# OLYMPIAD SOLUTIONS

*Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(10), p. 425–426.*

**OC256**. Prove that there exist infinitely many positive integers $n$ such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

*Originally problem 3 of the 2014 France Team Selection Test.*

*We present the solution by Steven Chow. There were no other submissions.*

Let $F(x) = x^2 + x + 1$ for all $x$. Let $P(x)$ be the greatest prime divisor of $x$ for all integers $x \geq 2$. Then, since

$$F(n^2) = n^4 + n^2 + 1 = (n^2 + n + 1)(n^2 - n + 1) = F(n)F(n-1)$$

we see that

$$P(F(n^2)) = P(F(n)F(n-1)) = \max\{P(F(n)), P(F(n-1))\}.$$

Therefore, $P(F(n^2)) = P(F((n+1)^2))$ if and only if

$$\max\{P(F(n)), P(F(n-1))\} = \max\{P(F(n+1)), P(F(n))\},$$

which is true if and only if either $P(F(n)) \geq \max\{P(F(n+1)), P(F(n-1))\}$ or $P(F(n-1)) = P(F(n+1)) \geq P(F(n))$.

Now, assume towards a contradiction that there exist finitely many integers $n \geq 1$ such that $P(F(n^2)) = P(F((n+1)^2))$. Then, the above implies that there exists an integer $k$ such that for all integers $n \geq k$,

$$P(F(n)) < \max\{P(F(n+1)), P(F(n-1))\}$$

and

$$(P(F(n-1)) \neq P(F(n+1)) \text{ or } P(F(n-1)) = P(F(n+1)) < P(F(n))).$$

Rearranging the above implies that for all integers $n \geq k$, $F(n)$ is either an increasing function or a decreasing function.

If $F(n)$ is an increasing function, then for any integer $a \geq k+1$, $a < a^2$ and so

$$P(F(a)) < P(F(a^2)) = \max\{P(F(a)), P(F(a-1))\} = P(F(a))$$

which is a contradiction. Therefore, $F(n)$ is a decreasing function. However, since $P(F(n)) \geq 2$ for all positive integers $n$, we have a contradiction by infinite descent.

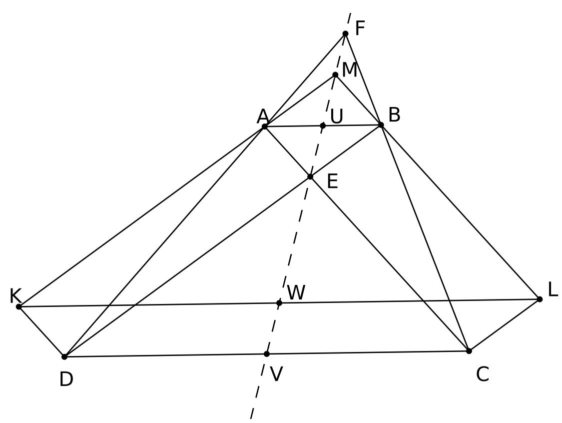
Thus, there exist infinitely many such $n$.

**OC257**. Let $ABCD$ be a trapezoid (quadrilateral with one pair of parallel sides) such that $AB < CD$. Suppose that $AC$ and $BD$ meet at $E$ and $AD$ and $BC$ meet at $F$. Construct the parallelograms $AEDK$ and $BECL$. Prove that $EF$ passes through the midpoint of the segment $KL$.

*Originally problem 3 from day 1 of the 2014 Indonesian Mathematical Olympiad.*

*We received 5 correct submissions. We present the solution by Michel Bataille.*

We assume that the parallel sides are $AB$ and $CD$ and we denote by $U$ and $V$ the midpoints of $AB$ and $CD$, respectively. We first recall that $U, V, E, F$ are collinear: indeed, the homothety $\mathrm{h}_F$ with centre $F$ transforming $A$ into $D$ also transforms $B$ into $C$, hence $\mathrm{h}_F(U) = V$ and $U, V, F$ are collinear. Similarly, the homothety $\mathrm{h}_E$ with centre $E$ such that $\mathrm{h}_E(B) = D$ and $\mathrm{h}_E(A) = C$ also satisfies $\mathrm{h}_E(U) = V$ so that $U, V, E$ are collinear.




We have

$$\overrightarrow{KL} = \overrightarrow{KD} + \overrightarrow{DC} + \overrightarrow{CL} = \overrightarrow{AE} + \overrightarrow{DC} + \overrightarrow{EB} = \overrightarrow{AB} + \overrightarrow{DC}.$$

Thus, $\overrightarrow{KL}$ and $\overrightarrow{AB}$ are collinear and $KL$ is parallel to $AB$ (and $CD$).

Let $M$ be the point of intersection of $KA$ and $BL$. Then, $MA \parallel BE$ and $MB \parallel AE$ so that $AMBE$ is a parallelogram and $U$ is also the midpoint of $ME$. It follows that the lines $MU$ and $UE$ coincide.

Now, consider the trapezoid $ABLK$ (with $AB \parallel KL$). From the property recalled above and applied to $ABLK$, the points $M, U$ and the midpoint $W$ of $KL$ are collinear. As a result, $W$ is on the line $MU = UE = EF$.

**OC258**. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x$, $y \in \mathbb{R}$ hold:

$$f(xf(y) - yf(x)) = f(xy) - xy$$

*Originally problem 1 of day 1 of the 2014 Serbian National Mathematical Olympiad.*

*We received 3 correct submissions and 1 incorrect submission. We present the solution by the Missouri State University Problem Solving Group.*

First, we will show that $f(x) = 0$ if and only if $x = 0$. If $f(0) \neq 0$, then for any real $z$, if $x = z/f(0)$ and $y = 0$ in the given equation, then

$$f(z) = f(xf(0)) = f(0) - 0,$$

which implies that $f$ is constant. This violates the given equation. Therefore, $f(0) = 0$. Now, if $x = y$, then the given equation becomes

$$f(x^2) - x^2 = f(xf(x) - xf(x)) = f(0) = 0$$

or equivalently, if $x = \sqrt{z}$, then $f(z) = z$ for all $z \geq 0$ (Equation 1).

Now, assume that $f(x) = 0$. Then using $y = 1$ and the fact that $f(1) = 1$ (seen by using $x = y = 1$ in the given equation), we see that

$$-x = f(x) - x = f(xf(1) - f(x)) = f(x) = 0.$$

Let $f(-1) = a$. Assume that $x \geq 0$. Then

$$f(f(-x) - ax) = f(-xf(-1) - (-1)f(-x)) = f(x) - x = 0.$$

Therefore, $f(-x) - ax = 0$ and hence, $f(-x) = ax$ for all $x \geq 0$ (Equation 2).

Now, if $x = 1$ and $y = -1$ or vice versa, then the given equation becomes

$$f(\pm(a+1)) = a + 1.$$

In particular, $a + 1 = f(|a+1|) = |a+1|$ which implies that $a + 1 \geq 0$. Then

$$a + 1 = f(-|a+1|) = a(a+1)$$

which implies that either $a = 1$ or $a = -1$. It follows from Equations 1 and 2 above that $f(x) = x$ or that $f(x) = |x|$. A quick check verifies that these two functions are indeed solutions.

**OC259**. If the polynomials $f(x)$ and $g(x)$ are written on a blackboard then we can also write down the polynomials $f(x) \pm g(x)$, $f(x)g(x)$, $f(g(x))$ and $cf(x)$, where $c$ is an arbitrary real constant. The polynomials $x^3 - 3x^2 + 5$ and $x^2 - 4x$ are written on the blackboard. Can we write a nonzero polynomial of the form $x^n - 1$ after a finite number of steps?

*Originally problem 3 from day 2 of the 2014 All Russian Mathematical Olympiad.*

*We present the solution by Arkady Alt. There were no other submissions.*

Let

$$f_0(x) = x^3 - 3x^2 + 5, \quad g_0(x) = x^2 - 4x.$$

Noting that

$$f_0'(x) = 3x^2 - 6x = 3x(x-2) \quad \text{and} \quad g_0'(x) = 2x - 4 = 2(x-2),$$

we conclude that the derivative of the two initial polynomials have common root $x = 2$, that is $g_0'(2) = f_0'(2) = 0$.

Using basic derivative rules, we see that the derivative of any polynomial, obtained from polynomials $f_0(x)$ and $g_0(x)$, must be zero if $x = 2$.

Since $(x^n - 1)' = nx^{n-1}, n \in \mathbb{N}$ cannot be zero if $x = 2$ , the answer is negative.

**OC260**. Find the maximum of

$$P = \frac{x^3y^4z^3}{(x^4+y^4)(xy+z^2)^3} + \frac{y^3z^4x^3}{(y^4+z^4)(yz+x^2)^3} + \frac{z^3x^4y^3}{(z^4+x^4)(zx+y^2)^3}$$

where $x, y, z$ are positive real numbers.

*Originally problem 2 from day 2 of the 2014 Vietnam National Olympiad.*

*We received 2 correct submissions. We present the solution by Titu Zvonaru.*

We will make use of the following inequalities:

$$\begin{aligned}
&(xy+z^2)^2 \geq 4xyz^2,\\
&\frac{1}{a+b} \leq \frac{1}{4}\cdot\left(\frac{1}{a}+\frac{1}{b}\right),\\
&x^2+y^2+z^2 \geq xy+yz+zx,\\
&x^2y^2+y^2z^2+z^2x^2 \geq xyz(x+y+z)
\end{aligned}$$

and lastly, $x^4 + y^4 \geq (2/3)xy(x^2+y^2+xy) \Leftrightarrow (x-y)^2(3x^2+4xy+3y^2) \geq 0$. We have

$$\begin{aligned}
\frac{x^3y^4z^3}{(x^4+y^4)(xy+z^2)^3} &\leq \frac{3}{8}\cdot\frac{x^3y^4z^3}{xy(x^2+y^2+xy)xyz^2(xy+z^2)}\\
&= \frac{3}{8}\cdot\frac{xy^2z}{(x^2+y^2+xy)(xy+z^2)}\\
&= \frac{3}{8}\cdot\frac{xy^2z}{x^2y^2+y^2z^2+z^2x^2+xy(x^2+y^2+z^2)}\\
&= \frac{3}{32}\cdot\left(\frac{xy^2z}{x^2y^2+y^2z^2+z^2x^2}+\frac{yz}{x^2+y^2+z^2}\right).
\end{aligned}$$

It follows that

$$P \leq \frac{3}{32}\left(\frac{xy^2z+xyz^2+x^2yz}{x^2y^2+y^2z^2+z^2x^2}+\frac{yz+zx+xy}{x^2+y^2+z^2}\right) \leq \frac{3}{32}(1+1) = \frac{3}{16}.$$

If $x = y = z > 0$, then $P = 3/16$ and hence the maximum value of $P$ is $3/16$.