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IN THIS ISSUE / DANS CE NUMÉRO

MATHEMATTIC

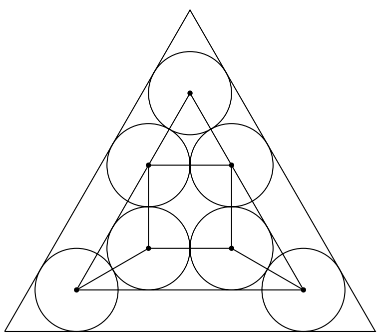
No. 9

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **December 30, 2019.**

MA41. The diagram shows the densest packing of seven circles in an equilateral triangle.



Determine the exact fraction of the area of the triangle that is covered by the circles.

MA42. Find all functions of the form $f(x) = \dfrac{a+bx}{b+x}$ where a and b are constants such that $f(2) = 2f(5)$ and $f(0) + 3f(-2) = 0$.

MA43. If n is not divisible by 4, prove that $1^n + 2^n + 3^n + 4^n$ is divisible by 5 for any positive integer n.

MA44. Find the largest positive integer which divides all expressions of the form $n^5 - n^3$ where n is a positive integer. Justify your answer.

MA45. A sequence $s_1, s_2, \ldots, s_n$ is harmonic if the reciprocals of the terms are in arithmetic sequence. Suppose $s_1, s_2, \ldots, s_{10}$ are in harmonic sequence. Given $s_1 = 1.2$ and $s_{10} = 3.68$, find $s_1 + s_2 + \cdots + s_{10}$.

. . . .

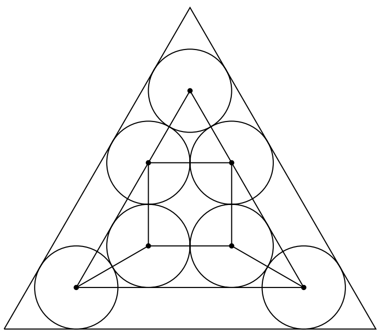
Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 décembre 2019.**

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA41. Le schéma suivant montre l'empilement le plus compact de sept cercles dans un triangle équilatéral.



Déterminer exactement la fraction de la surface du triangle qui est couverte par les cercles.

MA42. Déterminer toutes les fonctions de la forme $f(x) = \dfrac{a+bx}{b+x}$, où a et b sont des constantes, puis telles que $f(2) = 2f(5)$ et $f(0) + 3f(-2) = 0$.

MA43. Si n n'est pas divisible par 4, démontrer que $1^n + 2^n + 3^n + 4^n$ est divisible par 5 pour tout entier positif n.

MA44. Déterminer le plus grand entier positif qui divise toutes les expressions du genre $n^5 - n^3$, où n est un entier positif.

MA45. Une séquence $s_1, s_2, \ldots, s_n$ est dite harmonique si les réciproques des termes sont en progression arithmétique. Supposer que $s_1, s_2, \ldots, s_{10}$ est harmonique et que $s_1 = 1.2$ puis $s_{10} = 3.68$. Déterminer $s_1 + s_2 + \cdots + s_{10}$.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(4), p. 168–169.

MA16. Prove that every year must have a month in which the 13^{th} is a Friday. What is the maximum number of Friday the 13^{th}s in one year?

Problem proposed by John McLoughlin.

We received 3 submissions, all correct. We present the solution of Richard Hess, modified by the editor.

The 13^{th} of the month taken $\pmod 7$ for a normal year is

$$6, 2, 2, 5, 0, 3, 5, 1, 4, 6, 2, 4, \qquad (1)$$

for each month respectively. The 13^{th} of the month, mod(7) for a leap year is

$$6, 2, 3, 6, 1, 4, 6, 2, 5, 0, 3, 5, \qquad (2)$$

for each month respectively. In each of the above cases, 0 through 6 occur at least once. It follows that every year must have at least one Friday the 13^{th}.

In a normal year, (1) implies that the maximum number of Friday the 13^{th}s is 3: in February, March, and November. In a leap year, by (2) the maximum number of Friday the 13^{th}s is also 3, occurring in January, April, and July.

MA17. Let $f(n) = 25^n - 72n - 1$. Determine, with proof, the largest integer M such that $f(n)$ is divisible by M for every positive integer n.

Originally Problem 5 from the 2005 Iowa Collegiate Mathematics Competition.

We received 5 correct solutions. We present a composite of the similar solutions by Henry Ricardo and Amit Kumar Basistha, done independently.

We prove that $M = 48$ by noting that 48 is the largest divisor of $f(1)$ and by using induction to show that $48|f(n)$ for every positive integer n.

When $n = 1$, $f(1) = -48$, which indeed has 48 as its largest divisor, so $M \leq 48$. Now suppose that $48|f(k)$ for some $n = k \geq 1$. Then

$$\begin{aligned} f(k+1) &= 25^{k+1} - 72(k+1) - 1 \\ &= 25^k + 24(25^k) - 72k - 72 - 1 \\ &= (25^k - 72k - 1) + 24(25^k) - 72 \\ &= (25^k - 72k - 1) + 24(25^k - 3). \end{aligned}$$

In the last expression, the first summand is assumed divisible by 48; and the second term, as the product of 24 and an even number, is also a multiple of 48. Thus, $48|f(k+1)$, so $M = 48$.

MA18. Twenty calculus students are comparing grades on their first two quizzes of the year. The class discovers that every pair of students received the same grade on at least one of the two quizzes. Prove that the entire class received the same grade on at least one of the two quizzes.

No solutions were received. The problem remains open.

MA19. Choose N elements of $\{1, 2, 3, \ldots, 2N\}$ and arrange them in increasing order. Arrange the remaining N elements in decreasing order. Let D_i be the absolute value of the difference of the ith elements in each arrangement. Prove that

$$D_1 + D_2 + \cdots + D_N = N^2.$$

Originally Problem 6 from the 2009 Iowa Collegiate Mathematics Competition.

We received 3 submissions, all correct. We present the solution provided by Henry Ricardo.

Suppose we have

$$\{a_1, a_2, \ldots, a_N\} \cup \{b_1, b_2, \ldots, b_N\} = \{1, 2, \ldots, 2N\}$$

such that

$$a_1 < a_2 < \cdots < a_N \quad \text{and} \quad b_1 > b_2 > \cdots > b_N.$$

For each $i \in \{1, 2, \ldots, N\}$, one of the two numbers $a_i,\ b_i$ is less than or equal to N and the other is greater than N. If we assume the contrary, $a_i \leq N$ and $b_i \leq N$, then

$$a_1 < \cdots < a_i \leq N \quad \text{and} \quad b_N < \cdots < b_i \leq N.$$

Thus we have found

$$i + (N - i + 1) = N + 1$$

distinct positive integers which are less than or equal to N, which is a contradiction.

Similarly, we cannott have $a_i > N$ and $b_i > N$.

It follows that

$$\begin{aligned}
D &= \sum_{i=1}^{N} D_i \\
&= |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_N - b_N| \\
&= (N+1) + (N+2) + \cdots + (N+N) - 1 - 2 - \cdots - N \\
&= N^2.
\end{aligned}$$

MA20.

a) A line segment of length 11 is randomly cut into three pieces, each of integer length. What is the probability that the three pieces can be formed into a (non-degenerate) triangle?

b) Can you find the probability above for a line segment of any odd integer length $n = 2k + 1$?

Originally Problem 10 from the 2009 Iowa Collegiate Mathematics Competition.

We received 2 submissions, both correct. We present the solution by Richard Hess with extensive details by the editor.

b) We assume that all 3 pieces have positive integer length. Let

$$S = \{(x, y) \in \mathbb{Z}^2 \colon 1 \le x < y \le 2k\}.$$

A cutting of the line segment corresponds to a pair $(x, y) \in S$, where x and y are the locations of the cuts along the line segment, and so the three pieces will have lengths x, $y - x$, and $2k + 1 - y$. We need to determine how many elements of S result in pieces forming a triangle. Suppose $(a, b) \in S$ represents a cut with pieces forming a triangle. Note that the pieces form a triangle if and only if all pieces have length at most k. Thus

$$1 \le a,\ b - a,\ 2k + 1 - b \le k,$$

which is equivalent to $1 \le a \le k$ and $k + 1 \le b \le k + a$. It follows that the number of pairs representing a triangle is

$$\sum_{a=1}^{k} |\{b \in \mathbb{Z} \colon k + 1 \le b \le k + a\}| = \sum_{a=1}^{k} a = \frac{k(k+1)}{2}.$$

Since

$$|S| = \frac{2k(2k-1)}{2},$$

the probability of choosing a pair representing a triangle is

$$\frac{k(k+1)}{2k(2k-1)} = \frac{k+1}{4k-2}.$$

a) Using the above formula, the probability is $1/3$.

TEACHING PROBLEMS

No.6

John McLoughlin

Numerical Curiousities as Starting Points

Many numbers have curious properties that can become starting points for mathematical discussion including problem solving. This feature of *Teaching Problems* offers a range of examples for potential use in the development of numerical appreciation as well as problem solving.

What is meant by appreciating numbers more? Begin with the number 370 as an example. Perhaps you are unaware of the curious property of this number. The sum of the cubes of its digits is the number itself. That is,

$$370 = 3^3 + 7^3 + 0^3 = 27 + 343 + 0.$$

Clearly, it follows that 371 would share this property, as only the 0 is swapped for 1 with each of these digits being their own cubes. You are challenged to find two other three-digit numbers that also have this property.

Numerical appreciation can be advanced through skill building also, without necessarily engaging in problem solving. For instance, listing the factors of 220 and 284 (excluding the numbers themselves) we get $\{1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110\}$ and $\{1, 2, 4, 71, 142\}$, respectively. Summing the numbers in each of these lists we get the other number. The sum of the proper divisors of 220 is 284, and the sum of the proper divisors of 284 is 220. Hence, 220 and 284 are called an *amicable pair.*

Why does this work?

Likewise, another approach to enhancing wonder is to engage students in an activity or skill-related sequence of calculations that results in something surprising. The motivation is presented to figure out the "why" of the surprise. Consider the following sequence of steps:

- Take any three different digits.
- Arrange them to form the largest and smallest possible three-digit numbers.
- Subtract these two numbers to create the largest positive difference.
- Now take this result and add it to its reversal (the three digits ordered in the opposite way).

What is your result? It will always be 1089. The challenge is to figure out why does this work. You may find it helpful to consider using the three digits a, b, and c with $a > b > c$.

That number 1089 is an interesting number in itself, as $1089 \times 9 = 9801$. There is only one other four-digit number that when multiplied by a single digit results in a product that is its own reversal, aside from trivial cases such as $3333 \times 1 = 3333$.

Product Reversal

This particular example is one that I have used in many classes as a problem for addressing issues with number sense. The answer is far from obvious, thus ensuring adequate time for processing on the part of students. Also, it lends itself nicely to class discussion. Generally my approach is to let everyone play with the problem for just a few minutes, and then move into discussion where any student can offer only one insight that moves us forward collectively in solving the problem.

> Determine values for A, B, C, and D that make the following product correct:

$$\begin{array}{cccc} A & B & C & D \\ & & \times & 4 \\ \hline D & C & B & A \end{array}$$

(Note: Implicit to such a problem are two assumptions: the leading digit in a number is not 0, and each letter represents a distinct digit.)

Typically, the first observation concerns the value of A or the quantity "$ABCD$" as one notices the restriction imposed by the four digit product. (The use of quotation marks is to indicate that the number contains four digits and it is not the product of A, B, C, and D that is being referred to here.) Hence, it follows that A can not be greater than 2. In fact, A must be 2 as A is also the ones' digit in a multiple of 4, thus making it necessary for A to be an even number. So we have $A = 2$. This implies that $D = 8$ or 9, but in fact if we looked from the other point of view we see that $D \times 4$ ends in 2 making D either 3 or 8. So it follows that $D = 8$. Let us reconsider the product from this stage.

$$\begin{array}{cccc} 2 & B & C & 8 \\ & & \times & 4 \\ \hline 8 & C & B & 2 \end{array}$$

There are different ways of seeing this to home. One valid way is to make the observation that B is less than 2 since $2300 \times 4 > 9000$. Further, B must be odd since any multiple of 4 ending in 2 must have a tens' digit that is odd. Alternatively, note that $8 \times 4 = 32$ will produce a carry of 3 to be added to $4 \times C$, thus resulting in an odd value for B. So $B = 1$.

When the value of B has been determined, we can solve to get $C = 7$. Returning to that carrying idea we have $4 \times C + 3$ ending in 1. This makes it necessary for $4 \times C$ to end in 8, thus, giving $C = 2$ (inadmissible since $A = 2$) or $C = 7$. Checking we find that indeed $2178 \times 4 = 8712$.

A Factorial Problem

Find all three-digit numbers that equal the sum of the factorials of their digits. That is, find values of x, y and z (not necessarily different) for which the three-digit number "xyz" is equal to $x! + y! + z!$.

Before commencing to attack this problem, it is worth mentioning that the unusual presentation almost makes it a certainty that the problem is not going to be solved quickly. Again this is advantageous in teaching situations as it necessitates time for processing and understanding the problem. Further, it is unclear at first if there will be a unique solution, if any. The best way to begin is to make sure one understands the problem. Make up a three-digit number and try it out. For example, 281 does not work. However, something is learned quickly as the value of 8! is much too large. In fact, this points to the need to identify the values of the various factorials.

These are summarized here:

$$0! = 1, \quad 1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120, \quad 6! = 720$$

and $7! > 1000$. Our first significant insight is to note that the values of the factorials of 7, 8, and 9 preclude them from appearing as digits in our number. Further, we see that a 6 can not appear in the number either as the inclusion of the digit 6 would force one of the digits to be at least 7 due to the fact that $6! > 700$. Hence, our values of the digits are restricted to 0, 1, 2, 3, 4, and 5.

What do we notice next? If you observe that $4! = 24$ and $3 \times 24 < 100$, it becomes evident that the number must have at least one 5 in it to enable the sum of the factorials of the digits to be a three-digit number. We can quickly see that 555 is unacceptable as 3×120 is not equal to 555. So really there are two cases to consider, namely the number has one 5 or two 5's in it. Further, the maximum attainable value would be $5! + 5! + 4!$. It follows that the three-digit number is not greater than 500, and hence, any 5's do not appear in the hundreds' place.

Case 1: There are two 5's and the number is of the form $x55$.

This possibility can be dismissed by noting that since $5! + 5! = 240$, the first digit must be 2 and checking shows this does not work.

Case 2: There is only one 5.

Since $5! = 120$ and it is impossible to achieve a sum greater than 200, the number must begin with 1. Therefore, the number could be $15y$ or $1z5$. In either case, we have 121 as the sum of $1! + 5!$. It is impossible to reach 150 by adding 4! and hence the $15y$ case is not workable. The latter case of $1z5$ requires adding a value that ends in 4, thus leaving 24 as that possibility. Checking we find that 145 actually works as $1 + 24 + 120$ gives the required sum.

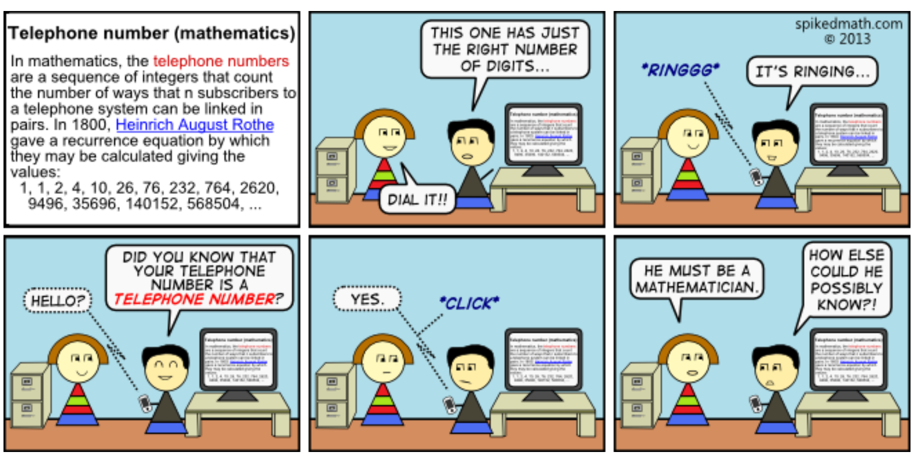
In conclusion, 145 is the unique three-digit number equal to the sum of the factorials of its digits.

Some questions to try.

Keeping with the theme of this feature, here are a couple of challenges for your consideration. In each case, find any numbers for which the property holds.

1. Three different digits a, b, and c form a three-digit number. Two of these digits can be selected and ordered to make six different two-digit numbers. Take each of these two-digit numbers and multiple them separately by the digit that is not part of that number. Sum these six values together. Lo and behold, the result is the original three-digit number.

2. There is a four-digit number with a special property. Suppose its digits are arranged in decreasing order to form the four-digit number M. Instead, if its digits are arranged in increasing order, we get the four-digit number N. The curious fact is that $M - N$ results in the original number.



OLYMPIAD CORNER

No. 377

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **December 30, 2019.**

OC451. Determine the least natural number a such that

$$a \geq \sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k)$$

for any nonzero natural number n and for any positive real numbers $a_1, a_2, \ldots, a_n$ whose sum is at most π.

OC452. Let $ABCD$ be a square. Consider the points $E \in AB$, $N \in CD$ and $F, M \in BC$ such that triangles AMN and DEF are equilateral. Prove that $PQ = FM$, where $\{P\} = AN \cap DE$ and $\{Q\} = AM \cap EF$.

OC453. Let $n \geq 2$ be an integer and let $A, B \in \mathcal{M}_n(\mathbb{C})$. If $(AB)^3 = O_n$, is it true that $(BA)^3 = O_n$? Justify your answer.

OC454. Find all the functions $f : \mathbb{N} \to \mathbb{N}$ having the following property for each natural number m: if $d_1, d_2, \ldots, d_n$ are all the divisors of the number m, then

$$f(d_1)f(d_2) \cdot \ldots \cdot f(d_n) = m.$$

OC455. Let D be a point on the base AB of an isosceles triangle ABC. Select a point E so that the $ADEC$ is a parallelogram. On the line ED, take a point F such that $E \in DF$ and $EB = EF$. Prove that the length of the chord that the line BE cuts on the circumcircle of triangle ABF is twice the length of the segment AC.

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 décembre 2019.**

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC451. Déterminer le plus petit nombre naturel a tel que

$$a \geq \sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k)$$

pour tout nombre naturel non nul n et pour tous nombres réels positifs $a_1, a_2, \ldots, a_n$ dont la somme est au plus π.

OC452. Soit $ABCD$ un carré. Considérer des points $E \in AB$, $N \in CD$, et $F, M \in BC$ tels que les triangles AMN et DEF sont équilatéraux. Démontrer que $PQ = FM$, où $\{P\} = AN \cap DE$ et $\{Q\} = AM \cap EF$.

OC453. Soit $n \geq 2$ un entier et soient $A, B \in \mathcal{M}_n(\mathbb{C})$. Si $(AB)^3 = O_n$, est-il toujours le cas que $(BA)^3 = O_n$? Justifier votre réponse.

OC454. Déterminer toutes les fonctions$f : \mathbb{N} \to \mathbb{N}$ ayant la propriété que pour tout nombre naturel m, si $d_1, d_2, \ldots, d_n$ sont tous les diviseurs de m, alors

$$f(d_1)f(d_2) \cdot \ldots \cdot f(d_n) = m.$$

OC455. Soit D un point sur la base du triangle isocèle ABC. Choisir un point E tel que $ADEC$ est un parallèlogramme. Sur la ligne ED, soit un point F tel que $E \in DF$ et $EB = EF$. Puis, soit la corde que la ligne BE taille dans le cercle circonscrit du triangle ABF ; démontrer que la longueur de cette corde est le double de la longueur du segment AC.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(4), p. 181–182.

OC426. Find all pairs of natural numbers a and k such that for every positive integer n relatively prime to a, the number $a^{k^n+1}-1$ is divisible by n.

Originally Moscow MO, 4th Problem, Grade 9, Final Round 2017.

We received 1 correct submission. We present the solution by Sushanth Sathish Kumar.

We claim that all solutions are of the form $a=1$, and k is an arbitrary natural number. Clearly all these work.

Next we assume that $a>1$. We show that this is not possible.

Assume k is even. Take $n=a+1$. We obtain $1\equiv a^{k^{a+1}+1}\equiv(-1)^{k^{a+1}+1}$ modulo $a+1$. This reduces to $1\equiv -1 \pmod{a+1}$, or $2\equiv 0 \pmod{a+1}$ which implies $a=1$; contradiction since we assumed $a>1$.

Assume that k is odd. Pick a prime $p>2$ satisfying $\gcd(p,a)=1$. Let g be the order of a modulo p. Since $a^{p-1}\equiv 1 \pmod p$ by Fermat's little theorem and $a^{k^p+1}\equiv 1 \pmod p$ as given, it follows that $g|p-1$ and $g|k^p+1$.

The latter condition implies

$$k^{2p}\equiv 1 \pmod g,$$

and that the possible orders of k modulo g are 1, p, 2, and $2p$.

If the order is 1, then $k\equiv 1 \pmod g$, and $k^p+1\equiv 2 \pmod g$. It follows that $g|2$, and $a^2=(a^g)^{2/g}\equiv 1 \pmod p$. Since k is odd, this shows $a^{k+1}\equiv 1 \pmod p$.

If the order is p, then $g|k^p-1$, which in combination with $g|k^p+1$ forces $g|2$. Using the same logic as in the above paragraph, we obtain $a^{k+1}\equiv 1 \pmod p$.

If the order is 2, then $k^{p-1}\equiv(k^2)^{(p-1)/2}\equiv 1 \pmod g$. So, $-1\equiv k(k^{p-1})\equiv k \pmod g$. Thus, $g|k+1$, which implies $a^{k+1}\equiv 1 \pmod p$.

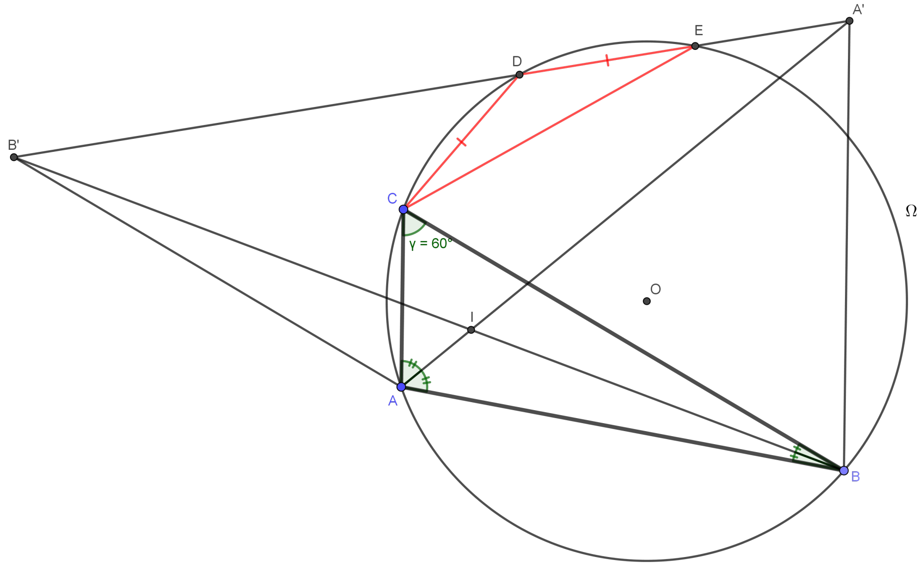
If the order is $2p$ then $2p|\phi(g)$, where ϕ is Euler's Totient function. Therefore $2p\le\phi(g)<g$, which is impossible since $g|p-1$.

Hence the order of k modulo g can be 1, 2, or p. As shown before $a^{k+1}\equiv 1 \pmod p$, or equivalently $p|a^{k+1}-1$ for every prime $p>2$ satisfying $\gcd(p,a)=1$. However, this is not possible because we can select such a prime p with $p>a^{k+1}-1$. In conclusion we cannot have $a>1$.

OC427. In a scalene triangle ABC, $\angle C = 60^\circ$ and Ω is its circumcircle. On the angle bisectors of $\angle A$ and $\angle B$ take two points A' and B', respectively such that $AB' \parallel BC$ and $BA' \parallel AC$. The line $A'B'$ intersects Ω at points D and E. Prove that triangle CDE is isosceles.

Originally Russia MO, 7th Problem, Grade 9, Final Round 2017.

We received 1 submission. We present the solution by Andrea Fanchini.



We use barycentric coordinates with reference to $\triangle ABC$.

Because $\angle C = 60^\circ$, we have that

$$c^2 = a^2 + b^2 - ab, \qquad c = R\sqrt{3}, \qquad s = \sqrt{3}(R + r).$$

The angle bisectors of $\angle A$ and $\angle B$ are

$$AI : cy - bz = 0, \qquad BI : cx - az = 0.$$

Then the points A' and B' are

$$A' = AI \cap BCA_\infty = (-c : b : c), \qquad B' = BI \cap ABC_\infty = (a : -c : c).$$

Therefore, the line $A'B' : c(b + c)x + c(a + c)y + (c^2 - ab)z = 0$ intersects the circumcircle of $\triangle ABC$ at points D and E given by

$$D\left(a(a + c)(b - a) : b(b + c)(a - b) : c(b + c)(a + c)\right),\ E\left(a(b - a) : b(a - b) : c^2\right).$$

Finally, we have

$$CD = DE = \sqrt{R(R - 2r)}.$$

OC428.

(a) Prove that there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g = g \circ f$, $f \circ f = g \circ g$ and $\forall x \in \mathbb{R}\ f(x) \neq g(x)$.

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f \circ g = g \circ f$ and $\forall x \in \mathbb{R}\ f(x) \neq g(x)$, then $\forall x \in \mathbb{R}\ (f \circ f)(x) \neq (g \circ g)(x)$.

Originally Romania MO, 2nd Problem, Grade 11, District Round 2017.

We received 2 submissions, both of which are correct. For part (a), we present each solver's example. For part (b), we present a summary of the similar solutions submitted by both solvers.

(a) *Solution 1, by Oliver Geupel.*

Let f be the identity function and let

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ 1/x & \text{if } x \notin \{0, 1\}. \end{cases}$$

Solution 2, by Nilendu Das.

Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 1 - x & \text{if } x \text{ is irrational,} \end{cases}$$

and

$$g(x) = \begin{cases} 1 - x & \text{if } x \text{ is rational,} \\ x & \text{if } x \text{ is irrational.} \end{cases}$$

In both examples, we can check that

$$(f \circ g)(x) = (g \circ f)(x),$$
$$(f \circ f)(x) = x = (g \circ g)(x),$$

and $f(x) \neq g(x)$ for all $x \in \mathbb{R}$.

(b) Suppose f and g are functions that satisfy the given conditions. Since f and g are continuous functions, so is their difference $h(x) = f(x) - g(x)$.

Since $\forall x \in \mathbb{R}\ f(x) \neq g(x)$, the function h has no root, and it never changes its sign by Bolzano's theorem. Hence,

$$\forall x \in \mathbb{R}\ f(x) > g(x) \quad \text{or} \quad \forall x \in \mathbb{R}\ g(x) > f(x).$$

By symmetry, there is no loss of generality in assuming that f is always greater than g. Hence,

$$f(f(x)) > g(f(x)) = f(g(x)) > g(g(x)).$$

Thus, $(f \circ f)(x) > (g \circ g)(x)$, from which $(f \circ f)(x) \neq (g \circ g)(x)$ is immediate.

OC429. Let $A \in \mathcal{M}_n(\mathbb{C})$ $(n \geq 2)$ with $\det A = 0$ and let A^* be its adjoint. Prove that $(A^*)^2 = \operatorname{tr}(A^*)A^*$, where $\operatorname{tr}(A^*)$ is the trace of the matrix A^*.

Originally Romania MO, 4th Problem, Grade 11, District Round 2017.

We received 2 correct submissions. We present the solution by Brian Bradie.

Let $A \in \mathcal{M}_n(\mathbb{C})$ $(n \geq 2)$ with $\det A = 0$. By A^*, the adjoint of A, we mean the transpose of the cofactor matrix of A. Because $\det A = 0$, $\operatorname{rank}(A) \leq n-1$. We consider two cases.

Case 1. If $\operatorname{rank}(A) = n-1$, then $\operatorname{rank}(A^*) = 1$, and there exist non-zero vectors $u, v \in \mathbb{C}^n$ such that $A^* = u\overline{v}^T$. It follows that

$$(A^*)^2 = (u\overline{v}^T)(u\overline{v}^T) = u(\overline{v}^T u)\overline{v}^T = (\overline{v}^T u)u\overline{v}^T = \operatorname{tr}(A^*)A^*.$$

Case 2. If $\operatorname{rank}(A) \leq n-2$, then $\operatorname{rank}(A^*) = 0$. It follows that $A^* = 0$ and $(A^*)^2 = \operatorname{tr}(A^*)A^* = 0$ is trivially satisfied.

Editor's comments. Oliver Geupel made us aware of the following: the adjoint of a matrix may refer to its conjugate transpose. If A^* is the conjugate transpose of A, then the desired result does not generally hold. A counterexample is the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $\operatorname{tr}(A^*) = 1$, $\det A = 0$, and

$$(A^*)^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \operatorname{tr}(A^*)A^*.$$

OC430.

(a) Prove that for any choice of n rational numbers a_i/b_i from the interval (0,1) with distinct pairs (a_i, b_i) of positive integers, the sum of the denominators is at least $\frac{2\sqrt{2}}{3} \cdot n^{\frac{3}{2}}$.

(b) Prove that if we add the restriction that the rational numbers are distinct, then the sum of the denominators is at least $2 \cdot \left(\frac{2}{3}n\right)^{\frac{3}{2}}$.

Originally Hungary MO, 3rd Problem, 2nd Category, Final Round 2017.

We received 1 correct submission for part (a) and none for part (b). We present the solution for part (a) by Oliver Geupel.

(a) Let m be the unique positive integer with the property that

$$1+2+\cdots+m \leq n < 1+2+\cdots+m+(m+1).$$

Then n has the form $n = \frac{1}{2}m(m+1)+\ell$ for some natural number $\ell \leq m$. In every choice of n rational numbers from $(0,1)$ with distinct pairs of numerator and denominator, there are at most k occurrences of the denominator $k+1$ for any $k \geq 1$. Hence, the sum S of denominators of the choice is at least

$$\begin{aligned}\sum_{k=1}^{m} k(k+1)+\ell(m+2) &= \sum_{k=1}^{m}(k^2+k)+\ell(m+2) \\ &= \frac{1}{6}m(m+1)(2m+1)+\frac{1}{2}m(m+1)+\ell(m+2) \\ &= \frac{1}{3}(m+2)(m(m+1)+3\ell).\end{aligned}$$

Thus,

$$\frac{(3S)^2}{(2n)^3} \geq \left(\frac{m(m+1)+3\ell}{m(m+1)+2\ell}\right)^2 \cdot \frac{(m+2)^2}{m(m+1)+2m} > 1.$$

Consequently, $3S > (2n)^{3/2}$, and the result is immediate.

FOCUS ON...

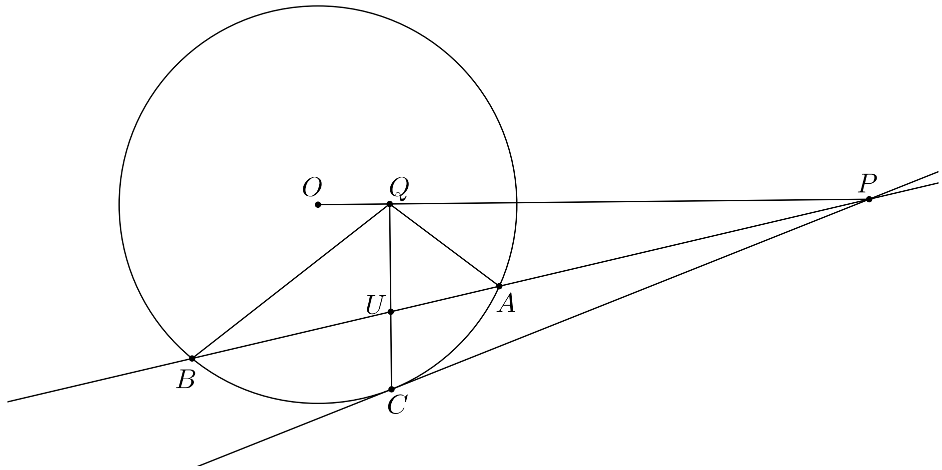
No. 38

Michel Bataille

Solutions to Exercises from Focus On... No. 32 – 36

From Focus On... No. 32

1. *Through a point P exterior to a given circle pass a secant and a tangent to the circle. The secant intersects the circle at A and B, and the tangent touches the circle at C on the same side of the diameter through P as A and B. The projection of C onto the diameter is Q. Prove that QC bisects $\angle AQB$.*



We observe that the line CQ is the polar of P with respect to the circle. It follows that if AB intersects CQ at U, then P, U divide A, B harmonically. As a result, the pencil of lines QP, QU, QA, QB is harmonic. Thus, QP and QU, which are perpendicular, are the bisectors of $\angle AQB$. In particular, QC bisects $\angle AQB$.

2. *The standard construction for bisecting a line segment involves the use of two arcs and one straight line. Show that it can, in fact, be done with straight lines and just one arc.*

Let AB be the given segment.

(1) Use the allowed arc to obtain the point C symmetrical of A about B;

(2) Join a point I (not on the line AB) to points A, B, C and obtain J such that IA, IC, IB, IJ is a harmonic pencil (see Figure 1). Note that IJ is parallel to AC (since IB is the median from I in ΔAIC).

(3) Draw the harmonic conjugate of IJ with respect to IA, IB. This line meets

AB at the midpoint O of the line segment AB (since IJ is parallel to AB). See Figure 2.

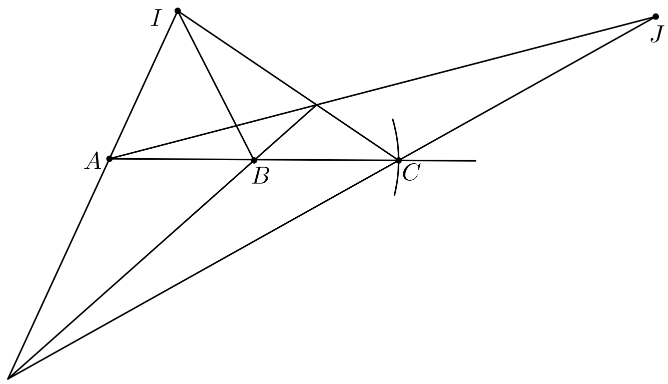


Figure 1

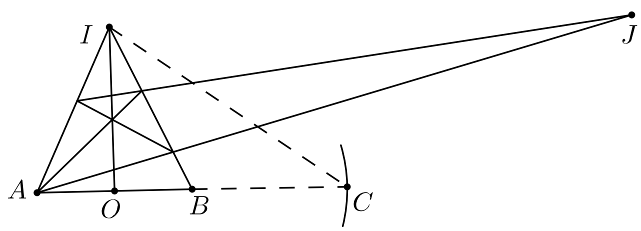


Figure 2

From Focus On... No. 34

1. *Evaluate* *a)* $\sin\dfrac{\pi}{14}\sin\dfrac{3\pi}{14}\sin\dfrac{5\pi}{14}$ *b)* $\tan\dfrac{6\pi}{7}+4\sin\dfrac{5\pi}{7}$.

a) Let p denote the given product. Then we have

$$p\cos\frac{\pi}{14}=\frac{1}{2}\sin\frac{\pi}{7}\sin\frac{3\pi}{14}\cos\frac{\pi}{7}=\frac{1}{4}\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}=\frac{1}{8}\sin\frac{4\pi}{7}=\frac{1}{8}\cos\frac{\pi}{14}$$

and $p=\frac{1}{8}$.

(Note that $\frac{5\pi}{14}=\frac{\pi}{2}-\frac{\pi}{7}$, $\frac{3\pi}{14}=\frac{\pi}{2}-\frac{2\pi}{7}$, $\frac{4\pi}{7}=\frac{\pi}{2}+\frac{\pi}{14}$.)

b) We show that $\tan\frac{6\pi}{7}+4\sin\frac{5\pi}{7}=\sqrt{7}$. Let $u=\exp(2\pi i/7)$. Then, we have

$$\begin{aligned}\tan\frac{6\pi}{7}=\frac{2i}{1+u^6}-i=i\cdot\frac{1+(u^6)^7}{1+u^6}-i&=i(-u^6+u^{12}-u^{18}+u^{24}-u^{30}+u^{36})\\&=i(-u^6+u^5-u^4+u^3-u^2+u)\end{aligned}$$

and

$$4\sin\frac{5\pi}{7} = 4\sin\frac{2\pi}{7} = -2i(u-u^6).$$

It follows that

$$\tan\frac{6\pi}{7} + 4\sin\frac{5\pi}{7} = i(-u-u^2+u^3-u^4+u^5+u^6) = i(-i\sqrt{7}) = \sqrt{7}$$

(by Gauss's Theorem since for $k\in\{1,2,3,4,5,6\}, \left(\frac{k}{7}\right) = 1$ if and only if $k\in\{1,2,4\}$).

2. *Prove that a)* $\cos\frac{7\pi}{15} = 4\sin\frac{2\pi}{15}\cos\frac{2\pi}{5}\cos\frac{13\pi}{30}$

b) $1+6\cos\frac{2\pi}{7} = 2\sqrt{7}\cos\left(\frac{1}{3}\cos^{-1}\frac{1}{2\sqrt{7}}\right)$

a) On the one hand,

$$\cos\left(\frac{7\pi}{15}\right) = \cos\left(\frac{2\pi}{3}-\frac{\pi}{5}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{5}\right)+\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right) \tag{1}$$

and on the other hand,

$$\begin{aligned}4\sin\left(\frac{2\pi}{15}\right)\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{13\pi}{30}\right) &= 2\sin\left(\frac{2\pi}{15}\right)\left(\cos\left(\frac{\pi}{30}\right)-\frac{\sqrt{3}}{2}\right)\\ &= -\sqrt{3}\sin\left(\frac{2\pi}{15}\right)+2\sin\left(\frac{2\pi}{15}\right)\cos\left(\frac{\pi}{30}\right)\\ &= -\sqrt{3}\sin\left(\frac{2\pi}{15}\right)+\frac{1}{2}+\sin\left(\frac{\pi}{10}\right)\\ &= \frac{1}{2}+\sin\left(\frac{\pi}{10}\right)-\frac{3}{2}\cos\left(\frac{\pi}{5}\right)+\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right).\end{aligned}$$

(note that $\frac{2\pi}{15} = \frac{\pi}{3}-\frac{\pi}{5}$).

Comparing with (1), we see that all boils down to proving that

$$\cos\left(\frac{\pi}{5}\right) = \frac{1}{2}+\sin\left(\frac{\pi}{10}\right) \quad \text{or} \quad 1-2\sin^2\left(\frac{\pi}{10}\right) = \frac{1}{2}+\sin\left(\frac{\pi}{10}\right).$$

But

$$\sin\left(\frac{\pi}{10}\right) = \cos\left(\frac{\pi}{2}-\frac{\pi}{10}\right) = \cos\left(\frac{2\pi}{5}\right)$$

and the latter follows from $4\cos^2\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{2\pi}{5}\right) = 1$ (see Focus On... No 34 p. 28).

b) Let $\theta = \cos^{-1}\frac{1}{2\sqrt{7}}$. Then, $\theta\in(0,\frac{\pi}{2})$ and

$$4\cos^3\frac{\theta}{3}-3\cos\frac{\theta}{3} = \cos\theta = \frac{1}{2\sqrt{7}}. \tag{2}$$

The numbers $\cos\frac{2\pi}{7} = -\cos\frac{5\pi}{7}, \cos\frac{4\pi}{7} = -\cos\frac{3\pi}{7}, \cos\frac{6\pi}{7} = -\cos\frac{\pi}{7}$ are the roots of the polynomial $P(x) = 8x^3 + 4x^2 - 4x - 1$ (*ibid.* p. 29) and it follows that $\dfrac{1+6\cos\frac{2\pi}{7}}{2\sqrt{7}}, \dfrac{1+6\cos\frac{4\pi}{7}}{2\sqrt{7}}, \dfrac{1+6\cos\frac{6\pi}{7}}{2\sqrt{7}}$ are the roots of

$$Q(x) = P\left(\frac{2x\sqrt{7}-1}{6}\right) = \frac{14\sqrt{7}}{27}\left(4x^3 - 3x - \frac{1}{2\sqrt{7}}\right).$$

Recalling (2), we see that $Q\left(\cos\frac{\theta}{3}\right) = 0$.

In addition, since $0 < \frac{\theta}{3} < \theta < \frac{\pi}{2}$, we have $\cos\frac{\theta}{3} > \cos\theta$, hence $2\sqrt{7}\cos\frac{\theta}{3} - 1 > 0$. Thus $\dfrac{2\sqrt{7}\cos\frac{\theta}{3}-1}{6}$ is a positive root of $P(x)$ and since the only positive root of $P(x)$ is $\cos\frac{2\pi}{7}$, we must have $\dfrac{2\sqrt{7}\cos\frac{\theta}{3}-1}{6} = \cos\frac{2\pi}{7}$. The result follows.

From Focus On... No. 35

1. *Let* $X_n = \displaystyle\int_0^{\frac{\pi}{4}} (\tan x)^n\,dx$. *Compute* $X_n + X_{n+2}$ *and deduce that* $X_n \sim \frac{1}{2n}$.

We calculate

$$X_n + X_{n+2} = \int_0^{\frac{\pi}{4}} (\tan x)^n(1+\tan^2 x)\,dx = \left[\frac{(\tan x)^{n+1}}{n+1}\right]_0^{\pi/4} = \frac{1}{n+1}.$$

For $0 \le x \le \frac{\pi}{4}$ we have $0 \le \tan x \le 1$, hence $(\tan x)^{n+1} \le (\tan x)^n$ and therefore $X_{n+1} \le X_n$ for $n \in \mathbb{N}$. For $n > 2$, we deduce that

$$\frac{1}{n+1} = X_{n+2} + X_n \le 2X_n \le X_n + X_{n-2} = \frac{1}{n-1}$$

so that X_n satisfies $\frac{n}{n+1} \le 2nX_n \le \frac{n}{n-1}$. The squeeze principle now gives $\lim\limits_{n\to\infty} 2nX_n = 1$, as desired.

2. *Let* $Y_n = \displaystyle\int_0^1 \frac{x^n \ln x}{x^n - 1}\,dx$. *Show that* $Y_n \sim \frac{\alpha}{n^2}$ *for some positive* α.

We observe that

$$Y_n = \frac{1}{n}\int_0^1 x^n \cdot \frac{\ln(x^n)}{x^n-1}\,dx = \frac{1}{n^2}\left(n \cdot \int_0^1 x^n \cdot f(x^n)\,dx\right)$$

where f is the continuous function defined on $[0,1]$ by $f(x) = \frac{\ln x}{x-1}$ if $x \neq 1$, $f(1) = 1$. Theorem 1(i) in Focus On... No 36 p. 138 then gives

$$\lim_{n\to\infty} n \cdot \int_0^1 x^n \cdot f(x^n)\,dx = \int_0^1 f(x)\,dx = \int_0^1 \frac{\ln x}{x-1}\,dx = \int_0^1 \frac{-\ln(1-u)}{u}\,du = \frac{\pi^2}{6}$$

and finally, $Y_n \sim \frac{\alpha}{n^2}$ with $\alpha = \frac{\pi^2}{6}$.

3. *Let* $f_1(t) = \dfrac{1}{1+t}$ *and for* $k \geq 2$, *let* $f_k(t) = \dfrac{d}{dt}(tf_{k-1}(t))$. *Prove that for any integer* $m \geq 1$,

$$\int_0^1 \frac{t^n}{1+t}\,dt = \frac{f_1(1)}{n} - \frac{f_2(1)}{n^2} + \cdots + (-1)^{m-1}\frac{f_m(1)}{n^m} + o(1/n^m)$$

as $n \to \infty$.

Integrating by parts, we obtain the recursion

$$\int_0^1 t^n f_{k-1}(t)\,dt = \frac{f_{k-1}(1)}{n} - \frac{1}{n}\int_0^1 t^n f_k(t)\,dt$$

from which we readily deduce that for $m \geq 1$,

$$\int_0^1 \frac{t^n}{1+t}\,dt = \frac{f_1(1)}{n} - \frac{f_2(1)}{n^2} + \cdots + (-1)^{m-1}\frac{f_m(1)}{n^m} + \frac{(-1)^m}{n^m}\int_0^1 t^n f_{m+1}(t)\,dt.$$

Since $\lim_{n\to\infty} n \cdot \int_0^1 t^n f_{m+1}(t)\,dt = f_{m+1}(1)$ (Theorem 1(ii), *ibid.*), we have

$$\frac{(-1)^m}{n^m}\int_0^1 t^n f_{m+1}(t)\,dt = o\left(\frac{1}{n^m}\right)$$

and the result follows.

4. *Let* $Z_n = \displaystyle\int_0^1 (ax^2+bx+c)^n\,dx$ *where* a, b *are negative numbers and* $a+b+c$ *is positive. Use this number of Focus On... to obtain* $Z_n \sim \frac{c^{n+1}}{-nb}$ *already found in No 10.*

Let ϕ be the function defined on $[0,1]$ by $\phi(x) = ax^2+bx+c$. Since $a, b < 0$, ϕ is strictly decreasing on $[0,1]$ with $\phi(0) = c, \phi(1) = a+b+c$. Thus, ϕ is a bijection from $[0,1]$ onto $[a+b+c, c]$. It is easy to obtain that for $y \in [a+b+c, c]$, we have

$$\phi^{-1}(y) = \frac{-b-\sqrt{b^2-4ac+4ay}}{2a} \quad \text{and} \quad (\phi^{-1})'(y) = \frac{-1}{\sqrt{b^2-4ac+4ay}}.$$

As a result, the change of variables $y = ax^2+bx+c = \phi(x)$ leads to

$$Z_n = \int_{a+b+c}^{c} y^n \cdot \frac{1}{\sqrt{b^2-4ac+4ay}}\,dy.$$

Now, we remark that $0 < a+b+c < c$ and that the function $y \mapsto \frac{1}{\sqrt{b^2-4ac+4ay}}$ is continuous on $[0, c]$. Thus, we can transform Z_n as follows:

$$\begin{aligned} Z_n &= \int_0^c y^n \cdot \frac{1}{\sqrt{b^2 - 4ac + 4ay}}\, dy - \int_0^{a+b+c} y^n \cdot \frac{1}{\sqrt{b^2 - 4ac + 4ay}}\, dy \\ &= c^{n+1} \int_0^1 u^n \cdot \frac{1}{\sqrt{b^2 - 4ac + 4acu}}\, du \\ &\quad - (a+b+c)^{n+1} \int_0^1 u^n \cdot \frac{1}{\sqrt{b^2 - 4ac + 4a(a+b+c)u}}\, du. \end{aligned}$$

Using Theorem 1(ii) again, we see that

$$\int_0^1 u^n \cdot \frac{1}{\sqrt{b^2 - 4ac + 4acu}}\, du \ \sim\ \frac{1}{-nb}$$

and

$$\int_0^1 u^n \cdot \frac{1}{\sqrt{b^2 - 4ac + 4a(a+b+c)u}}\, du \ \sim\ \frac{1}{-n(2a+b)}$$

and because

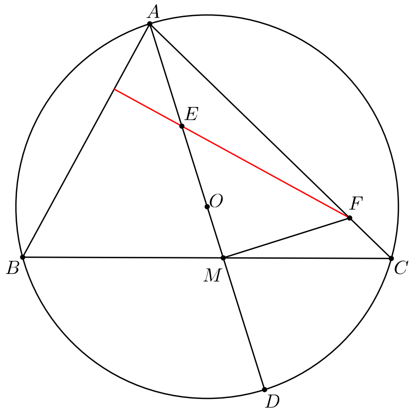
$$\frac{(a+b+c)^{n+1}}{n(2a+b)} = o\left(\frac{c^{n+1}}{nb}\right)$$

as $n \to \infty$, we can conclude that

$$Z_n \ \sim\ \frac{c^{n+1}}{-nb}.$$

From Focus On... No. 36

1. *Given an acute triangle ABC, let O be its circumcenter, let M be the intersection of lines AO and BC, and let D be the other intersection of AO with the circumcircle of ABC. Let E be that point on AD such that M is the midpoint of ED. Let F be the point at which the perpendicular to AD at M meets AC. Prove that EF is perpendicular to AB.*



(We use the conventions and notations set in the first part of Focus On... No. 36).

Clearly, $d = -a$ and since $d+e = 2m$, we have $e = 2m+a$. Since F is the point of intersection of the lines AC and the perpendicular to AO at M, f satisfies their respective equations $z + ac\overline{z} = a + c$ and $\dfrac{z-m}{a} + \dfrac{\overline{z}-\overline{m}}{\overline{a}} = 0$. We readily obtain

$$f = \frac{a^2(c\overline{m} - 1) + c(m-a)}{c-a}.$$

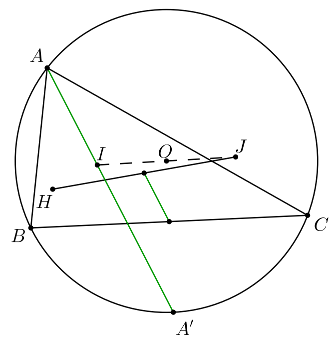
Now, lines EF and AB are perpendicular if and only if $f - e = ab(\overline{f} - \overline{e})$, that is,

$$a^2c\overline{m} + m(2a-c) - 2ac = 2ab - mb + ab(a-2c)\overline{m}.$$

Because M is on OA, m satisfies $\overline{a}m - a\overline{m} = 0$, that is, $m = a^2\overline{m}$. Using this expression of m, the above condition easily becomes $m + bc\overline{m} = b + c$. Since M is on BC, the latter holds and therefore EF and AB are perpendicular.

Note that the hypothesis "ABC acute" is not necessary.

2. *Let H, I, O be the orthocenter, incenter, circumcenter of a triangle ABC and let J be the reflection of I about O. Prove that the line through the midpoints of JH and BC is parallel to AI.*



(We use the conventions and notations set in the second part of Focus On... No 36).

Recalling that the affixes of A, B, C, I are $a^2, b^2, c^2, -(ab+bc+ca)$, we see that the affixes of J and H are $ab+bc+ca$ and $a^2+b^2+c^2$, respectively. Let m and n be the respective affixes of the midpoints of BC and JH. Since the bisector AI intersects the circumcircle at A and A' with affixes a^2 and $-bc$, we have to show that $n-m = a^2bc(\overline{n}-\overline{m})$. Because $2m = b^2+c^2$ and $2n = a^2+b^2+c^2+ab+bc+ca$, the result directly follows from $2n - 2m = a^2 + ab + bc + ca$ and

$$a^2bc(2\overline{n} - 2\overline{m}) = a^2bc\left(\frac{1}{a^2} + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) = bc + ca + a^2 + ab.$$

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **December 30, 2019**.

4481. *Proposed by Warut Suksompong.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y^2) = f(x + y)f(x - y) + 2f(y)y$$

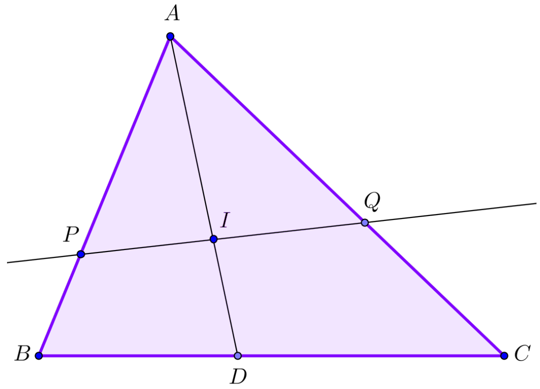
for all $x, y \in \mathbb{R}$.

4482. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be a triangle with incenter I. The line AI intersects BC at D. A line l passes through I and intersects the sides AB and AC at P and Q, respectively. Show that

$$AC \cdot \frac{[BDIP]}{[API]} + AB \cdot \frac{[CDIQ]}{[AQI]} = 2 \cdot BC + \frac{BC^2}{AB + AC},$$

where square brackets denote area.



4483. *Proposed by Paul Bracken.*

For non-negative integers m and n, evaluate the following sum in closed form

$$\sum_{j=0}^{m} j^2 \binom{j+n}{j}.$$

4484. *Proposed by Leonard Giugiuc and Michael Rozenberg.*

Let $a, b, c \in [0, 2]$ such that $a + b + c = 3$. Prove that

$$4(ab + bc + ac) \leq 12 - ((a - b)(b - c)(c - a))^2$$

and find when the equality holds.

4485. *Proposed by Jonathan Parker and Eugen J. Ionascu.*

For every square matrix with real entries $A = [a_{i,j}]_{i=1..n,j=1,2...n}$, we define the value

$$GM(A) = \max_{\pi \in S_n}\{\min\{a_{1\pi(1)}, a_{2\pi(2)}, ..., a_{n\pi(n)}\}\}$$

where S_n is the set of all permutations of the set $[n] := \{1, 2, 3, ..., n\}$.

Given the 6×6 matrix

$$A := \begin{bmatrix} 20 & 9 & 7 & 26 & 27 & 13 \\ 19 & 18 & 17 & 6 & 12 & 25 \\ 22 & 24 & 21 & 11 & 20 & 11 \\ 20 & 8 & 9 & 23 & 5 & 14 \\ 22 & 17 & 4 & 10 & 36 & 33 \\ 21 & 16 & 23 & 35 & 15 & 34 \end{bmatrix}$$

find the value $GM(A)$.

4486. *Proposed by Marian Cucoaneş and Marius Drăgan.*

Let $a, b > 0$, $c > 1$ such that $a^2 \geq b^2 c$. Compute

$$\lim_{n\to\infty}(a - b\sqrt{c})(a - b\sqrt[3]{c}) \cdots (a - b\sqrt[n]{c}).$$

4487. *Proposed by Martin Lukarevski.*

Let a, b, c be the sides of a triangle ABC, m_a, m_b, m_c the corresponding medians and R, r its circumradius and inradius respectively. Prove that

$$\frac{a^2}{m_b^2 + m_c^2} + \frac{b^2}{m_c^2 + m_a^2} + \frac{c^2}{m_a^2 + m_b^2} \geq \frac{4r}{R}.$$

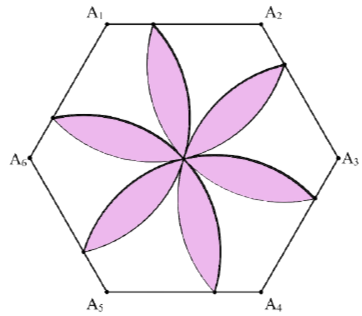
4488. *Proposed by George Apostolopoulos.*

Let ABC be an acute-angled triangle. Prove that

$$\sqrt{\cot A} + \sqrt{\cot B} + \sqrt{\cot C} \leq \sqrt{\cot\frac{A}{2} \cdot \cot\frac{B}{2} \cdot \cot\frac{C}{2}}.$$

4489. *Proposed by Arsalan Wares.*

Regular hexagon A has its vertices at points A_1, A_2, A_3, A_4, A_5 and A_6. Six circular congruent arcs are drawn inside hexagon A and all six pass through the center of A. The terminal points of each of the six arcs divide the sides of A in the ratio $3:7$. The six regions within A that are bounded only by circular arcs have been shaded. Find the ratio of the area of A to the area of the shaded region.



4490. *Proposed by Borislav Mirchev and Leonard Giugiuc.*

A line ℓ through the orthocenter H of the acute triangle ABC meets the circumcircle at points K on the smaller arc AC and L on the smaller arc BC. If M, N, and P are the feet of the perpendiculars to ℓ from the vertices A, B, and C, respectively, prove that $PH = |KM - LN|$.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 décembre 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4481. *Proposé par Warut Suksompong.*

Déterminer toute fonction $f : \mathbb{R} \to \mathbb{R}$ telle que

$$f(x^2 + y^2) = f(x + y)f(x - y) + 2f(y)y$$

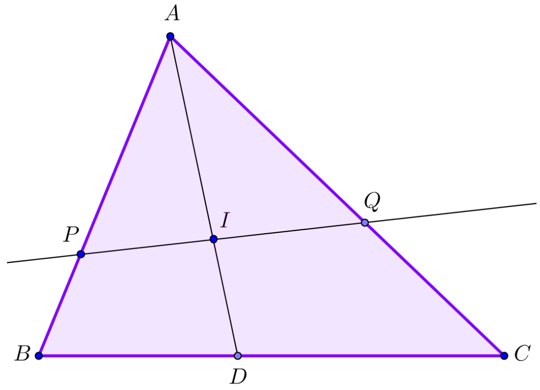
pour tous $x, y \in \mathbb{R}$.

4482. *Proposé par Miguel Ochoa Sanchez et Leonard Giugiuc.*

Soit ABC un triangle et I le centre de son cercle inscrit. La ligne AI intersecte BC en D. Une ligne l passe par I et coupe AB et AC en P et Q, respectivement. Démontrer que

$$AC \cdot \frac{[BDIP]}{[API]} + AB \cdot \frac{[CDIQ]}{[AQI]} = 2 \cdot BC + \frac{BC^2}{AB + AC},$$

où le crochet dénote la surface.



4483. *Proposé par Paul Bracken.*

Pour m et n des entiers non négatifs, évaluer en forme close la somme

$$\sum_{j=0}^{m} j^2 \binom{j+n}{j}.$$

4484. *Proposé par Leonard Giugiuc et Michael Rozenberg.*

Soient $a, b, c \in [0, 2]$ tels que $a + b + c = 3$. Démontrer que

$$4(ab + bc + ac) \leq 12 - ((a - b)(b - c)(c - a))^2$$

et déterminer toute condition donnant égalité.

4485. *Proposé par Jonathan Parker et Eugen J. Ionascu.*

Pour $A = [a_{i,j}]_{i=1..n, j=1,2...n}$ une matrice carrée avec éléments réels, on pose

$$GM(A) = \max_{\pi \in S_n} \{\min\{a_{1\pi(1)}, a_{2\pi(2)}, ..., a_{n\pi(n)}\}\}$$

où S_n est l'ensemble des permutations de l'ensemble $[n] := \{1, 2, 3, ..., n\}$. Pour A, matrice 6×6 donnée par

$$A := \begin{bmatrix} 20 & 9 & 7 & 26 & 27 & 13 \\ 19 & 18 & 17 & 6 & 12 & 25 \\ 22 & 24 & 21 & 11 & 20 & 11 \\ 20 & 8 & 9 & 23 & 5 & 14 \\ 22 & 17 & 4 & 10 & 36 & 33 \\ 21 & 16 & 23 & 35 & 15 & 34 \end{bmatrix},$$

déterminer $GM(A)$.

4486. *Proposé par Marian Cucoaneş et Marius Drăgan.*

Soient $a, b > 0$, $c > 1$, tels que $a^2 \geq b^2 c$. Calculer

$$\lim_{n \to \infty} (a - b\sqrt{c})(a - b\sqrt[3]{c}) \cdots (a - b\sqrt[n]{c}).$$

4487. *Proposé par Martin Lukarevski.*

Soient a, b, c les côtés du triangle ABC, m_a, m_b, m_c les médianes, puis R et r les rayons du cercle circonscrit et du cercle inscrit, respectivement. Démontrer que

$$\frac{a^2}{m_b^2 + m_c^2} + \frac{b^2}{m_c^2 + m_a^2} + \frac{c^2}{m_a^2 + m_b^2} \geq \frac{4r}{R}.$$

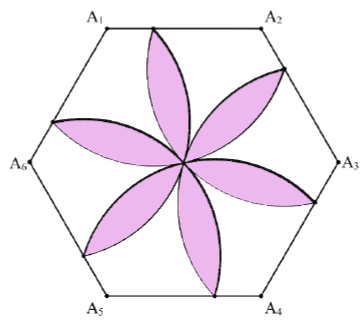
4488. *Proposé par George Apostolopoulos.*

Soit ABC un triangle acutangle. Démontrer que

$$\sqrt{\cot A} + \sqrt{\cot B} + \sqrt{\cot C} \leq \sqrt{\cot \frac{A}{2} \cdot \cot \frac{B}{2} \cdot \cot \frac{C}{2}}.$$

4489. *Proposé par Arsalan Wares.*

Un hexagone régulier A possède les sommets A_1, A_2, A_3, A_4, A_5 et A_6. Six arcs circulaires congrus sont tracés à l'intérieur de A, passant tous par le centre de A. Les points terminaux de ces six arcs divisent les côtés de A en ratio 3 : 7. Les 6 régions bornées par deux arcs circulaires voisins sont ombragées. Déterminer le ratio de cette surface par rapport à la surface ombragée.



4490. *Proposé par Borislav Mirchev et Leonard Giugiuc.*

Une ligne ℓ passant par l'orthocentre du triangle acutangle ABC intersecte le cercle circonscrit, au point K dans le petit arc AC et au point L dans le petit arc AC. Si M, N et P sont les pieds des perpendiculaires vers l à partir des sommets A, B et C, respectivement, démontrer que $PH = |KM - LN|$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2019: 45(4), p. 196–201.

4431. *Proposed by Şefket Arslanagić.*

Let $x, y \geq 0$ and $x + y = 2$. Prove that

$$\sqrt{x^2+8} + \sqrt{y^2+8} + \sqrt{xy+8} \geq 9.$$

We received 18 submissions, of which 17 were correct, and the other one incorrect. We present the solution by Kee-Wai Lau.

By the AM-GM inequality, we have

$$\sqrt{x^2+8} + \sqrt{y^2+8} + \sqrt{xy+8} \geq 3\sqrt[6]{(x^2+8)(y^2+8)(xy+8)},$$

so it suffices to show that

$$(x^2+8)(y^2+8)(xy+8) \geq 729. \qquad (1)$$

Since

$$(x^2+8)(y^2+8) = (xy)^2 + 8(x+y)^2 - 16xy + 64 = (xy)^2 - 16xy + 96,$$

we have

$$\begin{aligned}(x^2+8)(y^2+8)(xy+8) &= \left((xy)^2 - 16xy + 96\right)(xy+8)\\ &= (xy)^3 - 8(xy)^2 - 32xy + 768\\ &= \left((xy)^2 - 7xy - 39\right)(xy-1) + 729.\end{aligned}$$

Now,

$$0 \leq xy = \frac{(x+y)^2 - (x-y)^2}{4} = 1 - \frac{(x-y)^2}{4} \leq 1$$

gives $xy - 1 \leq 0$ and $(xy)^2 - 7xy - 39 < 0$, so that

$$\left((xy)^2 - 7xy - 39\right)(xy-1) + 729 \geq 729,$$

so (1) follows.

Editor's comments. Many submitted solutions used convexity and Jensen's Inequality involving tedious computations. The solution presented above is the only one using nothing more than elementary algebra.

4432. *Proposed by Marian Cucoaneş and Marius Drăgan.*

Let $x_1, x_2, \dots, x_k \geq 0$, $n \in \mathbb{N}$ be such that

$$(x_1 + x_2)(x_2 + x_3) \dots (x_k + x_1) \geq (x_1 + 1)(x_2 + 1) \dots (x_k + 1)$$

Prove that

$$\frac{x_1^{n+1}}{x_2+1} + \frac{x_2^{n+1}}{x_3+1} + \cdots + \frac{x_k^{n+1}}{x_1+1} \geq \frac{x_1^n}{x_2+1} + \frac{x_2^n}{x_3+1} + \cdots + \frac{x_k^n}{x_1+1}.$$

We received 3 correct solutions. One additional solution contained statements that needed justification. We present the solution of the proposers.

Setting $x_{k+1} = x_1$, using the arithmetic-geometric means inequality along with the hypothesis, we have that

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{x_i^{n+1}}{x_{i+1}+1} - \sum_{i=1}^{k} \frac{x_i^n}{x_{i+1}+1} \\
&= \sum_{i=1}^{k} \frac{(x_i - 1)(x_i^n - 1)}{x_{i+1}+1} + \sum_{i=1}^{k} \frac{x_i - 1}{x_{i+1}+1} \\
&= \left[\sum_{i=1}^{k} \frac{(x_i - 1)(x_i^n - 1)}{x_{i+1}+1} \right] + \left[\sum_{i=1}^{k} \frac{(x_i + x_{i+1}) - (x_{i+1}+1)}{x_{i+1}+1} \right] \\
&= \left[\sum_{i=1}^{k} \frac{(x_i - 1)(x_i^n - 1)}{x_{i+1}+1} \right] + \left[\sum_{i=1}^{k} \frac{x_i + x_{i+1}}{x_{i+1}+1} \right] - k \\
&\geq 0 + k \left[\prod_{i=1}^{k} \frac{x_i + x_{i+1}}{x_{i+1}+1} \right]^{1/k} - k \geq 0 + k - k = 0,
\end{aligned}$$

as desired.

Editor's comments. Variants of this solution came from Prithwijit De and Marian Dincă. Let

$$f(m) = \sum_{i=1}^{m} x_i^m (x_{i+1} + 1)^{-1}.$$

De used the Cauchy-Schwarz inequality to show that $f(m+1)f(m-1) \geq (f(m))^2$ for $m \geq 1$. From a proof of $f(1) \geq f(0)$, this allowed an induction argument to show that $f(m)$ increases with m. Dincă obtained $f(k+1) \geq f(k)$ from the weighted Chebyshev inequality

$$\left(\sum_{i=1}^{k} w_i \right) \left(\sum_{j=1}^{k} w_j a_j b_j \right) \geq \left(\sum_{i=1}^{k} w_i a_i \right) \left(\sum_{j=1}^{k} w_j b_j \right)$$

with $\{a_i\}$ and $\{b_i\}$ monotonic in the same direction. For $1 \leq i \leq k$, we take $w_i = (x_{i+1} + 1)^{-1}$, $a_i = x_i^n$ and $b_i = x_i$.

4433. *Proposed by Leonard Giugiuc.*

Let $ABCD$ be a tetrahedron and let M be an interior point of $ABCD$. If $AB \geq AC \geq AD \geq BC \geq BD \geq CD$, then prove that

$$MA + MB + MC + MD < AB + AC + AD.$$

We received only 1 correct submission, from the proposer, whose solution we present.

Let $\dfrac{V_{MBCD}}{V_{ABCD}} = x$, $\dfrac{V_{MACD}}{V_{ABCD}} = y$, $\dfrac{V_{MABD}}{V_{ABCD}} = z$, and $\dfrac{V_{MABC}}{V_{ABCD}} = t$.

Then x, y, z, $t > 0$ and $x + y + z + t = 1$. Further, for any point P from space we have

$$\overrightarrow{PM} = x\overrightarrow{PA} + y\overrightarrow{PB} + z\overrightarrow{PC} + t\overrightarrow{PD}.$$

Therefore,

$$\overrightarrow{AM} = y\overrightarrow{AB} + z\overrightarrow{AC} + t\overrightarrow{AD} \tag{1}$$

$$\overrightarrow{BM} = x\overrightarrow{BA} + z\overrightarrow{BC} + t\overrightarrow{BD} \tag{2}$$

$$\overrightarrow{CM} = x\overrightarrow{CA} + y\overrightarrow{CB} + t\overrightarrow{CD} \tag{3}$$

$$\overrightarrow{DM} = x\overrightarrow{DA} + y\overrightarrow{DB} + z\overrightarrow{DC} \tag{4}$$

We apply in (1) the triangle inequality. Because no two of $y\overrightarrow{AB}$, $z\overrightarrow{AC}$, $t\overrightarrow{AD}$ are collinear we obtain the strict inequalities

$$|\overrightarrow{AM}| = |y\overrightarrow{AB} + z\overrightarrow{AC} + t\overrightarrow{AD}| < y|\overrightarrow{AB}| + z|\overrightarrow{AC}| + t|\overrightarrow{AD}|,$$

and

$$MA < yAB + zAC + tAD.$$

Similarly, from relations (2), (3), and (4) we obtain

$$\begin{aligned} MB &< xAB + zBC + tBD, \\ MC &< xAC + yBC + tCD, \\ MD &< xAD + yBD + zCD. \end{aligned}$$

Adding the last four inequalities, we have

$$\begin{aligned} &MA + MB + MC + MD \\ &< x(AB + AC + AD) + y(BA + BC + BD) + z(CA + CB + CD) + t(DA + DB + DC) \\ &< x(AB + AC + AD) + y(AB + AC + AD) + z(AB + AC + AD) + t(AB + AC + AD) \\ &= (x + y + z + t)(AB + AC + AD) \\ &= AB + AC + AD, \end{aligned}$$

as required. Above we used the inequalities provided between the sides of the tetrahedron, $AB \geq AC \geq AD \geq BC \geq BD \geq CD$.

4434. *Proposed by Michel Bataille.*

Let ABC be a scalene triangle with $BC = a, CA = b, AB = c$. The internal bisectors of $\angle A, \angle B, \angle C$ meet the opposite sides at D, E, F and the perpendicular bisectors of AD, BE, CF intersect the lines BC, CA, AB at U, V, W, respectively. Prove that the sum

$$(b^2 - c^2)PU^2 + (c^2 - a^2)PV^2 + (a^2 - b^2)PW^2$$

is independent of the point P chosen in the plane of ABC and find its value in terms of a, b, c.

We received 5 submissions, all of which were correct, and present the solution by Leonard Giugiuc.

Let the external bisector of the angle at A of ΔABC intersect BC at A', and let M be the midpoint of AD. Because the angle bisectors at a vertex divide the opposite side externally and internally in the ratio $c : b$, we have

$$A' = \frac{bB - cC}{b - c} \quad \text{and} \quad D = \frac{bB + cC}{b + c}.$$

Since both AA' and MU are perpendicular to AD, M the midpoint of AD implies that U must be the midpoint of $A'D$; it follows that

$$U = \frac{A' + D}{2} = \frac{b^2 B - c^2 C}{b^2 - c^2},$$

and for any point P in the plane, therefore,

$$\overrightarrow{PU} = \frac{b^2\overrightarrow{PB} - c^2\overrightarrow{PC}}{b^2 - c^2}.$$

Applying the cosine law to ΔPBC, we find that

$$2\overrightarrow{PB} \cdot \overrightarrow{PC} = 2|\overrightarrow{PB}||\overrightarrow{PC}| \cos \angle BPC = PB^2 + PC^2 - a^2,$$

and we deduce that

$$\begin{aligned}
(b^2 - c^2)PU^2 &= (b^2 - c^2)\left(\frac{b^2\overrightarrow{PB} - c^2\overrightarrow{PC}}{b^2 - c^2}\right)^2 \\
&= \frac{b^4 PB^2 + c^4 PC^2 - b^2c^2(PB^2 + PC^2 - a^2)}{b^2 - c^2} \\
&= b^2 PB^2 - c^2 PC^2 + \frac{a^2b^2c^2}{b^2 - c^2}.
\end{aligned}$$

Analogously,

$$(c^2 - a^2)PV^2 = c^2 PC^2 - a^2 PA^2 + \frac{a^2b^2c^2}{c^2 - a^2},$$

$$(a^2 - b^2)PW^2 = a^2 PA^2 - b^2 PB^2 + \frac{a^2b^2c^2}{a^2 - b^2}.$$

Adding the three equations, we find that the sum we seek equals

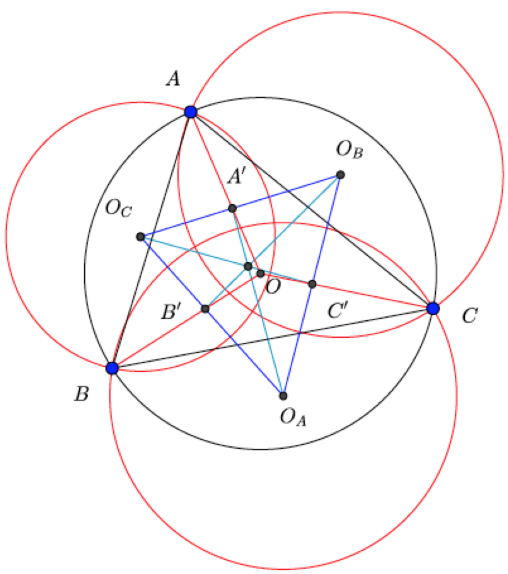
$$a^2b^2c^2\left(\frac{1}{a^2-b^2}+\frac{1}{b^2-c^2}+\frac{1}{c^2-a^2}\right),$$

which is independent of the choice of P, as claimed.

Editor's comment. The proposer added, in passing, that U, V, W are collinear by Menelaus's theorem.

4435. *Proposed by Dao Thanh Oai.*

Let ABC be a triangle with circumcenter O. Let C_{BC}, C_{CA}, C_{AB} be three arbitrary circles through B and C, C and A, A and B with centers O_A, O_B and O_C, respectively. Suppose OA, OB and OC meet O_BO_C, O_CO_A and O_AO_B at A', B' and C', respectively. Show that O_AA', O_BB' and O_CC' are concurrent.



We received 8 solutions. We present the solution by Prithwijit De.

Observe that OO_A, OO_B and OO_C are the perpendicular bisectors of BC, CA and AB respectively. Since O is the circumcenter of ABC, it follows that

$$\angle A'OO_C = \angle AOO_C = \frac{1}{2}\angle AOB = \angle ACB = \angle C,$$

$$\angle A'OO_B = \angle AOO_B = \frac{1}{2}\angle AOC = \angle ABC = \angle B.$$

Therefore,

$$\frac{O_BA'}{A'O_C} = \frac{[O_BOA']}{[O_COA']} = \frac{O_BO\cdot\sin\angle A'OO_B}{O_CO\cdot\sin\angle A'OO_C} = \left(\frac{O_BO}{O_CO}\right)\cdot\left(\frac{\sin\angle B}{\sin\angle C}\right).$$

Similarly, we show that

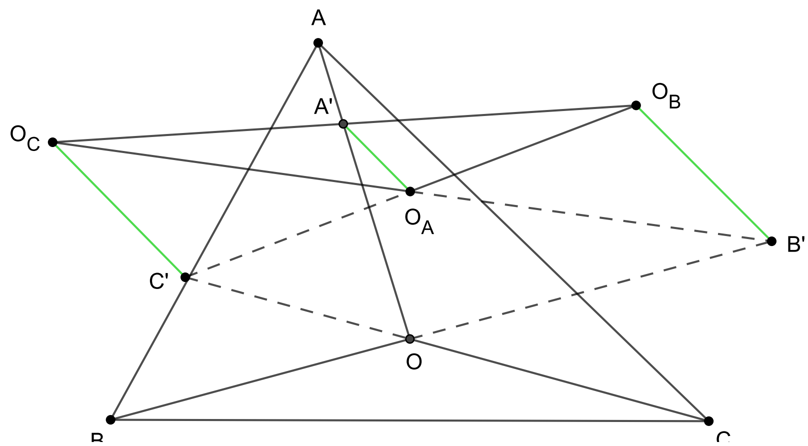
$$\frac{O_C B'}{B' O_A} = \left(\frac{O_C O}{O_A O}\right) \cdot \left(\frac{\sin \angle C}{\sin \angle A}\right),$$
$$\frac{O_A C'}{C' O_B} = \left(\frac{O_A O}{O_B O}\right) \cdot \left(\frac{\sin \angle A}{\sin \angle B}\right).$$

Thus we obtain

$$\frac{O_B A'}{A' O_C} \cdot \frac{O_C B'}{B' O_A} \cdot \frac{O_A C'}{C' O_B} = 1$$

and hence, by the converse of Ceva's theorem, $O_A A'$, $O_B B'$ and $O_C C'$ are either concurrent or parallel.

Editor's comment. As pointed out by Michel Bataille, it is possible to choose O_A, O_B and O_C such that the lines $O_A A'$, $O_B B'$ and $O_C C'$ are parallel instead of concurrent. An example is given below:



4436. *Proposed by Leonard Giugiuc.*

Find all positive solutions to the following equation:

$$(x+y+z)^2(10x+4y+z) = \frac{243xyz}{2}.$$

We received 8 solutions, all correct. We present the solution by Şefket Arslanagić.

Note first that $(x, y, z) = (2, 3, 4)$ is a solution since

$$(x+y+z)^2(10x+4y+z) = (81)(36) = (243)(12) = \frac{243xyz}{2}.$$

We show that all solutions are given by $k(2, 3, 4)$ for some $k > 0$.

Let $x = 2u$, $y = 3v$, and $z = 4w$. Then $u > 0$, $v > 0$, $w > 0$. By AM-GM inequality, we have

$$x + y + z = 2u + 3v + 4w = u + u + v + v + v + w + w + w + w \geq 9\sqrt[9]{u^2v^3w^4},$$

so

$$(x + y + z)^2 \geq 81\sqrt[9]{u^4v^6w^8}. \tag{1}$$

Similarly,

$$10x + 4y + z = 4(5u + 3v + w) \geq (4)(9)\sqrt[9]{u^5v^3w}. \tag{2}$$

Multiplying (1) and (2), we then obtain

$$\begin{aligned}(x + y + z)^2(10x + 4y + z) &\geq (81)(36)uvw \\ &= (3^6)(2^2)uvw \\ &= (3^6)(2^2)\left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\left(\frac{z}{4}\right) \\ &= \frac{243}{2}xyz.\end{aligned}$$

Since

$$(x + y + z)^2(10x + 4y + z) = \frac{243}{2}xyz,$$

we conclude that $u = v = w = k$ for some $k > 0$, so $(x, y, z) = k(2, 3, 4)$, completing the proof.

4437. *Proposed by Leonard Giugiuc and Shafiqur Rahman.*

Let a be a real number with $a > 1$. Find $\lim_{n\to\infty}(n\sqrt[n]{an} - n - \ln n)$.

We received 10 correct solutions. An additional solution involved a step that seemed as difficult as the problem posed. We present two solutions.

Solution 1, by Michel Bataille.

The result holds for all positive values of a. Since

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2) \quad \text{as} \quad x \to 0,$$

it follows that

$$\begin{aligned}n\sqrt[n]{an} &= ne^{\ln n/n}e^{\ln a/n} \\ &= n\left(1 + \frac{\ln n}{n} + \frac{\ln^2 n}{2n^2} + o\left(\frac{\ln^2 n}{n^2}\right)\right)\left(1 + \frac{\ln a}{n} + \frac{\ln^2 a}{2n^2} + o\left(\frac{1}{n^2}\right)\right) \\ &= n + \ln n + \ln a + o(1)\end{aligned}$$

as $n \to \infty$. Hence

$$\lim_{n\to\infty}(n\sqrt[n]{an} - n - \ln n) = \ln a.$$

Solution 2, by Henry Ricardo.

Note that $\lim_{x\to\infty} x^{1/x} = 1$, $\lim_{x\to 0}(a^x - 1)/x = \ln a$. Then

$$\begin{aligned} n\sqrt[n]{an} - n - \ln n &= n^{1/n} \cdot n(\sqrt[n]{a} - 1) + n(\sqrt[n]{n} - 1) - \ln n \\ &= n^{1/n}\left[\frac{a^{1/n}-1}{1/n}\right] + n(e^{\ln n/n} - 1) - \ln n \\ &= n^{1/n}\left[\frac{a^{1/n}-1}{1/n}\right] + \left(\ln n + \frac{\ln^2 n}{2n} + o\left(\frac{\ln^2 n}{n}\right)\right) - \ln n. \end{aligned}$$

Letting $n \to \infty$ yields the required limit, $\ln a$.

Editor's comments. Florentin Visescu gave a variant of the second solution in which he rendered the limit expression as

$$n^{1/n}\left[\frac{a^{1/n}-1}{1/n}\right] + \left(\frac{\ln^2 n}{n}\right)\left[\frac{e^{\ln n/n} - 1 - \frac{\ln n}{n}}{\left(\frac{\ln n}{n}\right)^2}\right],$$

and used the fact that $\lim_{x\to 0}(e^x - 1 - x)x^{-2} = \frac{1}{2}$.

4438. *Proposed by Michel Bataille.*

Let a, b, c be nonzero distinct real numbers and let A be the matrix

$$\begin{pmatrix} 2(a^2-bc) & b^2-a^2 & (c+b)(c-a) \\ (a+c)(a-b) & 2(b^2-ca) & c^2-b^2 \\ a^2-c^2 & (b+a)(b-c) & 2(c^2-ab) \end{pmatrix}.$$

Prove that A is invertible.

The statement as written is not true.

We received 8 submissions. Of those, 2 were incorrect proving the result as stated, 3 provided a counterexample to the claim, 2 found an additional condition that ensures the conclusion holds. We present the solution by Marie-Nicole Gras.

We will prove that A is invertible if a, b and c are of the same sign and not necessarily invertible otherwise.

Let a, b, c be nonzero distinct real numbers; the matrix A is invertible if and only if its determinant Δ is nonzero. We have

$$\Delta = \begin{vmatrix} 2(a^2-bc) & b^2-a^2 & (c+b)(c-a) \\ (a+c)(a-b) & 2(b^2-ca) & c^2-b^2 \\ a^2-c^2 & (b+a)(b-c) & 2(c^2-ab) \end{vmatrix}.$$

We first add the last two columns to the first one, and we obtain

$$\begin{aligned} 2(a^2-bc) + (b^2-a^2) + (c+b)(c-a) &= a^2+b^2+c^2-ab-bc-ca \\ (a+c)(a-b) + 2(b^2-ca) + c^2-b^2 &= a^2+b^2+c^2-ab-bc-ca \\ a^2-c^2 + (b+a)(b-c) + 2(c^2-ab) &= a^2+b^2+c^2-ab-bc-ca\,. \end{aligned}$$

Then $\Delta = (a^2+b^2+c^2-ab-bc-ca)D$, with

$$D = \begin{vmatrix} 1 & b^2-a^2 & (c+b)(c-a) \\ 1 & 2(b^2-ca) & c^2-b^2 \\ 1 & (b+a)(b-c) & 2(c^2-ab) \end{vmatrix} = \begin{vmatrix} 0 & -b^2-a^2+2ac & b^2+bc-ac-ab \\ 1 & 2b^2-2ac & c^2-b^2 \\ 0 & -b^2+ab+ac-bc & c^2+b^2-2ab \end{vmatrix}$$

and we deduce

$$\begin{aligned} D &= (b^2+a^2-2ac)(b^2+c^2-2ab)-(b^2+bc-ab-ac)^2 \\ &= (b^2+a^2)(b^2+c^2)-(b^2+b^2c^2+a^2b^2+a^2c^2)-2ac(b^2+c^2) \\ &\quad -2ab(b^2+a^2)+4a^2bc-2b^2(bc-ab-ac)-2(-ab^2c-abc^2+a^2bc) \\ &= -2(a^3b+b^3c+c^3a)+2abc(a+b+c). \end{aligned}$$

Then we obtain

$$\Delta = -\Big((a-b)^2+(b-c)^2+(c-a)^2\Big)\Big((a^3b+b^3c+c^3a)-abc(a+b+c)\Big).$$

We put $\delta = a^3b+b^3c+c^3a-abc(a+b+c)$ and we compute

$$\begin{aligned} (a+b+c)\,\delta &= (a+b+c)(a^3b+b^3c+c^3a)-abc(a+b+c)^2 \\ &= a^4b+ab^3c+a^2c^3+a^3b^2+b^4c+abc^3+a^3bc+b^3c^2+ac^4 \\ &\qquad -a^3bc-ab^3c-abc^3-2abc(a+b+c) \\ &= a^4b+b^4c+c^4a+a^3b^2+b^2c^3+c^3a^2-2a^2bc-2ab^2c-2abc^2 \\ &= b(a^2-bc)^2+c(b^2-ac)^2+a(c^2-ab)^2. \end{aligned}$$

If $a>0$, $b>0$, $c>0$ or $a<0$, $b<0$, $c<0$, then $a+b+c\neq 0$; whence, from the above identity,

$$\delta = 0 \iff a^2=bc,\ b^2=ac,\ c^2=ab \iff a=b=c, \text{ since } a,b,c\in\mathbb{R}\,.$$

However, by assumption, a, b, c are distinct; then $\Delta\neq 0$, and the matrix A is invertible.

If a, b and c are not of the same sign, it is possible that $\Delta = 0$, with distinct a,b,c; for example, if $a=1$, $b=2$ and c is the real root of x^3-2x^2+2x+2, $c\approx -0.5747430$, then $\Delta = 0$.

4439★. *Proposed by Cristi Savescu.*

Let M be a set of at least six points in the plane such that any of its six point subsets represents, in a certain order, the vertices of two triangles with the same centroid. Prove that M has exactly six points.

We received one solution by Roy Barbara, which is presented below, modified by the editor.

It suffices to show that M cannot have seven points. Suppose to the contrary that M is a set of seven points $A_0, A_1, \ldots, A_6$ in the plane. Since the property in the question is invariant under translations and rotations of the plane, we may assume without loss of generality that A_0 is the origin O and that all the x-coordinates of the points are distinct.

The centroid of a triangle ABC can be calculated as $\frac{1}{3}(A+B+C)$. Therefore the hypothesis can be formulated as seven equations

$$\varepsilon_{k0}A_0 + \varepsilon_{k1}A_1 + \cdots + \varepsilon_{k6}A_6 = O$$

with $k \in \{0, 1, \ldots, 6\}$, $\varepsilon_{kk} = 0$, and the property that in each equation three of the coefficients are $+1$ and three are -1. Setting a_i to be the first coordinate of A_i (note that $a_0 = 0$) we obtain that $x_i = a_i, i = 1, \ldots, 6$ is a solution to the system (S) of seven linear equations in six variables:

$$\varepsilon_{k1}x_1 + \cdots + \varepsilon_{k6}x_6 = 0$$

for $k \in \{0, 1, \ldots, 6\}$. We will show that (S) does not have a non-zero solution, contradicting the hypothesis.

Suppose (S) has a non-zero solution. Since all the coefficients in (S) are integers, we can express the solution set (after using Gauss-Jordan reduction) in terms of rational constants and some parameters (at least one). Assigning a rational value to each parameter then gives a rational solution to (S). We can multiply this solution by the least common multiple of its denominators to obtain an integer solution to (S). Finally, we divide this solution by the greatest common divisor of its values to obtain an integer solution $x_i = n_i$ to (S) such that $\gcd(n_1, \ldots, n_6) = 1$. We observe that the first equation of (S)

$$\varepsilon_{01}x_1 + \varepsilon_{02}x_2 + \cdots + \varepsilon_{06}x_6 = 0$$

implies that $\sum_{i=1}^{6} n_i$ is even, since all the coefficients are ± 1. The second equation implies that $\sum_{i=2}^{6} n_i$ is even and thus n_1 must also be even. Similarly, n_k is even for $k = 2, \ldots, 6$, contradicting that the greatest common divisor of the n_i is 1.

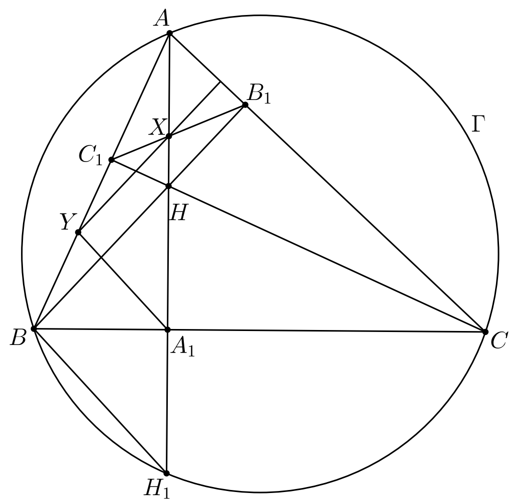
4440. *Proposed by Mihaela Berindeanu.*

Let H be the orthocenter of triangle ABC and let AA_1, BB_1, CC_1 be the altitudes; define the points X to be the intersection of AA_1 with B_1C_1 and Y to be where the perpendicular from X to AC intersects AB. Prove that the line YA_1 passes through the midpoint of BH.

We received 11 submissions, of which one was flawed. We feature two solutions.

Solution 1, by Michel Bataille.

Let H_1 be the reflection of H in BC; because $\angle H_1BC = \angle H_1AC = 90° - \angle C$, H_1 lies on the circumcircle Γ of ΔABC (see figure on the next page).



We first prove that $\dfrac{AX}{AH} = \dfrac{AA_1}{AH_1}$. To this aim, we consider the inversion $\mathbf{I}$ with centre A such that $\mathbf{I}(A_1) = H$. This inversion transforms the line BC into the circle with diameter AH. Observing that B_1 and C_1 lie on this circle (since $\angle AB_1H = \angle AC_1H = 90^\circ$), we see that $\mathbf{I}(B) = C_1$ and $\mathbf{I}(C) = B_1$ and, therefore, the line B_1C_1 inverts into the circle Γ. It follows that $\mathbf{I}(X) = H_1$, and so $AX \cdot AH_1 = AH \cdot AA_1$, as desired.

Now, the line YX being perpendicular to AC is parallel to BB_1, hence $\dfrac{AY}{AB} = \dfrac{AX}{AH}$. As a result, we have $\dfrac{AY}{AB} = \dfrac{AA_1}{AH_1}$, and so the lines YA_1 and BH_1 are parallel. Since A_1 is the midpoint of HH_1, the line A_1Y intersects HB in its midpoint.

Solution 2, a generalization by Chenxu Li.

The result is, in fact, a theorem of affine geometry:

> If P is an arbitrary point in the plane of ΔABC but not on any of the lines joining a pair of vertices, define A_1, B_1, C_1 to be the feet of the cevians AP, BP, CP (on the lines BC, CA, AB). As in the problem, define X to be the intersection of AA_1 with B_1C_1, and Y to be where the line parallel to BB_1 through X intersects AB. Then the line YA_1 passes through the midpoint of BP.

Proof. Let M be the intersection of YA_1 and BB_1; we are to prove that M is the midpoint of BP. Because A and P are diagonal points of the complete quadrangle B_1C_1BC, A_1 and X are harmonic conjugates with respect to A and P. (See, for example, [*Crux* **39**:4, "Harmonic Sets Part 2: Quadrangles and Quadrilaterals" (Apr. 2013), pages 174-177] or [**44**:7 (Sep. 2018) "Harmonic Ranges and Pencils", pages 291-296].)

The points A, A_1, P, X project from Y to the points B, M, P, ∞ of the line BB_1. Thus M and ∞ are harmonic conjugates with respect to B and P, which implies that M is the midpoint of BP, as desired. In the original problem, of course, P is taken to be the orthocenter H. (When $P = H$, both lines XY and BB_1 are perpendicular to AC, so they are parallel, as required by the generalization.)

Editor's comments. Zvonaru reports that our problem 4440 appeared as a contest problem in Hungary. He saw it on Andrei Eckstein's site:

`https://pregatirematematicaolimpiadejuniori.wordpress.com/problema-saptamanii`

(problem of the week nr. 100: July 9, 2018). The site provides five solutions (in Romanian), two of which come with English translations; all five are different from any of the solutions submitted to *Crux*. One is Zvonaru's own solution to which he added the same generalization as our solution 2 (featured above).

SNAPSHOT

The home of Michel Bataille is Rouen. Situated on river Seine and engulfed in greenery, Rouen is the capital of the northern French region of Normandy.