CMB = \angle CEB - \angle ECM = (\alpha + 2\theta) - \varphi = \alpha + 2\varphi - \varphi = \alpha + \varphi.$$

Therefore, we conclude that $\angle BCM = \angle CMB$, so that

$$AC = CB = MB = \frac{1}{2}BK,$$

proving the claim.

Focus On . . . Index

Algebra

1 Integer Part and Periodicity

38:3 (Mar 2012) 99-100

A periodic function that vanishes over a period is the zero function. This obvious property can lead to elegant proofs for some identities involving the integer part function.

3 From Linear Recurrences to a Polynomial Identity

38:7 (Sep 2012) 276-277

To determine the set of all sequences U_n satisfying the linear recurrence $U_{n+2} = xU_{n+1} - yU_n$, a direct approach (instead of the classical method) leads to a general polynomial identity.

7 Decomposition into Partial Fractions

39:5 (May 2013) 218-221

Some examples involving minimal calculations to display the usefulness of this algebraic tool.

15 A Formula of Euler

41:1 (Jan 2015) 16-20

Euler observed that the sum $S(n, m) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^m$ equals 0 if $m = 0, 1, \ldots, n-1$ and equals $n!$ when $m = n$.

17 Congruences (I)

41:5 (May 2015) 202-205

In many problems involving integers, an ingenious appeal to congruences can be most helpful. Choosing the appropriate modulus is often the key to a short solution.

18 Congruences (II)

41:7 (Sep 2015) 295-298

Here, the focus is on congruences modulo a prime number, with emphasis on the consequences of three simple, but useful, theorems.

23 Vieta's Formulas

42:7 (Sep 2016) 303-307

The formulas that relate the coefficients of a polynomial to sums and products of its roots can help solve algebra problems and establish inequalities.

Inequalities

Geometry

2 The Geometry Behind the Scene

38:5 (May 2012) 183-185

An algebra problem can sometimes be simplified if a connection to a corresponding geometry problem can be found.

4 The Barycentric Equation of a Line

38:9 (Nov 2012) 367-368

A geometric look at the coefficients of the barycentric equation of a line in the Euclidean plane, and some applications.

6 Glide Reflections in the Plane

39:3 (Mar 2013) 133-135

A few situations where glide reflections provide insight.

8 Generalized Inversion in the Plane

39:7 (Sep 2013) 307-310

We extend the notion of inversion to include the commutative product of a classical inversion in a circle and the half turn about its center. This provides a unique inversion with center O exchanging points A and B no matter the relative position of the three collinear points.

12 Intersecting Circles and Spiral Similarities

40:5 (May 2014) 203-206

Let the circles C_1, C_2 intersect at points U, V. Among the spiral similarities transforming C_1 into C_2, the one with center U provides a simple way to obtain the image of any point P of C_1 — it is the second point of intersection of the circle C_2 with the line through P and V.

13 The Dot Product

40:7 (Sep 2014) 289-292

The purpose of this essay is to show the dot product at work through alternative solutions to several past geometry problems.

16 Leibniz's and Stewart's Relations

41:3 (Mar 2015) 110-113

Leibniz's relation involves the center of mass of n weighted points in d-dimensional Euclidean space. It deserves to be better known; among other things, Stewart's theorem is an easy consequence.

Calculus

5 Inequalities via Lagrange Multipliers

39:1 (Jan 2013) 24-26

(See under Inequalities.)

10 Some Sequences of Integrals

40:1 (Jan 2014) 21-24

A study of the sequences $I_n = \int_0^1 (ax^2 + bx + c)^n dx$ with the goal of finding a sequence (ω_n) such that $\lim_{n\to\infty} = \frac{I_n}{\omega_n} = 1$.

11 The Partial Sums of Some Divergent Series

40:3 (Mar 2014) 112-115

For a sequence (a_n) of positive real numbers such that $\lim_{n\to\infty} = \frac{a_{n+1}}{a_n} = \omega > 1$, and the corresponding sequence of its partial sums $A_n = \sum_{k=1}^{n} A_k$, we establish a few results about the behavior of the sequences (A_n) and $\left(\frac{A_{n+1}}{A_n}\right)$ in comparison with the sequences (a_n) and $\left(\frac{a_{n+1}}{a_n}\right)$ and offer some applications.

15 A Formula of Euler

41:1 (Jan 2015) 16-20

(See under Algebra.)

30 Cauchy's Functional Equation

44:3 (Mar 2018) 106-109

(See under Algebra.)

31 Mean Value and Rolle's Theorems

44:5 (May 2018) 202-206

The Mean Value Theorem establishes a link between a function and its derivative. Here we will see it at work in various problems, sometimes rather unexpectedly.

35 The Asymptotic Behavior of Integrals

45:3 (Mar 2019) 137-143

We keep the same goal as in Number 10 [2014: 21-24] (namely, to determine the asymptotic behavior of a given sequence of integrals), but here, restricting ourselves to elementary problems and methods, present simple ways to obtain such an asymptotic behavior.

Combinatorics

15 A Formula of Euler

Trigonometry

34 Some Trigonometric Relations

We consider a selection of problems involving the values of the circular functions at $\frac{m\pi}{n}$ for various natural numbers m and n. Solutions depend on complex numbers and polynomials in addition to the classical trigonometric identities.

Solutions to Exercises

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **June 15, 2020**.

4531. *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Let a, b and c be positive real numbers and let x, y and z be real numbers. Suppose that $a+b+c=2$ and $xa+yb+zc=1$. Prove that

$$x+y+z-(xy+yz+zx) \geq \frac{3}{4}.$$

4532. *Proposed by Marius Stănean.*

Let ABC be a triangle with circumcircle Γ and let M, N, P be points on the sides BC, CA, AB, respectively. Let M', N', P' be the intersections of AM, BN, CP with Γ different from the vertices of the triangle. Prove that

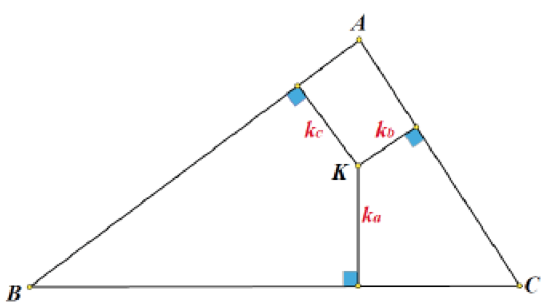
$$MM' \cdot NN' \cdot PP' \leq \frac{R^2 r}{4},$$

where R and r are the circumradius and the inradius of triangle ABC.

4533. *Proposed by Leonard Giugiuc and Kadir Altintas.*

Let K be the symmedian point of ABC. Let k_a, k_b and k_c be the lengths of the altitudes from K to the sides BC, AC and AB, respectively. If r is the inradius and s is the semiperimeter, prove that

$$\left(\frac{1}{r}\right)^2 + \left(\frac{3}{s}\right)^2 \geq \frac{2}{k_a^2+k_b^2+k_c^2}.$$



4534. *Proposed by Michel Bataille.*

For $n \in \mathbb{N}$, evaluate

$$\frac{\sum\limits_{k=0}^{\infty} \frac{1}{k!(n+k+1)}}{\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(n+k+1)!}}.$$

4535. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with orthocenter H, and let E be the reflection of H in the midpoint D of side BC. If the perpendicular to DE at H intersects AB at X and AC at Y, prove that $HX \cdot EC + YC \cdot HE = EX \cdot BE$.

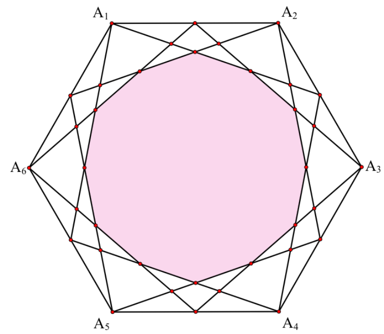
4536. *Proposed by Leonard Giugiuc and Rovsan Pirkuliev.*

Let ABC be a triangle with $\angle ABC = 60^\circ$. Consider a point M on the side AC. Find the angles of the triangle, given that

$$\sqrt{3}BM = AC + \max\{AM, MC\}.$$

4537. *Proposed by Arsalan Wares.*

Let A be a regular hexagon with vertices A_1, A_2, A_3, A_4, A_5 and A_6. The six midpoints on the six sides of hexagon A are connected to the six vertices with 12 line segments as shown. The dodecagon formed by these 12 line segments has been shaded. What part of hexagon A has been shaded?



4538. *Proposed by Nguyen Viet Hung.*

Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Prove that

$$\sum_{1 \le i \le n} \sqrt{1 + {a_i}^2} + \sum_{1 \le i < j \le n} a_i a_j \ge n - 1 + \sqrt{1 + (\sum_{1 \le i \le n} a_i)^2}.$$

When does equality occur?

4539. *Proposed by Leonard Giugiuc.*

Let ABC be a triangle with centroid G, incircle ω, circumradius R and semiperimeter s. Show that $24R\sqrt{6} \geq 25s$ given that G lies on ω.

4540. *Proposed by Prithwijit De.*

Given a prime p and an odd natural number k, do there exist infinitely many natural numbers n such that p divides $n^k + k^n$? Justify your answer.

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4531. *Proposé par Leonard Giugiuc et Dan Stefan Marinescu.*

Soient a, b et c des nombres réels positifs et soient x, y et z des nombres réels. De plus, supposer que $a + b + c = 2$ et $xa + yb + zc = 1$. Démontrer que

$$x + y + z - (xy + yz + zx) \geq \frac{3}{4}.$$

4532. *Proposé par Marius Stănean.*

Soit ABC un triangle de cercle circonscrit Γ et soient M, N, P des points situés sur les côtés BC, CA, AB, respectivement. Soient M', N', P' les intersections de AM, BN, CP avec Γ, mais distincts des sommets du triangle. Démontrer que

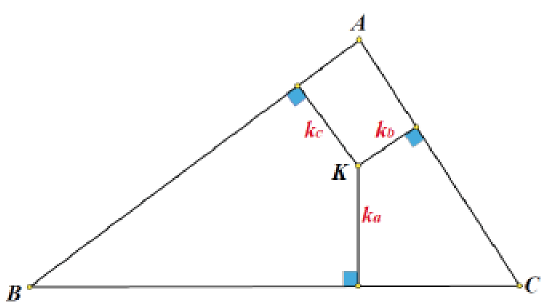
$$MM' \cdot NN' \cdot PP' \leq \frac{R^2 r}{4},$$

où R et r sont les rayons du cercle circonscrit, puis du cercle inscrit, du triangle ABC.

4533. *Proposé par Leonard Giugiuc et Kadir Altintas.*

Soit K un point symédian de ABC. Soient aussi k_a, k_b et k_c les longueurs des altitudes de K vers les côtés BC, AC et AB, respectivement. Si r est le rayon du cercle inscrit et s est le demi périmètre, démontrer que

$$\left(\frac{1}{r}\right)^2 + \left(\frac{3}{s}\right)^2 \geq \frac{2}{k_a^2 + k_b^2 + k_c^2}.$$



4534. *Proposé par Michel Bataille.*

Pour $n \in \mathbb{N}$, évaluer

$$\frac{\sum_{k=0}^{\infty} \frac{1}{k!(n+k+1)}}{\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k+1)!}}.$$

4535. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle acutangle d'orthocentre H et soit E la réflexion de H par rapport au mi point D du côté BC. Si la perpendiculaire vers DE au point H intersecte AB en X et AC en Y, démontrer que $HX \cdot EC + YC \cdot HE = EX \cdot BE$.

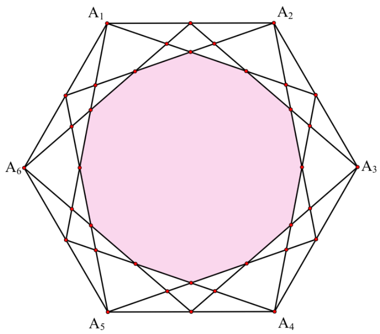
4536. *Proposé par Leonard Giugiuc et Rovsan Pirkuliev.*

Soit ABC un triangle tel que $\angle ABC = 60°$ et soit M un point sur le côté AC. Déterminer les angles du triangle, étant donné que

$$\sqrt{3}BM = AC + \max\{AM, MC\}.$$

4537. *Proposé par Arsalan Wares.*

Soit A un hexagone régulier de sommets A_1, A_2, A_3, A_4, A_5 et A_6. Les six mi points des six côtés de l'hexagone sont reliés aux six sommets, à l'aide de 12 segments, tels qu'indiqués ci-bas, où le dodécagone formé par ces 12 segments est coloré. Quelle fraction de l'hexagone A est ainsi colorée ?



4538. *Proposé par Nguyen Viet Hung.*

Soient $a_1, a_2, \ldots, a_n$ des nombres réels non négatifs. Démontrer que

$$\sum_{1 \le i \le n} \sqrt{1 + {a_i}^2} + \sum_{1 \le i < j \le n} a_i a_j \ge n - 1 + \sqrt{1 + (\sum_{1 \le i \le n} a_i)^2}.$$

Quand est-ce que l'égalité tient ?

4539. *Proposé par Leonard Giugiuc.*

Soit ABC un triangle de centroïde G, de cercle inscrit ω, de rayon de cercle circonscrit R et de demi périmètre s. Démontrer que $24R\sqrt{6} \geq 25s$, supposant que G se situe sur ω.

4540. *Proposé par Prithwijit De.*

À partir d'un nombre premier p et un entier naturel impair k, existe-t-il un nombre infini de nombres naturels n tels que p divise $n^k + k^n$? Justifier votre réponse.

BONUS PROBLEMS

*These problems appear as a bonus. Their solutions will **not** be considered for publication.*

B1. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let $ABCD$ be a cyclic quadrilateral. Let P be a point on the arc BC and let L and Q be the feet of perpendiculars dropped from P on the sides AD and BC, respectively. Let M and N be the feet of perpendiculars dropped from P on the lines AB and DC, respectively. Prove that

$$\frac{AM}{MB} \cdot \frac{BQ}{QC} \cdot \frac{CN}{ND} \cdot \frac{DL}{LA} = 1.$$

B2. *Proposed by Leonard Giugiuc.*

Find the real numbers x, y, z and t such that

$$xt - yz = -1 \quad \text{and} \quad x^2 + y^2 + z^2 + t^2 - xz - yt = \sqrt{3}.$$

B3. *Proposed by Leonard Giugiuc.*

Let a, b and c be positive real numbers such that $ab + bc + ca = 3$. Prove the inequality

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq 1.$$

B4. *Proposed by Leonard Giugiuc.*

Let ABC be a triangle with no angle exceeding 120° with $BC = a, AC = b$ and $AB = c$. Let T be its Fermat-Torricelli point, that is the point such that the total distance from the three vertices of ABC to T is minimum possible. Prove that

$$(b+c)|TA| + (c+a)|TB| + (a+b)|TC| \geq \sqrt{3}(|TA| + |TB| + |TC|)^2 - 4\text{Area}(ABC).$$

B5. *Proposed by Leonard Giugiuc.*

Let n be an integer such that $n \geq 4$. Consider real numbers $a_k, 1 \leq k \leq n$ such that $2 \geq a_1 \geq 1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n$ and $\sum_{k=1}^n a_k = n$. Prove that

a) $\sum_{k=1}^n a_k^2 \leq n + 2$.

b) $\sum_{1 \leq i < j \leq n}^n a_i a_j \geq \frac{(n-2)(n+1)}{2}$.

B6. *Proposed by Leonard Giugiuc.*

Let ABC be a triangle such that $\angle BAC \geq \frac{2\pi}{3}$. Prove that

$$\frac{r}{R} \leq \frac{2\sqrt{3}-3}{2},$$

where r is the inradius and R is the circumradius of ABC.

B7. *Proposed by Leonard Giugiuc.*

Let a, b, c and d be real numbers such that $2 \geq a \geq 1 \geq b \geq c \geq d \geq 0$ and $a + b + c + d = 4$. Prove that

$$\frac{2}{a^3 + b^3 + c^3 + d^3} + \frac{9}{ab + bc + cd + da + ac + bd} \leq 2.$$

B8. *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Let $ABCD$ be a rectangle with center O. Let M and P be two points in the plane (not necessarily distinct) such that O lies on the line MP and $OM = 3 \cdot OP$. Prove that

$$MA + MB + MC + MD \geq PA + PB + PC + PD.$$

B9. *Proposed by Leonard Giugiuc.*

Let $a, b, c \geq 1$ and $0 \leq d, e, f \leq 1$ such that $a + b + c + d + e + f = 6$. Prove that

$$6 \leq a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \leq 18.$$

B10. *Proposed by Leonard Giugiuc.*

Let ABC be a nonobtuse triangle with smallest angle A. Prove that

$$\cos(B - C) \geq \cos B + \cos C$$

and determine when equality holds.

B11. *Proposed by Michael Rozenberg and Leonard Giugiuc.*

Prove that if a, b, c and d are non-negative real numbers such that $a+b+c+d = 4$, then

$$ab + bc + cd + da + ac + bd \geq 3\sqrt{(a^2 + b^2 + c^2 + d^2)abcd}.$$

B12. *Proposed by Leonard Giugiuc.*

Let x, y and z be positive real numbers such that $xyz = 512$. Prove that

$$\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1+y}} + \frac{1}{\sqrt{1+z}} \geq 1.$$

B13$\star$. *Proposed by Leonard Giugiuc.*

Let n be an integer with $n \geq 4$. Prove or disprove that for any positive real numbers a_i, $i = 1, 2, \ldots, n$ that sum up to 1, we have:

$$\sqrt[n]{(1 - a_1^n)(1 - a_2^n) \cdots (1 - a_n^n)} \geq (n^n - 1)a_1 a_2 \ldots a_n.$$

B14$\star$. *Proposed by Leonard Giugiuc.*

Let k be a real number with $k > \frac{7+3\sqrt{5}}{2}$. Prove or disprove that for any non-negative real numbers x, y, z no two of which are zero, we have

$$\sqrt{\frac{x}{ky + z}} + \sqrt{\frac{y}{kz + x}} + \sqrt{\frac{z}{kx + y}} \geq \frac{3}{\sqrt{k + 1}}.$$

B15. *Proposed by Leonard Giugiuc.*

Let a, b and c be real numbers such that $a \geq b \geq 1 \geq c \geq 0$ and $a + b + c = 3$.

a) Show that $2 \leq ab + bc + ca \leq 3$.

b) Prove that $a^3 + b^3 + c^3 + \frac{45}{a^2+b^2+c^2} \leq 18$ and study the equality cases.

B16. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Let $ABCD$ be a cyclic quadrilateral. Prove that the following two statements are equivalent:

a) $AC \geq BD$,

b) $AB \cdot AD + CB \cdot CD \geq BA \cdot BC + DA \cdot DC$.

B17. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Let $ABCD$ be a cyclic quadrilateral. Prove that

$$AB + AC + AD + BC + BD + CD \leq 4R(\sqrt{2} + 1),$$

where R is the circumradius of $ABCD$.

B18. *Proposed by Leonard Giugiuc and Dorin Marghidanu.*

Let $n \geq 2$ be a natural number, and a_k be real numbers such that $0 < a_k < 2$ for all $k = 1, 2, \ldots, n$ with $\prod_{k=1}^{n} a_k = 1$. Prove that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1 + a_k}} \leq \frac{n}{\sqrt{2}}.$$

Prove further that the condition $a_k < 2$ can be dropped when $n = 2$ or $n = 3$.

B19. *Proposed by Leonard Giugiuc.*

Find the maximum value k such that

$$a^2 + b^2 + c^2 + k(ab + bc + ca) \geq 3 + k(a + b + c)$$

for any positive numbers a, b and c such that $abc = 1$.

B20$\star$. *Proposed by Leonard Giugiuc.*

Let $x, y \in (0, 3/2)$ be real numbers that satisfy $(x - 2)(y - 2) = 1$. Prove or disprove that

$$x^3 + y^3 \geq 2.$$

B21. *Proposed by Marian Cucoanes and Leonard Giugiuc.*

Consider an arbitrary triangle ABC with medians m_a, m_b, m_c, circumradius R, inradius r and exradii r_a, r_b, r_c. Show that

$$m_a + m_b + m_c \leq \sqrt{16R^2 + 4rR + 9r^2} \leq r_a + r_b + r_c.$$

B22. *Proposed by Leonard Giugiuc.*

Let a, b, c, d, e, f be non-negative real numbers such that $a+b+c+d+e+f = 4$. If $a \geq b \geq c \geq 1 \geq d \geq e \geq f \geq 0$, prove that

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + 180abcdef \leq 10.$$

B23. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Given a triangle ABC, let the tangent to its circumcircle at A intersect the line BC at D, and let the circle through A that is tangent to BC at D intersect the circumcircle again at E. Prove that $\dfrac{EB}{EC} = \left(\dfrac{AB}{AC}\right)^3$.

B24. *Proposed by Ruben Dario Auqui and Leonard Giugiuc.*

Let $ABCD$ be a square. Let ω be the circle centered at A with radius AB. A point M lies inside the square on ω; the line BM intersects the side CD at N. Prove that $CM = 2MN$ if and only if CM and BN are perpendicular.

B25. *Proposed by Ruben Dario Auqui and Leonard Giugiuc.*

Let ABC be a triangle with semiperimeter s. The A-excircle of the triangle touches the side BC at Q and the lines AB and AC at M and N, respectively. Suppose that AQ intersects MN at P. Prove that

$$AP = \frac{s\sqrt{a(s-a)(as + (b-c)^2)}}{b(s-c) + c(s-b)}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2019: 45(9), p. 518–523.

4481. *Proposed by Warut Suksompong.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y^2) = f(x + y)f(x - y) + 2f(y)y$$

for all $x, y \in \mathbb{R}$.

We received 16 submissions, of which 7 were correct and complete. We present the solution by Michel Bataille.

There are three functions that satisfy the equation:

- $f_0(x) = 0$ for all $x \in \mathbb{R}$,
- $f_1(x) = x$,
- $f_2(x) = 0$ for $x \neq 0$ and $f_2(0) = 1$.

The functions f_0, f_1 are obvious solutions. As for the function f_2, consider first the case when $x^2 + y^2 = 0$. Then $x = y = 0$ and

$$f_2(x^2 + y^2) = f_2(0) = 1 = f_2(0)f_2(0) + 2f_2(0) \cdot 0 = f_2(x + y)f_2(x - y) + 2f_2(y)y.$$

If $x^2+y^2 \neq 0$, then $x+y$ and $x-y$ cannot equal 0 both, hence $f_2(x+y)f_2(x-y) = 0$ and $2f_2(y)y = 0$ (since $f_2(y) = 0$ if $y \neq 0$) so that the required functional equation still holds and f_2 is also a solution.

Conversely, let f be an arbitrary solution. For convenience, we let $E(x, y)$ denote the equality $f(x^2+y^2) = f(x+y)f(x-y)+2f(y)y$. From $E(0, 0)$ we deduce that $f(0) = (f(0))^2$, hence $f(0) = 0$ or $f(0) = 1$.

First, suppose that $f(0) = 0$. From $E(x, x)$ and $E(x, -x)$ we obtain

$$2xf(x) = -2xf(-x) = f(2x^2),$$

so that $f(-x) = -f(x)$ if $x \neq 0$ and f is an odd function. With the help of this result and comparing $E(x, y)$ and $E(y, x)$, we get that

$$f(x + y)f(x - y) = xf(x) - yf(y)$$

for all x, y. Taking $y = 0$, this yields $(f(x))^2 = xf(x)$ and so $f(x) = 0$ or $f(x) = x$ for all x. If $f \neq f_0$, we have $f(a) = a \neq 0$ for some a. Now, we consider $x \neq 0$ and assume that $f(x) = 0$. Then, $f(x + a)f(x - a) = xf(x) - a^2 = -a^2 \neq 0$

and we must have $f(x+a) = x+a$ and $f(x-a) = x-a$. But it follows that $(x+a)(x-a) = -a^2$, contradicting $x \neq 0$. In consequence $f(x) = x$ and we can conclude that $f = f_0$ or $f = f_1$ in the case when $f(0) = 0$.

Second, suppose $f(0) = 1$. Again $E(x,x)$ and $E(x,-x)$ give $xf(x) = -xf(-x)$ so that $f(-x) = -f(x)$ whenever $x \neq 0$. Comparing $E(x,y)$ and $E(y,x)$ now gives $f(x+y)f(x-y) = xf(x) - yf(y)$ when $x \neq y$. To conclude, consider $x \neq 0$. Then $y = -\frac{x}{2} \neq \frac{x}{2}$, hence

$$f\left(\frac{x}{2}+\frac{x}{2}\right)f\left(\frac{x}{2}-\frac{x}{2}\right) = \frac{x}{2}f\left(\frac{x}{2}\right) + \frac{x}{2}f\left(-\frac{x}{2}\right),$$

that is, $f(x) = 0$. Thus, $f = f_2$ when $f(0) = 1$ and the proof is complete.

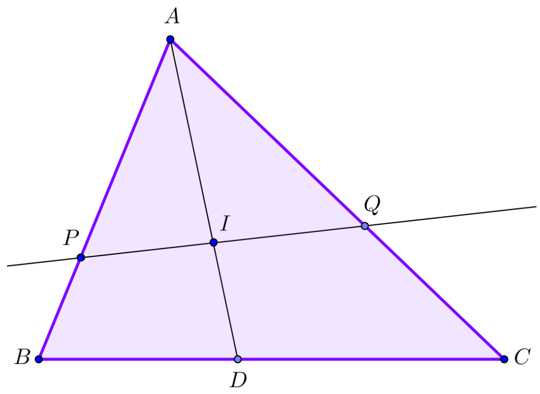
Editor's comments. With respect to incorrect solutions, there were two common errors: to note that if $f(0) = 0$ then for $x \neq 0$ it must be true that either $f(x) = x$ or $f(x) = 0$, but to omit the explanation of why only one of these statements must hold for all $x \neq 0$, and to miss the solution which has $f(0) = 1$.

4482. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be a triangle with incenter I. The line AI intersects BC at D. A line l passes through I and intersects the sides AB and AC at P and Q, respectively. Show that

$$AC \cdot \frac{[BDIP]}{[API]} + AB \cdot \frac{[CDIQ]}{[AQI]} = 2 \cdot BC + \frac{BC^2}{AB+AC},$$

where square brackets denote area.



All 18 of the submissions we received provided complete solutions; we feature the solution by Marie-Nicole Gras.

We put

$$a = BC, b = CA, c = AB, u = AP, v = AQ,$$

and use the label A to also denote the angles $\angle BAC = \angle PAQ$. The desired result is a consequence of the following known facts:

$$AD = \frac{2bc}{b+c}\cos\frac{A}{2},\ AI = \frac{2bc}{a+b+c}\cos\frac{A}{2}, \quad \text{so that } \frac{AD}{AI} = \frac{a+b+c}{b+c}.$$

Moreover, since PQ passes through I, AI is the internal bisector of the angle at A in ΔPAQ, so that

$$AI = \frac{2uv}{u+v}\cos\frac{A}{2} = \frac{2bc}{a+b+c}\cos\frac{A}{2}, \text{ which implies } \frac{u+v}{uv} = \frac{a+b+c}{bc}.$$

Putting these facts together, we have

$$\begin{aligned}
b\,\frac{[BDIP]}{[API]} + c\,\frac{[CDIQ]}{[AQI]} &= b\,\frac{[ABD]-[API]}{[API]} + c\,\frac{[ACD]-[AQI]}{[AQI]}\\
&= b\,\frac{cAD\sin\frac{A}{2} - uAI\sin\frac{A}{2}}{uAI\sin\frac{A}{2}} + c\,\frac{bAD\sin\frac{A}{2} - vAI\sin\frac{A}{2}}{vAI\sin\frac{A}{2}}\\
&= \frac{bc}{u}\frac{AD}{AI} + \frac{bc}{v}\frac{AD}{AI} - b - c\\
&= bc\frac{AD}{AI}\left(\frac{u+v}{uv}\right) - b - c\\
&= bc\,\frac{a+b+c}{b+c}\;\frac{a+b+c}{bc} - b - c\\
&= \frac{(a+b+c)^2}{b+c} - (b+c)\\
&= \frac{(a+b+c)^2-(b+c)^2}{b+c}\\
&= \frac{a(a+2b+2c)}{b+c} = 2a + \frac{a^2}{b+c} = 2BC + \frac{BC^2}{AB+AC}.
\end{aligned}$$

4483. *Proposed by Paul Bracken.*

For non-negative integers m and n, evaluate the following sum in closed form

$$\sum_{j=0}^{m} j^2\binom{j+n}{j}.$$

We received 15 solutions, all correct. We present the solution by Marie-Nicole Gras.

We will use the hockey-stick identity

$$\sum_{k=0}^{m}\binom{n+k}{n} = \binom{m+n+1}{n+1}.$$

For all $m, n \geq 0$ let $S_{m,n} = \sum_{j=0}^{m} j^2 \binom{j+n}{j}$. We have

$$\begin{aligned}
S_{m,n} &= \sum_{j=0}^{m} [j + j(j-1)] \binom{n+j}{n} \\
&= \sum_{j=1}^{m} j \binom{n+j}{n} + \sum_{j=2}^{m} j(j-1) \binom{n+j}{n} \\
&= \sum_{j=1}^{m} (n+1) \binom{n+j}{n+1} + \sum_{j=2}^{m} (n+1)(n+2) \binom{n+j}{n+2} \\
&= (n+1) \sum_{k=0}^{m-1} \binom{n+1+k}{n+1} + (n+1)(n+2) \sum_{k=0}^{m-2} \binom{n+2+k}{n+2}.
\end{aligned}$$

Apply the hockey-stick identity to each sum to get

$$S_{m,n} = (n+1) \binom{m+n+1}{n+2} + (n+1)(n+2) \binom{m+n+1}{n+3}.$$

4484. *Proposed by Leonard Giugiuc and Michael Rozenberg.*

Let $a, b, c \in [0, 2]$ such that $a + b + c = 3$. Prove that

$$4(ab + bc + ac) \leq 12 - ((a-b)(b-c)(c-a))^2$$

and find when the equality holds.

We received nine submissions, eight of which are correct and the other is incomplete. We present the solution by Digby Smith, modified slightly by the editor.

Without loss of generality we may assume that $a \geq b \geq c$.

Let $x = a - b$, $y = b - c$, and $z = a - c$. Then $x, y, z \geq 0$ and $x + y = z \leq 2$. Hence, $4xy \leq (x + y)^2 \leq 4$ so $xy \leq 1$, with equality if and only if $x = y = 1$.

Next, note that

$$\begin{aligned}
xy - (x+y)^2 &= xy - z(x+y) = xy - yz - zx \\
&= (a-b)(b-c) + (b-c)(c-a) + (c-a)(a-b) \\
&= (ab + bc + ca) - (a^2 + b^2 + c^2) \\
&= (ab + bc + ca) - ((a+b+c)^2 - 2(ab + bc + ca)) \\
&= 3((ab + bc + ca) - 3). \qquad (1)
\end{aligned}$$

By (1), the given inequality is equivalent in succession to

$$\begin{aligned} 12\left((ab+bc+ca)-3\right)+3(a-b)^2(b-c)^2(c-a)^2 &\le 0,\\ 4(xy-(x+y)^2)+3x^2y^2(x+y)^2 &\le 0,\\ (x+y)^2(4-3x^2y^2)-4xy &\ge 0. \end{aligned} \tag{2}$$

Finally, since $xy \le 1$, we have

$$\begin{aligned} (x+y)^2(4-3x^2y^2)-4xy &\ge 4xy(4-3x^2y^2)-4xy\\ &= 4xy(3-3x^2y^2)\\ &= 12xy(1-xy)(1+xy) \ge 0, \end{aligned}$$

so (2) holds and the proof of the inequality is complete. Clearly, equality holds if and only if $x=y=0$ or $x=y=1$ which implies that $(a,b,c)=(1,1,1)$ or any permutation of $(2,1,0)$.

4485. *Proposed by Jonathan Parker and Eugen J. Ionascu.*

For every square matrix with real entries $A = [a_{i,j}]_{i=1..n,j=1,2...n}$, we define the value

$$GM(A) = \max_{\pi\in S_n}\{\min\{a_{1\pi(1)}, a_{2\pi(2)}, ..., a_{n\pi(n)}\}\}$$

where S_n is the set of all permutations of the set $[n] := \{1,2,3,...,n\}$.

Given the 6×6 matrix

$$A := \begin{bmatrix} 20 & 9 & 7 & 26 & 27 & 13\\ 19 & 18 & 17 & 6 & 12 & 25\\ 22 & 24 & 21 & 11 & 20 & 11\\ 20 & 8 & 9 & 23 & 5 & 14\\ 22 & 17 & 4 & 10 & 36 & 33\\ 21 & 16 & 23 & 35 & 15 & 34 \end{bmatrix}$$

find the value $GM(A)$.

There were 11 correct solutions, most straightforward.

Let $\pi = (15)(263)$. Then

$$\begin{aligned} \min\{a_{1,\pi(1)}, a_{2,\pi(2)}, \ldots, a_{6,\pi(6)}\} &= \min\{a_{15}, a_{26}, a_{32}, a_{44}, a_{51}, a_{63}\}\\ &= \min\{27, 25, 24, 23, 22, 23\}\\ &= 22. \end{aligned}$$

Therefore $GM(A) \ge 22$.

On the other hand, since for each permutation π, the set being minimized contains a number $a_{\pi^{-1}(1),1}$ from the first column, the minimum cannot exceed 22. Thus $GM(A) \le 22$. Therefore $GM(A) = 22$.

4486. *Proposed by Marian Cucoaneş and Marius Drăgan.*

Let $a, b > 0$, $c > 1$ such that $a^2 \geq b^2 c$. Compute

$$\lim_{n\to\infty} (a - b\sqrt{c})(a - b\sqrt[3]{c}) \cdots (a - b\sqrt[n]{c}).$$

We received 12 solutions, one of which was incomplete and one of which was incorrect. We present the solution by Michel Bataille.

Let $P_n = \prod_{k=2}^{n} (a - b\sqrt[k]{c})$. From the hypotheses, we have $a \geq b\sqrt{c}$. We show that if $a = b\sqrt{c}$, then $\lim\limits_{n\to\infty} P_n = 0$ while if $a > b\sqrt{c}$, then $\lim\limits_{n\to\infty} P_n = \infty$ if $a > b + 1$ and $\lim\limits_{n\to\infty} P_n = 0$ if $a \leq b + 1$.

If $a = b\sqrt{c}$, then $P_n = 0$ for all $n \geq 2$ and so $\lim\limits_{n\to\infty} P_n = 0$.

From now on, we suppose that $a > b\sqrt{c}$. Note that $a > b$ and that $a - b\sqrt[k]{c} > a - b\sqrt{c} > 0$ for all $k \geq 2$ (so that $P_n > 0$ for all $n \geq 2$).

If $a > b + 1$, we choose m such that $a - b > m > 1$. Since

$$\frac{P_{n+1}}{P_n} = a - b\sqrt[n+1]{c} \quad \text{and} \quad \lim_{n\to\infty} \sqrt[n+1]{c} = 1,$$

we have $\lim\limits_{n\to\infty} \frac{P_{n+1}}{P_n} = a - b$. We deduce that for some $N \in \mathbb{N}$, we have $\frac{P_{n+1}}{P_n} > m$ for all $n \geq N$. Then, for any $r \in \mathbb{N}$, we obtain

$$\frac{P_{N+r}}{P_N} = \frac{P_{N+1}}{P_N} \cdot \frac{P_{N+2}}{P_{N+1}} \cdot \cdots \cdot \frac{P_{N+r}}{P_{N+r-1}} > m^r$$

and therefore $P_{N+r} > m^r \cdot P_N$ for all $r \geq 1$. Since $\lim\limits_{r\to\infty} m^r = \infty$, we must have $\lim\limits_{n\to\infty} P_n = \infty$.

If $a < b + 1$, similarly, we choose q such that $a - b < q < 1$ and then for some $N \in \mathbb{N}$, we have $\frac{P_{n+1}}{P_n} < q$ for all $n \geq N$. As above we deduce that $P_{N+r} < q^r P_N$ and so $\lim\limits_{n\to\infty} P_n = 0$ (since $\lim\limits_{r\to\infty} q^r = 0$).

Lastly, suppose that $a = b + 1$. Then $a - b\sqrt[n]{c} = 1 - b(\sqrt[n]{c} - 1)$ and when $n \to \infty$, we have

$$\ln(a - b\sqrt[n]{c}) \;\sim\; -b(\sqrt[n]{c} - 1) \;\sim\; -\frac{b\ln(c)}{n}$$

(since $\ln(1 - x) \;\sim\; -x$ as $x \to 0$ and $\sqrt[n]{c} - 1 = \exp((\ln(c))/n) - 1 \;\sim\; \frac{\ln(c)}{n}$ as $n \to \infty$).

The series $\sum_{n\geq 2} \ln(a - b\sqrt[n]{c})$ (whose terms are all negative for n large enough) is divergent (since $\sum_{n\geq 2} \frac{1}{n}$ is divergent). This means that $\lim\limits_{n\to\infty} \ln(P_n) = -\infty$ and so $\lim\limits_{n\to\infty} P_n = 0$.

4487. *Proposed by Martin Lukarevski.*

Let a, b, c be the sides of a triangle ABC, m_a, m_b, m_c the corresponding medians and R, r its circumradius and inradius respectively. Prove that

$$\frac{a^2}{m_b^2+m_c^2}+\frac{b^2}{m_c^2+m_a^2}+\frac{c^2}{m_a^2+m_b^2}\geq\frac{4r}{R}.$$

We received 20 correct solutions. We present the solution by Ioannis Sfikas.

By inequality on page 52 of *Geometric Inequalities* by O. Bottema and R. Z. Djordjevic, we have

$$9R^2\geq a^2+b^2+c^2 \quad \text{or} \quad \frac{1}{3}\left(a^2+b^2+c^2\right)\geq\frac{1}{27R^2}$$

and

$$2s^2\geq 27Rr \quad \text{or} \quad 2(a+b+c)^2\geq 108Rr,$$

where s is the semiperimeter of the triangle ABC. Also, we will use the well-known relation

$$m_a^2+m_b^2+m_c^2=\frac{3}{4}\left(a^2+b^2+c^2\right).$$

By Titu's lemma (a special case of the Cauchy-Schwarz inequality),

$$\begin{aligned}\frac{a^2}{m_b^2+m_c^2}+\frac{b^2}{m_c^2+m_a^2}+\frac{c^2}{m_a^2+m_b^2} &\geq\frac{(a+b+c)^2}{2\left(m_a^2+m_b^2+m_c^2\right)}\\ &=\frac{2(a+b+c)^2}{3\left(a^2+b^2+c^2\right)}\\ &\geq\frac{108Rr}{27R^2}=\frac{4r}{R}.\end{aligned}$$

4488. *Proposed by George Apostolopoulos.*

Let ABC be an acute-angled triangle. Prove that

$$\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}\leq\sqrt{\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}}.$$

We received 14 submissions, all correct, and we present the solution by Leonard Giugiuc.

We set $x=\tan\frac{A}{2}$, $y=\tan\frac{B}{2}$, and $z=\tan\frac{C}{2}$. Then $0<x,y,z<1$ and it is well known that $xy+yz+zx=1$.

Since

$$\frac{1-x^2}{2x}=\frac{1-\tan^2 A/2}{2\tan A/2}=\frac{1}{\tan A}=\cot A, \quad \frac{1-y^2}{2y}=\cot B, \quad \text{and} \quad \frac{1-z^2}{2z}=\cot C,$$

the given inequality is equivalent to

$$\sum_{cyc}\sqrt{\frac{1-x^2}{2x}} \le \frac{1}{\sqrt{xyz}} \quad \text{or} \quad \sum_{cyc}\sqrt{\frac{yz(1-x^2)}{2}} \le 1. \qquad (1)$$

By the AM-GM inequality, we have

$$\sqrt{\frac{yz(1-x^2)}{2}} \le \frac{1}{2}\left(yz + \frac{1-x^2}{2}\right) = \frac{1}{4}(2yz+1-x^2).$$

Similarly,

$$\sqrt{\frac{zx(1-y^2)}{2}} \le \frac{1}{4}(2zx+1-y^2),$$

and

$$\sqrt{\frac{xy(1-z^2)}{2}} \le \frac{1}{4}(2xy+1-z^2).$$

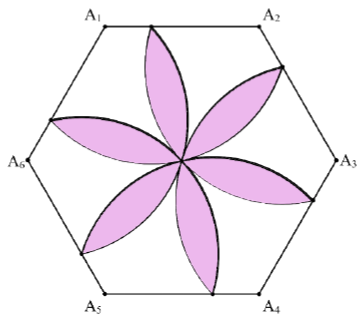
Adding up the three inequalities above then yields

$$\begin{aligned}\sum_{cyc}\sqrt{\frac{yz(1-x^2)}{2}} &\le \frac{1}{4}\left(2(xy+yz+zx)+3-(x^2+y^2+z^2)\right)\\ &= \frac{1}{4}(5-(x^2+y^2+z^2))\\ &\le \frac{1}{4}(5-(xy+yz+zx)) = 1,\end{aligned}$$

so (1) holds and the proof is complete.

4489. *Proposed by Arsalan Wares.*

Regular hexagon A has its vertices at points A_1, A_2, A_3, A_4, A_5 and A_6. Six circular congruent arcs are drawn inside hexagon A and all six pass through the center of A. The terminal points of each of the six arcs divide the sides of A in the ratio $3:7$. The six regions within A that are bounded only by circular arcs have been shaded. Find the ratio of the area of A to the area of the shaded region.



We received 17 correct solutions. We present the solution by Jason Smith.

Let the side length of the hexagon be 10. The area of the entire hexagon is

$$A_{\text{hex}} = 6 \cdot \frac{1}{2} \cdot 10^2 \sin \frac{\pi}{3} = 150\sqrt{3}.$$

Let C denote the center of the hexagon and B_1, B_2, ..., B_6 the tips of the flower petals. For the squared distance r^2 between consecutive tips, the law of cosines applied to triangle $B_1 A_2 B_2$ gives

$$r^2 = 3^2 + 7^2 - 2 \cdot 3 \cdot 7 \cdot \cos \frac{2\pi}{3} = 79.$$

We can use this value of r^2 to find the area of the of the two circular segments with vertices B_1 and C. Since the area of one of these circular segments is the area of a sector of the circle minus the area of a triangle, we have

$$A_{\text{seg}} = \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin\theta = \frac{1}{2} \cdot 79 \left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = \frac{79}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right).$$

There are twelve such identical circular segments comprising the shaded region, so

$$A_{\text{sh}} = 12 \cdot \frac{79}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = 79(2\pi - 3\sqrt{3}).$$

Therefore, the ratio we seek is equal to

$$\frac{A_{\text{hex}}}{A_{\text{sh}}} = \frac{150\sqrt{3}}{79(2\pi - 3\sqrt{3})} \approx 3.025.$$

4490. *Proposed by Borislav Mirchev and Leonard Giugiuc.*

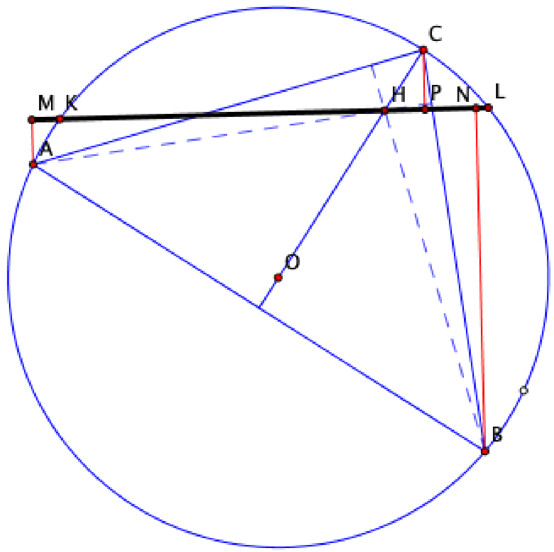
A line ℓ through the orthocenter H of the acute triangle ABC meets the circumcircle at points K on the smaller arc AC and L on the smaller arc BC. If M, N, and P are the feet of the perpendiculars to ℓ from the vertices A, B, and C, respectively, prove that $PH = |KM - LN|$.

All 7 of the solutions we received were complete and correct, although four of them failed to express clearly the result that they were proving. We feature a composite of the independent solutions by Michel Bataille and by J. Chris Fisher.

As stated, the problem is not correct: see the accompanying figure where $PH = KM + LN$. Moreover, the statement is unnecessarily restrictive. We shall show the following result, which remains true independent of the figure:

> A line ℓ through the orthocenter H of an *arbitrary* triangle ABC meets the circumcircle at points K and L. If M, N, and P are the feet of the perpendiculars to ℓ from the vertices A, B, and C, respectively, then
>
> $$PH = \|\overrightarrow{KM} + \overrightarrow{LN}\|.$$

Therefore, we have that $PH = KM + LN$ if $\overrightarrow{KM}$ and $\overrightarrow{LN}$ have the same sense, and $PH = |KM - LN|$ otherwise.



For the proof, we can assume a unit circumradius, place the circumcenter at the origin, and rotate the figure about O so that ℓ is parallel to the x-axis. If the coordinates of vertices are

$$A(a, a'),\ B(b, b'), \quad \text{and} \quad C(c, c'),$$

where $a^2 + a'^2 = b^2 + b'^2 = c^2 + c'^2 = 1$, then the orthocenter has coordinates

$$H(a + b + c,\ a' + b' + c').$$

By assumption, ℓ is the line $y = a' + b' + c'$. The x-coordinates of M, N, and P will therefore be a, b, and c, respectively, while the x-coordinates of K and L have values k and $-k$, which we have no need to calculate. Thus,

$$PH = |(a + b + c) - c| = |a + b|.$$

When M and L are both inside or both outside the circumcircle, the vectors $\overrightarrow{KM}$ and $\overrightarrow{LN}$ point in opposite directions, whence the quantities $k - a$ and $-k - b$ have opposite signs. It follows that

$$|KM - LN| = |k - a + (-k - b)| = |a + b| = PH.$$

When one of M or N is inside the circumcircle and the other outside, then $k - a$ and $-k - b$ have the same sign, and

$$KM + LN = |k - a + (-k - b)| = |a + b| = PH,$$

as claimed.

Editor's comments. When the editors revised the statement of the proposers' problem, their assumption that both M and N lie inside the circumcircle was unfortunately omitted. There seems to be no obvious criteria for predicting the relative behaviour of M and N. It is easy to check that the assumption of acute angles is not sufficient. Consider, for example, an isosceles triangle ABC with $CA = CB$. When $\angle C = 60^\circ$ (and $H = O$) then for all choices of the point K, ℓ is a diameter, and both M and N are on or inside the circumcircle. But should $\angle C > 60^\circ$ (and H be between C and O as in the above figure) then K can be chosen sufficiently close to A on the arc AC that omits B so that M is outside the circumcircle while N is inside. (This serves as a counterexample to the problem as revised by the editors.) Note that the value of the y-coordinate plays no role in the argument so that an analogous result holds for lines parallel to ℓ. Note, finally, that when $\angle C = 90^\circ$, we have a rather nice, easily proved result, namely $KM = CN$ (which is a consequence of our result for right triangles since $PH = 0$ and $C = L$).