

CMO Qualifying Repêchage 2018

Official Solutions

1. Determine all solutions to the following system of equations

$$\begin{cases} y = 4x^3 + 12x^2 + 12x + 3 \\ x = 4y^3 + 12y^2 + 12y + 3 \end{cases}$$

Solution: We can rewrite the two equations as:

$$\begin{cases} y + 1 = 4(x + 1)^3 \\ x + 1 = 4(y + 1)^3 \end{cases}$$

Substituting the second equation into the first, we get

$$(y + 1) = 256(y + 1)^9$$

By inspection we see that $y = -1$ satisfies the equation. Assume $y \neq -1$, then the equation becomes:

$$\frac{1}{2^8} = (y + 1)^8$$

The only real solutions to this are $y + 1 = \pm \frac{1}{2}$.

This gives $y = -\frac{1}{2}, -1, -\frac{3}{2}$ as possible solutions.

Substituting into the above given equations, we get the following three solutions:

$$\left(-\frac{1}{2}, -\frac{1}{2}\right), (-1, -1), \left(-\frac{3}{2}, -\frac{3}{2}\right).$$

□

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2. We call a pair of polygons, p and q , *nesting* if we can draw one inside the other, possibly after rotation and/or reflection; otherwise we call them *non-nesting*.

Let p and q be convex polygons. Prove that we can find a polygon r , which is similar to q , such that r and p are non-nesting if and only if p and q are not similar.

Solution: Suppose p and q are similar and let r be any polygon similar to q . Choose an identified vertex on p and r , and draw the two polygons oriented the same as each other with the identified vertices in the same location. Assume without loss of generality that p has smaller area than r . For any point m on the boundary of p , there is a line segment containing m the image of m on r and the identified vertex of the two polygons. Since p has smaller area than r , the point m is on the interior of this segment. Thus, m is a point in r . This holds for every point on the boundary of p , so the boundary of p is contained in r . Since r is convex, this means that p is contained in r . The result holds similarly if p has larger area than r .

Suppose p and q are not similar. Let r be a polygon similar to q which has the same area as p . Suppose r can be drawn inside p . Then since r and p have the same area, r and p must be similar. But since r and q are similar, then p and q would also have to be similar. Thus r cannot be drawn inside p . Similarly p cannot be drawn inside r . Thus, we can find an r such that p and r are non-nesting. \square

3. Let ABC be a triangle with $AB = BC$. Prove that $\triangle ABC$ is an obtuse triangle if and only if the equation

$$Ax^2 + Bx + C = 0$$

has two distinct real roots, where A, B, C are the angles in radians.

Solution: A degree 2 polynomial has two distinct real roots if and only if the discriminant is positive. The discriminant of the given polynomial is:

$$B^2 - 4AC.$$

Since $AB = BC$, we have $A = C$ and the discriminant is:

$$B^2 - 4C^2.$$

The discriminant is positive if and only if:

$$B^2 > 4C^2.$$

Since B and C are angles of a triangle, both are positive, so we can write this as:

$$B > 2C.$$

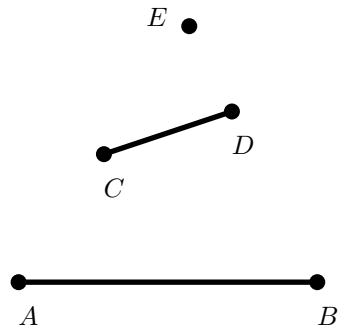
Since $A = C$, this is equivalent to:

$$B > A + C,$$

which happens if and only if the triangle is obtuse. \square

4. Construct a convex polygon such that each of its sides has the same length as one of its diagonals and each diagonal has the same length as one of its sides, or prove that such a polygon does not exist.

Solution: Suppose such a polygon exists. Let AB be the longest side of the polygon and CD the shortest diagonal. Let E be a vertex of the polygon on the side of CD opposite to AB , as seen in the figure below.



Since AB is the longest side in the polygon, AE and BE are at most as long as AB . Thus, AB is the longest side of triangle AEB , and so $\angle AEB \geq 60^\circ$.

Since CD is the shortest diagonal in the polygon, CE and DE are at least as long as CD . Thus, CD is the shortest side of triangle CDE , and so $\angle CED \leq 60^\circ$.

Since the polygon is convex, neither C nor D can lie inside or on the boundary of triangle ABE unless $C = A$ or $D = B$. Therefore, $\angle CED > \angle AEB$ which contradicts the previous paragraphs, and thus no such polygon can exist. □

5. A palindrome is a number that remains the same when its digits are reversed. Let n be a product of distinct primes not divisible by 10. Prove that the set $\{nk : k \in \mathbb{Z}\}$ contains infinitely many palindromes.

Solution: Consider a prime p other than 2, 3 or 5 that divides n . By Fermat's Little Theorem, $10^{p-1} - 1$ is divisible by p and since $p \neq 3$, $(10^{p-1} - 1)/9$ is also divisible by p .

Thus, the number consisting of $p - 1$ consecutive 1s is divisible by p . Also note that when we have 111 is divisible by 3.

Let P be the set of all primes dividing n other than 2, 3, and 5.

Let m be the product of $\{p - 1\}$ for all primes $p \in P$.

For every positive integer s , let d_s be the number consisting of $3ms$ consecutive 1s.

If n is not divisible by 2 or 5, then by construction, d_s is divisible by n for every s . If s is divisible by 2 or 5, then $2d_s$ or $5d_s$ will be divisible by s for every positive integer s . Thus, there are infinitely many multiples of n which are palindromes. □

6. Let $n \geq 2$ be a positive integer. Determine the number of n -tuples (x_1, x_2, \dots, x_n) such that $x_k \in \{0, 1, 2\}$ for $1 \leq k \leq n$ and $\sum_{k=1}^n x_k - \prod_{k=1}^n x_k$ is divisible by 3.

Solution:

Let $S(n)$ be the set of all n -tuples $\vec{x} = (x_1, x_2, \dots, x_n)$ such that $x_k \in \{0, 1, 2\}$ for $1 \leq k \leq n$. For $\vec{x} \in S(n)$, let $N_0(\vec{x})$ be the number of 0's in the sequence \vec{x} , and let

$$F(\vec{x}) = \sum_{k=1}^n x_k - \prod_{k=1}^n x_k \pmod{3}.$$

Our task is to evaluate the cardinality of

$$S^*(n) = \{\vec{x} \in S(n) : F(\vec{x}) = 0\}.$$

Observe that $\prod_{k=1}^n x_k = 0$ if and only if $N_0(\vec{x}) > 0$. Let

$$\begin{aligned} S_1(n) &= \{\vec{x} \in S(n) : N_0(\vec{x}) > 0, F(\vec{x}) = 0\} \\ &= \{\vec{x} \in S(n) : N_0(\vec{x}) > 0, \sum_{k=1}^n x_k = 0\} \\ Zero(n) &= \{\vec{x} \in S(n) : N_0(\vec{x}) = 0, \sum_{k=1}^n x_k = 0\} \\ Same(n) &= \{\vec{x} \in S(n) : N_0(\vec{x}) = 0, \sum_{k=1}^n x_k = \prod_{k=1}^n x_k\} \\ &= \{\vec{x} \in S(n) : N_0(\vec{x}) = 0, F(\vec{x}) = 0\} \\ Diff(n) &= \{\vec{x} \in S(n) : N_0(\vec{x}) = 0, \sum_{k=1}^n x_k = -\prod_{k=1}^n x_k\} \end{aligned}$$

Note that $N_0(\vec{x}) = 0$ if and only if every x_k is either 1 or 2. Also note that we have

$$|S^*(n)| = |S_1(n)| + |Same(n)|.$$

Let us first consider only the condition that $\sum_{k=1}^n x_k = 0$. There are clearly 3^{n-1} n -tuples which satisfy this requirement, as there is a unique x_n that will complete such a tuple. Thus

$$|S_1(n)| + |Zero(n)| = 3^{n-1}.$$

There are 2^n sequences with $N_0(\vec{x}) = 0$, so

$$|Zero(n)| + |Same(n)| + |Diff(n)| = 2^n.$$

It is easily verified by induction that when n is a multiple of 2,

$$|Zero(n)| = (2^n - 1)/3 + 1.$$

For the other $2 * (2^n - 1)/3$ sequences, half will have $\sum_{k=1}^n x_k = \prod_{k=1}^n x_k$ and half will have $\sum_{k=1}^n x_k = -\prod_{k=1}^n x_k$.

For any sequence with $N_0(\vec{x}) = 0$, we see that the sequence given by $3 - x_k$ has the negative of the summation term but has the same product. Thus

$$|Same(n)| = |Diff(n)| = (2^n - 1)/3.$$

Thus, for even n

$$\begin{aligned} |S^*(n)| &= |S_1(n)| + |Same(n)| \\ &= (3^{n-1} - |Zero(n)|) + |Same(n)| = 3^{n-1} - 1 \end{aligned}$$

When n is odd, we claim that

$$\begin{aligned} |Zero(n)| &= \frac{2^n - 2}{3} \\ |Same(n)| &= \frac{2^n + 1}{3} + (-3)^{(n-1)/2} \\ |Diff(n)| &= \frac{2^n + 1}{3} - (-3)^{(n-1)/2} \end{aligned}$$

Taking a base case of $n = 1$ it is easy to numerically verify the result. Inductively, we consider what happens when we add $\{11, 12, 21, 22\}$ to each type of sequence (i.e. Zero, Same, and Diff sequences).

Adding 12 and 21 to a Zero sequence gives another Zero sequence. One of 11 and 22 will give a Diff sequence and the other will give a Same sequence.

Adding 12 and 21 to a Same sequence gives a Diff sequence. One of 11 and 22 will give a Diff sequence and the other will give a Zero sequence.

Adding 12 and 21 to a Diff sequence gives a Same sequence. One of 11 and 22 will give a Same sequence and the other will give a Zero sequence.

Using these relations, it is easy to verify the recursive case of the induction.

For odd n this gives:

$$\begin{aligned} |S^*(n)| &= |S_1(n)| + |Same(n)| \\ &= (3^{n-1} - |Zero(n)|) + |Same(n)| \\ &= 3^{n-1} + 1 + (-3)^{(n-1)/2}. \end{aligned}$$

□

7. Let n be a positive integer, with prime factorization

$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

for distinct primes p_1, \dots, p_r , and e_i positive integers. Define

$$rad(n) = p_1 p_2 \cdots p_r,$$

the product of all distinct prime factors of n .

Find all polynomials $P(x)$ with rational coefficients such that there exist infinitely many positive integers n with $P(n) = rad(n)$. f

Solution: Note that $0 < rad(n) \leq n$ for all n and so we have $0 < P(n) \leq n$ for infinitely many positive integers n . This tells us that the leading coefficient is positive and the degree of $P(x)$ is 0 or 1.

If the degree of $P(x)$ is 0, then $P(x) = c$ for some positive integer x and there must be infinitely many solutions to the equation $rad(n) = c$. Since $rad(n)$ is square-free for any n , c must also be square free. For any square-free $c = p_1 p_2 \cdots p_k$, we have $rad(p_1^m p_2 \cdots p_k) = c$, which gives infinitely many values.

If the degree of $P(x)$ is 1, we can write $P(x) = \frac{ax+b}{c}$, where a, b, c are integers with $a, c > 0$. If $P(n) = rad(n)$ then $c \cdot rad(n) = an + b$. Assume $b \neq 0$, and note that $rad(n) | n$. Thus $rad(n) | b$ and so there are finitely many possible values of $rad(n)$. But each value of $rad(n)$ determines n uniquely as $\frac{c \cdot rad(n) - b}{a}$, contradicting there being an infinite number of values for n . thus $b = 0$, and $P(x) = \frac{ax}{c}$. When $P(n) = rad(n)$ we have $c = a \frac{n}{rad(n)}$, and so $a | c$. Thus, we can write $P(x) = x/c$ for some integer c .

For any integer $c = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, let $d = p_1^{\alpha_1+1} \cdots p_k^{\alpha_k+1}$. For any prime p distinct from p_1, \dots, p_k , we have $rad(dp) = dp/c$ and so there are infinitely many solutions to $P(n) = rad(n)$.

Thus, the solutions are $P(x) = c$ and $P(x) = x/d$ where c is any square-free positive integer and d is any positive integer. \square

8. Let n and k be positive integers with $1 \leq k \leq n$. A set of cards numbered 1 to n are arranged randomly in a row from left to right. A person alternates between performing the following moves:
- The leftmost card in the row is moved $k - 1$ positions to the right while the cards in positions 2 through k are each moved one place to the left.
 - The rightmost card in the row is moved $k - 1$ positions to the left while the cards in positions $n - k + 1$ through $n - 1$ are each moved one place to the right.

Determine the probability that after some move the cards end up in order from 1 to n , left to right.

Solution: We will show that the probability is:

$$\begin{cases} \frac{1}{n!} & \text{when } k = 1 \\ \frac{2k}{n!} & \text{when } 1 < k \leq \frac{n}{2} \\ \frac{4(n-k)+2}{n!} & \text{when } k > \frac{n}{2} \end{cases}$$

When $k = 1$ no movement of cards happens, so the cards will eventually end up in order if and only if they started in order.

When $k \leq \frac{n}{2}$ moves of type a only affect the first half of the cards and moves of type b only affect the second half of the cards. After performing $2k$ moves, it is clear that we end up back to the starting permutation and we have seen $2k$ different permutations of the cards.

We label the positions of cards from left to right, with the left most card being in position 1. When $k > \frac{n}{2}$ consider the movement of the card in the position 1. The first a move moves it to position k . While it is not in the position n , a b move will move it 1 position to the right and an a move will not move it at all. It takes $n - k$ b moves for the card to end up in position n . This gives us $2(n - k)$ moves so far. After 2 more moves, the card will be moved to position $n - k + 1$. It will then take $n - k$ more a moves to move the card back up to position 1, and then we do 1 final b move, which does not affect its position.

It proceeds similarly to verify that all the other cards have all returned to their starting positions. \square