

1. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}^+$ be a function, and define $h : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}^+$ by $h(x, y) = \gcd(f(x), f(y))$. If $h(x, y)$ is a two-variable polynomial in x and y , prove that it must be constant.

Solution

Since $h > 0$, write $h(x, y) = P_n(y)x^n + \cdots + P_1(y)x + P_0(y)$ where $n \geq 0$, $\{P_0, P_1, \dots, P_n\}$ are polynomials, and $P_n \neq 0$. As P_n has finitely many roots, choose a $y_0 > 0$ with $P_n(y_0) \neq 0$. Then for every x , $h(x, y_0) = \gcd(f(x), f(y_0)) \mid f(y_0)$, hence $h(x, y_0) \leq f(y_0)$ for all x .

But now as $x \rightarrow \infty$, $h(x, y_0) \rightarrow P_n(y_0)x^n$. Since $P_n(y_0)$ is constant, and $h(x, y_0)$ is bounded (below by 0 and above by $f(y_0)$), this then implies that $n \geq 1$ is impossible. Hence $n = 0$ so $h(x, y) = P_0(y)$.

Thus the highest power of x that occurs is x^0 , i.e. $h(x, y)$ does not depend on x . But we can switch x and y to obtain the analogous result of $h(x, y)$ does not depend on y either, hence it is the constant polynomial.

2. Alphonse and Beryl play a game involving n safes. Each safe can be opened by a unique key and each key opens a unique safe. Beryl randomly shuffles the n keys, and after placing one key inside each safe, she locks all of the safes with her master key. Alphonse then selects m of the safes (where $m < n$), and Beryl uses her master key to open just the safes that Alphonse selected. Alphonse collects all of the keys inside these m safes and tries to use these keys to open up the other $n - m$ safes. If he can open a safe with one of the m keys, he can then use the key in that safe to try to open any of the remaining safes, repeating the process until Alphonse successfully opens all of the safes, or cannot open any more. Let $P_m(n)$ be the probability that Alphonse can eventually open all n safes starting from his initial selection of m keys.

- (a) Show that $P_2(3) = \frac{2}{3}$.
- (b) Prove that $P_1(n) = \frac{1}{n}$.
- (c) For all integers $n \geq 2$, prove that

$$P_2(n) = \frac{2}{n} \cdot P_1(n - 1) + \frac{n - 2}{n} \cdot P_2(n - 1).$$

- (d) Determine a formula for $P_2(n)$.

Solution

- (a) All 3 safes can be open if the key for the third safe is in one of the two opened safes. This occurs with probability $\frac{2}{3}$.
- (b) When we have a single key, we can open all the safes if and only if we can order the safes as s_1, s_2, \dots, s_n , where the key inside safe s_i opens safe s_{i+1} for each i . There are $(n - 1)!$ assignments of keys for which this can occur. There are a total of $n!$ ways the keys can be assigned to the safes, so $P_1(n) = \frac{(n-1)!}{n!} = \frac{1}{n}$.
- (c) Consider the key inside the first chosen safe. With probability $\frac{2}{n}$ it opens one of the 2 selected safes. In this case, there is a $P_1(n - 1)$ chance to open the remaining safes. With probability $\frac{n-2}{n}$ the key opens one of the other $n - 2$ safes. In this case, there is a $P_2(n - 1)$ chance to open the remaining safes. Thus $P_2(n) = \frac{2}{n}P_1(n - 1) + \frac{n-2}{n}P_2(n - 1)$.
- (d) $P_2(n) = \frac{2}{n}$, which easily follows by induction from part (c).

3. Let $1000 \leq n = ABCD_{10} \leq 9999$ be a positive integer whose digits ABCD satisfy the divisibility condition:

$$1111 \mid (ABCD + AB \times CD).$$

Determine the smallest possible value of n .

Solution

Define integers $n = ABCD$, $x = AB$ and $y = CD$. Then $n = 100x + y$, is the number that we want to find. The condition that n has four decimal digit positive integer means $1000 \leq n \leq 9999$, which is equivalent $10 \leq x \leq 99$ and $0 \leq y \leq 99$.

Adding 100 to the the condition (3) gives the equivalent condition

$$100 + n + xy = (x + 1)(y + 100) = 100 + 1111z \tag{1}$$

for some integer z .

If $x = 10$, then we would have $11 \mid 100 + 1111z$, which is impossible. Therefore $x \geq 11$ and $(x + 1) \geq 12$. Because $n > 0$ and $x, y \geq 0$, it follows that $1111z = n + xy > 0$, and thus $z > 0$.

Suppose that $z = 1$. In this case, $(x + 1)(y + 100) = 1211$, and it follows that $x + 1 = 1211/(y + 100) \leq 1211/100 = 12.11$. Since, $x + 1$ is integer, we have $(x + 1) \leq 12$ and from above, $(x + 1) \geq 12$, which combined say that $x + 1 = 12$. But 12 does not divide 1211, contradicting condition (1) and supposition $z = 1$. Therefore $z \geq 2$.

Suppose that $z = 2$. The right hand side of condition (1) becomes $100 + 1111z = 2322$. This is even. Dividing by two gives 1161. The sum of the digits of this is 9, so it is divisible by 3 giving 387. This is again divisible by 3, giving 129, which is a factor of 2322, namely $2322/(2 \times 3 \times 3)$. Because factor 129 is between 100 and 199, try it as a candidate $y + 100$. The other factor $2 \times 3 \times 3 = 18$ can serve as $x + 1$, and is in the right range. This gives a possible solution satisfying (1) with $x = 17$ and $y = 29$ and $n = 1729$.

Suppose henceforth that a smaller solution $n < 1729$ to condition (1) existed. Suppose that $x > 17$. Then $n > 1729$, a contradiction. Suppose $x = 17$. Then $z \geq 2$ gives $100 + 1111z \geq 2322$, which gives $y \geq 29$, and $n \geq 1729$, a contradiction. Therefore, $x < 17$.

Consequently, $(x + 1) \leq 17$ substituted into (1) gives $100 + 1111z = (x + 1)(y + 100) \leq 17 \times 199 = 3383$. It follows that $z \leq 3283/1111 < 3$. Therefore $z = 2$, since $z \geq 2$ was proved above.

Consequently, $100 + 1111z = 2322$, whose prime factorization, started above, completes to:

$$2322 = 2 \times 3 \times 3 \times 3 \times 43. \tag{2}$$

The number $x + 1$ belongs to the set $\{12, 13, 14, 15, 16, 17\}$ because the bounds $12 \leq (x + 1) \leq 17$ established above. Each number in this set is divisible either by 4 or by a prime strictly between 3 and 43, so it cannot divide 2322, a contradiction.

4. In $\triangle ABC$, the interior sides of which are mirrors, a laser is placed at point A_1 on side BC . A laser beam exits the point A_1 , hits side AC at point B_1 , and then reflects off the side. (Because this is a laser beam, every time it hits a side, the angle of incidence is equal to the angle of reflection). It then hits side AB at point C_1 , then side BC at point A_2 , then side AC again at point B_2 , then side AB again at point C_2 , then side BC again at point A_3 , and finally, side AC again at point B_3 .

(a) Prove that $\angle B_3A_3C = \angle B_1A_1C$.

(b) Prove that such a laser exists if and only if all the angles in $\triangle ABC$ are less than 90° .

Solution

(a) Let $a = \angle A, b = \angle B, c = \angle C, \alpha = \angle B_1A_1C$.

$$\angle AB_1C_1 = \alpha \tag{1}$$

$$\angle AC_1B_1 = \pi - a - \alpha \tag{2}$$

$$\angle BC_1A_2 = \pi - a - \alpha$$

$$\angle BA_2C_1 = a + \alpha - b \tag{3}$$

$$\angle CA_2B_2 = a + \alpha - b$$

$$\angle CB_2A_2 = \pi - c - a + b - \alpha \tag{4}$$

$$\angle AB_2C_3 = \pi - c - a + b - \alpha$$

$$\angle AC_3B_2 = c - b + \alpha \tag{5}$$

$$\angle BC_3A_3 = c - b + \alpha$$

$$\angle BA_3C_3 = \pi - c - \alpha \tag{6}$$

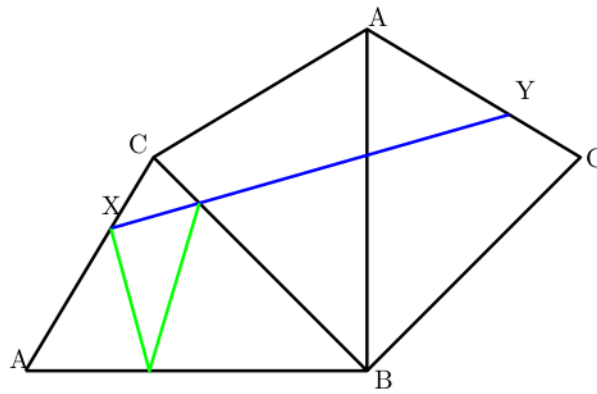
$$\angle CA_3B_3 = \pi - c - \alpha$$

$$\angle CB_3A_3 = \alpha$$

(b) We observe that if the angle of incidence of the laser with the side is greater than $\frac{\pi}{2}$ that the laser will bounce back to the side it came from. Thus, for such a laser to exist, we must have all of the angles on the right side of the above being less than $\frac{\pi}{2}$.

Taking lines (1) and (4), we have that $\pi - c - a + b < \pi$, which gives $b < c + a$. We similarly get $c < a + b$ from (2) and (5), and $a < b + c$ from (3) and (6). This tells us that all angles of $\triangle ABC$ are less than 90° .

To see that when all angles are less than 90° such a laser must exist, we take 3 copies of $\triangle ABC$ as shown in the figure below. We choose a point X on side AC and label Y , the same point in the last copy of the triangle. Then the line between these two points defines 3 line segments inside $\triangle ABC$ which will give us a laser.



5. Let $f(x) = x^4 + 2x^3 - x - 1$.

- (a) Prove that $f(x)$ cannot be written as the product of two non-constant polynomials with integer coefficients.
- (b) Find the exact values of the 4 roots of $f(x)$.

Solution

- (a) If $f(x)$ was reducible, then it factors either as a product of a linear and a cubic, or two quadratics. But for the linear and cubic case, this implies that it has a root, so we use the rational roots test and check that $f(\pm 1) \neq 0$ so this is impossible. Hence let $f(x) = (x^2 + ax + b)(x^2 + cx + d)$ where a, b, c, d are integers. Then:

$$\begin{aligned} x^4 + 2x^3 - x - 1 &= x^4 + (a + c)x^3 + (b + d + ac)x^2 + (ad + bc)x + bd \\ a + c &= 2; \\ b + d + ac &= 0; \quad ad + bc = -1; \quad bd = -1 \end{aligned}$$

From $bd = -1$ we get $b = 1, d = -1$ or $b = -1, d = 1$. In either case $b + d = 0$, hence $0 = b + d + ac = ac$ so $a = 0$ or $c = 0$. But $a + c = 2$, so $a = 0, c = 2$ or $a = 2, c = 0$. But then $2 \mid a, c$ so $2 \mid ad + bc = -1$, contradiction. Thus $f(x)$ is irreducible as claimed.

- (b) Note that $f(x - 1) = x^4 - 2x^3 + x - 1$, hence $f(x) = f(-x - 1)$, and so $f(x - 1/2) = f(-x - 1/2)$. This encourages us to look at $g(x) = f(x - 1/2) = x^4 - \frac{3}{2}x^2 - \frac{9}{16}$; this is

obviously a quadratic in x^2 ! The roots of $g(x)$ satisfy $x^2 = \frac{3/2 \pm \sqrt{(-3/2)^2 + 4(9/16)}}{2}$
 $= \frac{3 \pm 3\sqrt{2}}{4}$. Hence $g(x)$ has roots $\pm \sqrt{\frac{3 \pm 3\sqrt{2}}{4}}$, and using $0 = g(x) = f(x - 1/2)$ we see

that the 4 roots of $f(x)$ are $\frac{1}{2} \pm \sqrt{\frac{3 \pm 3\sqrt{2}}{4}}$.

6. Given a triangle A, B, C , X is on side AB , Y is on side AC , and P and Q are on side BC such that $AX = AY$, $BX = BP$ and $CY = CQ$. Let XP and YQ intersect at T . Prove that AT passes through the midpoint of PQ .

Solution

Let AT intersect BC at M . Let R, S be on sides AB, AC , respectively such that $MR \parallel XP$ and $MS \parallel YQ$. Then since $BX = BP$, $BR = BM$. Therefore, $XR = PT$. Similarly, $YS = QT$. We need to show that $PT = QT$. It suffices to show that $XR = YS$.

Since $RM \parallel XT$ and $SM \parallel YT$, $\frac{AR}{AX} = \frac{AM}{AT} = \frac{AS}{AY}$. Since $AX = AY$, $AR = AS$. Therefore, $RX = SY$, as desired.

7. A bug is standing at each of the vertices of a regular hexagon $ABCDEF$. At the same time each bug picks one of the vertices of the hexagon, which it is not currently in, and immediately starts moving towards that vertex. Each bug travels in a straight line from the vertex it was in originally to the vertex it picked. All bugs travel at the same speed and are of negligible size. Once a bug arrives at a vertex it picked, it stays there. In how many ways can the bugs move to the vertices so that no two bugs are ever in the same spot at the same time?

Solution

Label the vertices $0, 1, 2, 3, 4, 5$ in order around the hexagon and let b_i be the bug starting at vertex i . We have the following restrictions on the bugs:

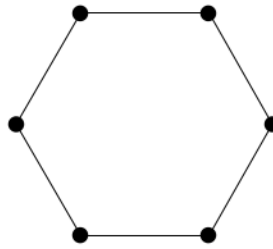
1. We cannot have two bugs travel to the same vertex, since they will be at the same point at the end.
2. If b_i goes to vertex $i + 3$ then for $j \neq i$ bug b_j does not go to vertex $j + 3$.
3. If bug b_i goes to vertex b_{i+2} then bug b_{i+1} cannot go to vertex b_{i-1} .
4. If bug b_i goes to vertex b_{i-2} then bug b_{i-3} cannot go to vertex b_{i-1} .

The first restriction tells us that the movement of the bugs must be a permutation of the vertices $0, 1, 2, 3, 4, 5$. The last 3 restrictions tell us which permutations will have bugs meeting at interior points of the hexagon and dictate which permutations are valid.

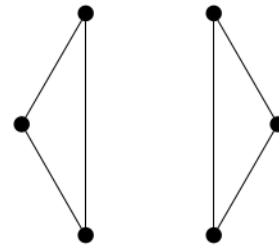
In the below figure, we show the permutations which will work.

- A There are 2 permutations that look like this. There are 2 choices for the direction of the cycle.
- B There are 12 permutations that look like this. There are 3 choices for which pair of 3-cycles are used, and 2 choices for the direction of each cycle.
- C There are 6 permutations that look like this. There are 3 rotations of the cycle, and 2 choices for the direction.
- D There are 2 permutations that look like this. There are 2 choices for the direction of one cycle, and the other cycle must have the same direction.
- E There are 12 permutations that look like this. There are 2 choices for the direction of the cycle and 6 rotations of the cycle.
- F There are 12 permutations that look like this. There are 2 choices for the direction of the cycle and 6 rotations of the cycle.
- G There are 12 permutations that look like this. There are 2 choices for the direction of the cycle and 6 rotations of the cycle.

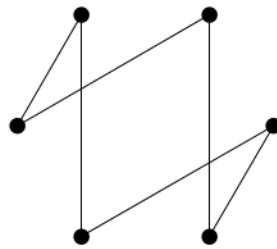
This gives a total of $2 + 12 + 6 + 2 + 12 + 12 + 12 = 56$ ways.



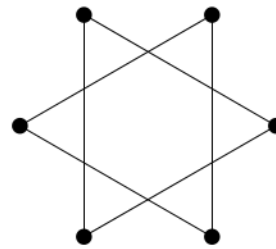
A



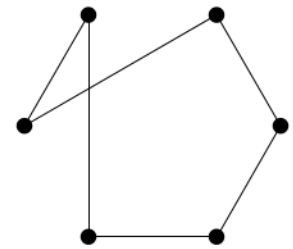
B



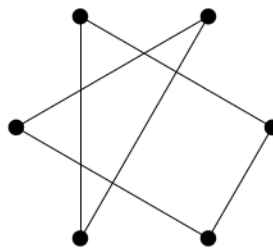
C



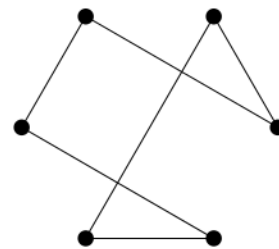
D



E



F



G

8. For any given non-negative integer m , let $f(m)$ be the number of 1's in the base 2 representation of m . Let n be a positive integer. Prove that the integer

$$\sum_{m=0}^{2^n-1} \left((-1)^{f(m)} \cdot 2^m \right)$$

contains at least $n!$ positive divisors.

Solution

Note that every integer from 0 to $2^n - 1$ (inclusively) can be written uniquely as the sum of the elements of a subset of $\{2^0, 2^1, \dots, 2^{n-1}\}$. By the definition of $f(m)$, the summation in the problem statement is equal to

$$(-1)^n \cdot (2^{2^0} - 1)(2^{2^1} - 1)(2^{2^2} - 1) \cdots (2^{2^{n-1}} - 1).$$

Note that for each $k \in \{0, 1, 2, \dots, n-1\}$,

$$2^{2^k} - 1 = (2^{2^1} - 1)(2^{2^2} + 1)(2^{2^3} + 1) \cdots (2^{2^{k-1}} + 1) = (2^{2^0} + 1)(2^{2^1} + 1)(2^{2^2} + 1) \cdots (2^{2^{k-1}} + 1).$$

Therefore,

$$\begin{aligned} & (-1)^n \cdot (2^{2^0} - 1)(2^{2^1} - 1)(2^{2^2} - 1) \cdots (2^{2^{n-1}} - 1) \\ &= (-1)^n (2^{2^0} + 1)^{n-1} (2^{2^1} + 1)^{n-2} \cdots (2^{2^{n-2}} + 1)^2 (2^{2^{n-1}} + 1). \end{aligned}$$

It suffices to show that this term has at least $n!$ positive divisors. To show this, it suffices to show that these $n-1$ factors are pairwise relatively prime. This is because then $(-1)^n (2^{2^0} + 1)^{n-1} (2^{2^1} + 1)^{n-2} \cdots (2^{2^{n-2}} + 1)^2 (2^{2^{n-1}} + 1)$ has a factor of the form $p_1^{n-1} p_2^{n-2} \cdots p_{n-1}^2 p_n$ where p_1, \dots, p_n are pairwise distinct primes. This factor has $n(n-1)(n-2) \cdots 3 \cdot 2 = n!$ positive divisors, which will imply that $(-1)^n (2^{2^0} + 1)^{n-1} (2^{2^1} + 1)^{n-2} \cdots (2^{2^{n-2}} + 1)^2 (2^{2^{n-1}} + 1)$ has $n!$ positive divisors. This will complete the problem.

It suffices to show that for all pairs of non-negative integers a, b satisfying $a \neq b$, we have $\gcd(2^{2^a} + 1, 2^{2^b} + 1) = 1$. Without loss of generality, suppose $a < b$. Note that $(2^{2^0} + 1)(2^{2^1} + 1) \cdots (2^{2^{b-1}} + 1) = 2^0 + 2^1 + \cdots + 2^{2^{b-1}} = 2^{2^b} - 1 = (2^{2^b} + 1) - 2$. Then suppose $\gcd(2^{2^a} + 1, 2^{2^b} + 1) > 1$. Let p be a prime number that divides both $2^{2^a} + 1$ and $2^{2^b} + 1$. Then p divides 2. Hence, $p = 2$. This is impossible since $2^{2^a} + 1$ is odd. Therefore, $\gcd(2^{2^a} + 1, 2^{2^b} + 1) = 1$. This solves the problem.