THE OLYMPIAD CORNER

No. 350

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er octobre 2017**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC316. Soit ABC un triangle rectangle en B et soit BD la hauteur abaissée depuis B jusqu'à AC. Soit P, Q et I les centres des cercles inscrits dans les triangles respectifs ABD, CBD et ABC. Démontrer que le centre du cercle circonscrit au triangle PIQ est situé sur l'hypoténuse AC.

OC317. On a accueilli 110 équipes lors d'un tournoi de volleyball. Chaque équipe a rencontré chaque autre équipe une fois (il n'y a aucun match nul au volleyball). Or, dans n'importe quel ensemble de 55 équipes, il y a une équipe qui n'a pas perdu plus de 4 matchs contre les 54 autres équipes de l'ensemble. Démontrer que dans le tournoi, il y a une équipe qui n'a pas perdu plus de 4 des matchs contre les 109 autres équipes.

OC318. Soit n un entier strictement positif et k un entier de 1 à n. On considère un quadrillage blanc de dimensions $n \times n$ et on procède comme suit:

> On trace k rectangles délimités par les lignes du quadrillage de manière que chaque rectangle recouvre le coin supérieur droit 1×1 du quadrillage $n \times n$ et on peint les k rectangles en noir. De cette manière, il reste toujours de l'espace blanc sur le quadrillage.

Combien de figures blanches différentes peut-on former avec k rectangles de manière que ces figures ne puissent être formées avec moins de k rectangles?

OC319. Soit p un nombre premier supérieur à 30. Démontrer qu'un des nombres suivants,

$$p + 1, 2p + 1, 3p + 1, \dots, (p - 3)p + 1,$$

est la somme des carrés de deux entiers.

OC320. Soit n un entier supérieur à 1. On écrit d'abord n ensembles au tableau, puis on leur fait subir une série de manoeuvres. Une *manoeuvre* se fait comme suit:

> On choisit deux ensembles A et B au tableau de manière que l'un ne soit pas un sous-ensemble de l'autre et on les remplace par $A \cap B$ et $A \cup B$.

Déterminer le nombre maximal de manoeuvres pour tous ensembles possibles présentés au départ.

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OC316. Let ABC be a right-angled triangle with $\angle B = 90°$. Let BD be the altitude from B. Let P, Q and I be the incenters of triangles ABD, CBD and ABC respectively. Show that the circumcenter of PIQ lies on the hypotenuse AC.

OC317. In a recent volleyball tournament, 110 teams participated. Every team has played every other team exactly once (there are no ties in volleyball). It turns out that in any set of 55 teams, there is one which has lost to no more than 4 of the remaining 54 teams. Prove that in the entire tournament, there is a team that has lost to no more than 4 of the remaining 109 teams.

OC318. Let n be a positive integer and let k be an integer between 1 and n inclusive. Given an $n \times n$ white board, we do the following process.

> We draw k rectangles with integer side lengths and sides parallel to the sides of the $n \times n$ board, and such that each rectangle covers the top-right corner of the $n \times n$ board. Then, the k rectangles are painted black. This process leaves a white figure in the board.

How many different white figures can be formed with k rectangles that cannot be formed with less than k rectangles?

OC319. Let $p > 30$ be a prime number. Prove that one of the following numbers

$$p + 1, 2p + 1, 3p + 1, \ldots, (p - 3)p + 1$$

is the sum of two integer squares $x^2 + y^2$ for integers x and y.

OC320. Let $n \geq 2$ be a given integer. Initially, we write n sets on the blackboard and do a sequence of *moves* as follows:

> choose two sets A and B on the blackboard such that neither of them is a subset of the other, and replace A and B by $A \cap B$ and $A \cup B$.

Find the maximum number of moves in a sequence for all possible initial sets.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(10), p. 425–426.

OC256. Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

Originally problem 3 of the 2014 France Team Selection Test.

We present the solution by Steven Chow. There were no other submissions.

Let $F(x) = x^2 + x + 1$ for all x. Let $P(x)$ be the greatest prime divisor of x for all integers $x \geq 2$. Then, since

$$F(n^2) = n^4 + n^2 + 1 = (n^2 + n + 1)(n^2 - n + 1) = F(n)F(n-1)$$

we see that

$$P(F(n^2)) = P(F(n)F(n-1)) = \max\{P(F(n)), P(F(n-1))\}.$$

Therefore, $P(F(n^2)) = P(F((n+1)^2))$ if and only if

$$\max\{P(F(n)), P(F(n-1))\} = \max\{P(F(n+1)), P(F(n))\},$$

which is true if and only if either $P(F(n)) \geq \max\{P(F(n+1)), P(F(n-1))\}$ or $P(F(n-1)) = P(F(n+1)) \geq P(F(n))$.

Now, assume towards a contradiction that there exist finitely many integers $n \geq 1$ such that $P(F(n^2)) = P(F((n+1)^2))$. Then, the above implies that there exists an integer k such that for all integers $n \geq k$,

$$P(F(n)) < \max\{P(F(n+1)), P(F(n-1))\}$$

and

$$(P(F(n-1)) \neq P(F(n+1)) \text{ or } P(F(n-1)) = P(F(n+1)) < P(F(n))).$$

Rearranging the above implies that for all integers $n \geq k$, $F(n)$ is either an increasing function or a decreasing function.

If $F(n)$ is an increasing function, then for any integer $a \geq k+1$, $a < a^2$ and so

$$P(F(a)) < P(F(a^2)) = \max\{P(F(a)), P(F(a-1))\} = P(F(a))$$

which is a contradiction. Therefore, $F(n)$ is a decreasing function. However, since $P(F(n)) \geq 2$ for all positive integers n, we have a contradiction by infinite descent.

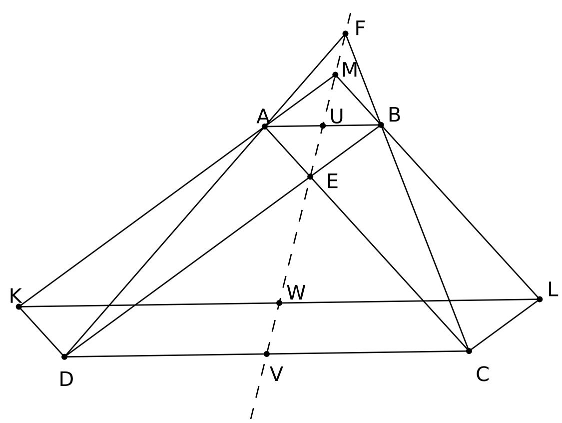
Thus, there exist infinitely many such n.

OC257. Let $ABCD$ be a trapezoid (quadrilateral with one pair of parallel sides) such that $AB < CD$. Suppose that AC and BD meet at E and AD and BC meet at F. Construct the parallelograms $AEDK$ and $BECL$. Prove that EF passes through the midpoint of the segment KL.

Originally problem 3 from day 1 of the 2014 Indonesian Mathematical Olympiad.

We received 5 correct submissions. We present the solution by Michel Bataille.

We assume that the parallel sides are AB and CD and we denote by U and V the midpoints of AB and CD, respectively. We first recall that U, V, E, F are collinear: indeed, the homothety h_F with centre F transforming A into D also transforms B into C, hence $\mathrm{h}_F(U) = V$ and U, V, F are collinear. Similarly, the homothety h_E with centre E such that $\mathrm{h}_E(B) = D$ and $\mathrm{h}_E(A) = C$ also satisfies $\mathrm{h}_E(U) = V$ so that U, V, E are collinear.



We have

$$\overrightarrow{KL} = \overrightarrow{KD} + \overrightarrow{DC} + \overrightarrow{CL} = \overrightarrow{AE} + \overrightarrow{DC} + \overrightarrow{EB} = \overrightarrow{AB} + \overrightarrow{DC}.$$

Thus, $\overrightarrow{KL}$ and $\overrightarrow{AB}$ are collinear and KL is parallel to AB (and CD).

Let M be the point of intersection of KA and BL. Then, $MA \parallel BE$ and $MB \parallel AE$ so that $AMBE$ is a parallelogram and U is also the midpoint of ME. It follows that the lines MU and UE coincide.

Now, consider the trapezoid $ABLK$ (with $AB \parallel KL$). From the property recalled above and applied to $ABLK$, the points M, U and the midpoint W of KL are collinear. As a result, W is on the line $MU = UE = EF$.

OC258. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x, $y \in \mathbb{R}$ hold:

$$f(xf(y) - yf(x)) = f(xy) - xy$$

Originally problem 1 of day 1 of the 2014 Serbian National Mathematical Olympiad.

We received 3 correct submissions and 1 incorrect submission. We present the solution by the Missouri State University Problem Solving Group.

First, we will show that $f(x) = 0$ if and only if $x = 0$. If $f(0) \neq 0$, then for any real z, if $x = z/f(0)$ and $y = 0$ in the given equation, then

$$f(z) = f(xf(0)) = f(0) - 0,$$

which implies that f is constant. This violates the given equation. Therefore, $f(0) = 0$. Now, if $x = y$, then the given equation becomes

$$f(x^2) - x^2 = f(xf(x) - xf(x)) = f(0) = 0$$

or equivalently, if $x = \sqrt{z}$, then $f(z) = z$ for all $z \geq 0$ (Equation 1).

Now, assume that $f(x) = 0$. Then using $y = 1$ and the fact that $f(1) = 1$ (seen by using $x = y = 1$ in the given equation), we see that

$$-x = f(x) - x = f(xf(1) - f(x)) = f(x) = 0.$$

Let $f(-1) = a$. Assume that $x \geq 0$. Then

$$f(f(-x) - ax) = f(-xf(-1) - (-1)f(-x)) = f(x) - x = 0.$$

Therefore, $f(-x) - ax = 0$ and hence, $f(-x) = ax$ for all $x \geq 0$ (Equation 2).

Now, if $x = 1$ and $y = -1$ or vice versa, then the given equation becomes

$$f(\pm(a+1)) = a + 1.$$

In particular, $a + 1 = f(|a + 1|) = |a + 1|$ which implies that $a + 1 \geq 0$. Then

$$a + 1 = f(-|a + 1|) = a(a + 1)$$

which implies that either $a = 1$ or $a = -1$. It follows from Equations 1 and 2 above that $f(x) = x$ or that $f(x) = |x|$. A quick check verifies that these two functions are indeed solutions.

OC259. If the polynomials $f(x)$ and $g(x)$ are written on a blackboard then we can also write down the polynomials $f(x) \pm g(x)$, $f(x)g(x)$, $f(g(x))$ and $cf(x)$, where c is an arbitrary real constant. The polynomials $x^3 - 3x^2 + 5$ and $x^2 - 4x$ are written on the blackboard. Can we write a nonzero polynomial of the form $x^n - 1$ after a finite number of steps?

Originally problem 3 from day 2 of the 2014 All Russian Mathematical Olympiad.

We present the solution by Arkady Alt. There were no other submissions.

Let

$$f_0(x) = x^3 - 3x^2 + 5, \quad g_0(x) = x^2 - 4x.$$

Noting that

$$f_0'(x) = 3x^2 - 6x = 3x(x-2) \quad \text{and} \quad g_0'(x) = 2x - 4 = 2(x-2),$$

we conclude that the derivative of the two initial polynomials have common root $x = 2$, that is $g_0'(2) = f_0'(2) = 0$.

Using basic derivative rules, we see that the derivative of any polynomial, obtained from polynomials $f_0(x)$ and $g_0(x)$, must be zero if $x = 2$.

Since $(x^n - 1)' = nx^{n-1}, n \in \mathbb{N}$ cannot be zero if $x = 2$, the answer is negative.

OC260. Find the maximum of

$$P = \frac{x^3y^4z^3}{(x^4+y^4)(xy+z^2)^3} + \frac{y^3z^4x^3}{(y^4+z^4)(yz+x^2)^3} + \frac{z^3x^4y^3}{(z^4+x^4)(zx+y^2)^3}$$

where x, y, z are positive real numbers.

Originally problem 2 from day 2 of the 2014 Vietnam National Olympiad.

We received 2 correct submissions. We present the solution by Titu Zvonaru.

We will make use of the following inequalities:

$$\begin{aligned}
&(xy+z^2)^2 \geq 4xyz^2, \\
&\frac{1}{a+b} \leq \frac{1}{4} \cdot \left(\frac{1}{a} + \frac{1}{b}\right), \\
&x^2 + y^2 + z^2 \geq xy + yz + zx, \\
&x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x+y+z)
\end{aligned}$$

and lastly, $x^4 + y^4 \geq (2/3)xy(x^2 + y^2 + xy) \Leftrightarrow (x-y)^2(3x^2 + 4xy + 3y^2) \geq 0$. We have

$$\begin{aligned}
\frac{x^3y^4z^3}{(x^4+y^4)(xy+z^2)^3} &\leq \frac{3}{8} \cdot \frac{x^3y^4z^3}{xy(x^2+y^2+xy)xyz^2(xy+z^2)} \\
&= \frac{3}{8} \cdot \frac{xy^2z}{(x^2+y^2+xy)(xy+z^2)} \\
&= \frac{3}{8} \cdot \frac{xy^2z}{x^2y^2+y^2z^2+z^2x^2+xy(x^2+y^2+z^2)} \\
&= \frac{3}{32} \cdot \left(\frac{xy^2z}{x^2y^2+y^2z^2+z^2x^2} + \frac{yz}{x^2+y^2+z^2}\right).
\end{aligned}$$

It follows that

$$P \leq \frac{3}{32}\left(\frac{xy^2z+xyz^2+x^2yz}{x^2y^2+y^2z^2+z^2x^2} + \frac{yz+zx+xy}{x^2+y^2+z^2}\right) \leq \frac{3}{32}(1+1) = \frac{3}{16}.$$

If $x = y = z > 0$, then $P = 3/16$ and hence the maximum value of P is $3/16$.