About the Side and Diagonals of the Regular Heptagon

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In a regular heptagon $ABCDEFG$, we can distinguish two kinds of diagonals: a short one such as AC and a long one such as AD. In this article, we are interested in various relationships between the lengths $a = AB$ of the side of the heptagon and $b = AC, c = AD$ of the diagonals. Since $AC = BD$, a, b, c are the sidelengths of the triangle ABD (Figure 1). It is easily seen that the angles of this *heptagonal* triangle are $\angle ADB = \frac{\pi}{7}$, $\angle BAD = \frac{2\pi}{7}$ and $\angle ABD = \frac{4\pi}{7}$.

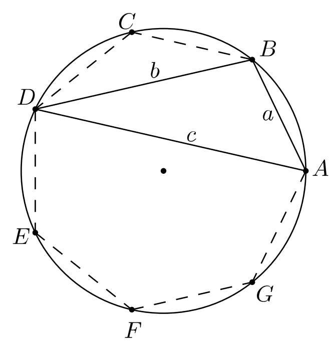

Figure 1

The heptagonal triangle was the subject of a celebrated 1973 article by Leon Bankoff and Jack Garfunkel ([**1**]). The two great problemists were gathering scattered results discovered by another great problemist, Victor Thébault. Here, guided by a closely related polynomial, we will re-obtain some of the numerous results of [**1**] and offer new relations between a, b, c.

Basic relationships

Let Γ be the circumscribing circle of the heptagon and R its radius. The Sine Law gives

$$a = 2R\sin\frac{\pi}{7}, \quad b = 2R\sin\frac{2\pi}{7}, \quad c = 2R\sin\frac{4\pi}{7}. \tag{1}$$

From $b = 4R\sin\frac{\pi}{7}\cos\frac{\pi}{7} = 2a\cos\frac{\pi}{7}$ and similar equalities, we deduce

$$\cos\frac{\pi}{7} = \frac{b}{2a}, \quad \cos\frac{2\pi}{7} = \frac{c}{2b}, \quad \cos\frac{4\pi}{7} = -\frac{a}{2c}. \tag{2}$$

Applying the Cosine Law, we readily obtain

$$a^2 = b^2 + c^2 - 2bc\cdot\frac{b}{2a} \quad \text{or} \quad a(b^2 + c^2) = a^3 + b^2c.$$

Similarly, we get

$$b(a^2+c^2) = b^3 + c^2a \quad \text{and} \quad c(a^2+b^2) = c^3 - a^2b.$$

However, simpler relations link the lengths a, b, c, e.g. the series

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c}, \quad c^2 - a^2 = bc, \quad b^2 - a^2 = ca, \quad c^2 - b^2 = ab. \tag{3}$$

These four equalities result from judicious applications of Ptolemy's Theorem (see **[2]** if necessary), but elementary proofs may be of interest. For example, we can derive the first one from the following geometrical argument (Figure 2): The chords AB and CG are parallel and so are the chords BC and AD. If H is the point of intersection of AD and CG, it follows that $ABCH$ is a parallelogram, even a rhombus since $AB = BC$. But AC is parallel to DG, hence the triangle DHG is homothetic to the triangle AHC. Thus, $\frac{AB}{AC} = \frac{AH}{AC} = \frac{DH}{DG}$. Since $DG = DA$, we obtain

$$AB \cdot AD = AC \cdot DH = AC(AD - AH) = AC(AD - AB),$$

that is, $ac = b(c-a)$, the desired equality.

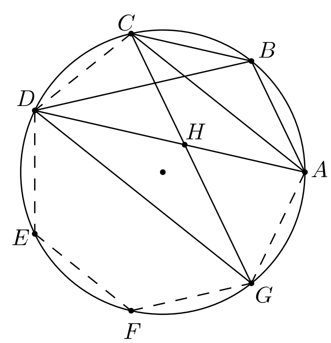

Figure 2

Let us prove the second relation by means of trigonometry. Note that using (1) and since $\sin\frac{4\pi}{7} = \sin\frac{3\pi}{7}$, we may rewrite the equality $c^2 - a^2 = bc$ as

$$\sin^2\frac{3\pi}{7} - \sin^2\frac{\pi}{7} = \sin\frac{2\pi}{7}\sin\frac{4\pi}{7},$$

which is directly obtained from the following calculation of the left-hand side:

$$\left(\sin\frac{3\pi}{7} - \sin\frac{\pi}{7}\right)\left(\sin\frac{3\pi}{7} + \sin\frac{\pi}{7}\right) = 2\sin\frac{\pi}{7}\cos\frac{2\pi}{7}\cdot 2\sin\frac{2\pi}{7}\cos\frac{\pi}{7} = \sin\frac{2\pi}{7}\sin\frac{4\pi}{7}.$$

Interestingly, we can remark that relations (3) lead to the proportions

$$a : b : c = b : a+c : b+c = c : b+c : a+b+c,$$

to be compared to the well-known golden proportion $a : b = b : a+b$ linking the side a and the diagonal b of a regular pentagon. Actually, these examples are particular cases of a remarkable, general pattern valid for all regular n-gons, the diagonal product formula, discovered and studied by P. Steinbach in [**3**].

To conclude this section, we propose the following exercise: using relations (1), (2) and (3), prove the beautiful formulas

$$a^2+b^2+c^2=7R^2, \quad a^4+b^4+c^4=21R^4 \quad \text{and} \quad a^2b^2+b^2c^2+c^2a^2=14R^4.$$

A useful polynomial

We introduce the polynomial

$$\begin{aligned} P(x) &= \left(x-\cos\frac{2\pi}{7}\right)\left(x-\cos\frac{4\pi}{7}\right)\left(x-\cos\frac{6\pi}{7}\right) \\ &= \left(x-\cos\frac{2\pi}{7}\right)\left(x-\cos\frac{4\pi}{7}\right)\left(x+\cos\frac{\pi}{7}\right). \end{aligned}$$

Let

$$\begin{aligned} s_1 &= \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}, \\ s_2 &= \cos\frac{2\pi}{7}\cos\frac{4\pi}{7}+\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}+\cos\frac{6\pi}{7}\cos\frac{2\pi}{7}, \\ s_3 &= \cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}. \end{aligned}$$

Using the formulas

$$\begin{aligned} 2\cos u\cos v &= \cos(u+v)+\cos(u-v), \\ 2\sin u\cos v &= \sin(u+v)+\sin(u-v), \end{aligned}$$

we successively obtain

$$2s_2=2s_1, \quad (2s_1)\sin\frac{\pi}{7}=-\sin\frac{\pi}{7} \quad \text{and} \quad (2s_3)\sin\frac{2\pi}{7}=\frac{1}{4}\sin\frac{2\pi}{7}.$$

Thus, the roots

$$x_1=\cos\frac{2\pi}{7}=\frac{c}{2b},\ x_2=\cos\frac{4\pi}{7}=-\frac{a}{2c},\ x_3=\cos\frac{6\pi}{7}=-\cos\frac{\pi}{7}=-\frac{b}{2a} \tag{4}$$

satisfy

$$x_1+x_2+x_3=x_1x_2+x_2x_3+x_3x_1=-\frac{1}{2}, \quad x_1x_2x_3=\frac{1}{8}.$$

We deduce that

$$P(x)=x^3+\frac{1}{2}x^2-\frac{1}{2}x-\frac{1}{8}$$

and two more formulas for a,b,c:

$$-\frac{1}{2}=\frac{c}{2b}-\frac{a}{2c}-\frac{b}{2a} \quad \text{and} \quad -\frac{1}{2}=-\frac{c}{2b}\cdot\frac{a}{2c}+\frac{a}{2c}\cdot\frac{b}{2a}-\frac{b}{2a}\cdot\frac{c}{2b},$$

that is,

$$\frac{a}{c} + \frac{b}{a} - \frac{c}{b} = 1, \qquad \frac{a}{b} + \frac{c}{a} - \frac{b}{c} = 2.$$

But we can go further! The fact that any symmetric polynomial in the roots x_1, x_2, x_3 can be expressed as a polynomial in s_1, s_2, s_3 opens the way to the discovery of a lot of relations between a, b, c. We give a few examples.

Consider first $x_1^2 + x_2^2 + x_3^2 = s_1^2 - 2s_2 = \frac{5}{4}$. Back to a, b, c, this yields a formula mentioned in [**1**]:

$$\frac{a^2}{c^2} + \frac{b^2}{a^2} + \frac{c^2}{b^2} = 5. \tag{5}$$

Similarly, from the well-known

$$x_1^3 + x_2^3 + x_3^3 = 3x_1x_2x_3 + (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - (x_1x_2 + x_2x_3 + x_3x_1)),$$

we get

$$\frac{a^3}{c^3} + \frac{b^3}{a^3} - \frac{c^3}{b^3} = 4$$

and from $x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2 = s_2^2 - 2s_3s_1$, we obtain

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6, \tag{6}$$

a companion to (5). More complicated formulas can appear; for example, the identity

$$x_1x_2(x_1 + x_2) + x_2x_3(x_2 + x_3) + x_3x_1(x_3 + x_1) = s_1s_2 - 3s_3$$

leads to

$$\frac{a^2}{bc} + \frac{bc}{a^2} - \left(\frac{b^2}{ca} + \frac{ca}{b^2}\right) - \left(\frac{c^2}{ab} + \frac{ab}{c^2}\right) = -1.$$

However, with the help of previous relations, this result simplifies into

$$b^3 + c^3 - a^3 = 4abc.$$

With the help of (5) and (6), one can obtain the very simple formula $abc = \sqrt{7}R^3$ (so that the area of our heptagonal triangle is $F = \frac{\sqrt{7}R^2}{4}$) (exercise!).

The reader is also encouraged to choose her/his favorite symmetric expressions in x_1, x_2, x_3 and discover a new formula for a, b, c!

Exploiting $P(x)$ in another way

The roots of $P(x)$ being simple roots, the decomposition of $\frac{1}{P(x)}$ into partial fractions is of the form

$$\frac{1}{P(x)} = \frac{\alpha_1}{x - x_1} + \frac{\alpha_2}{x - x_2} + \frac{\alpha_3}{x - x_3}. \tag{7}$$

By differentiation, we obtain another decomposition:

$$\frac{P'(x)}{(P(x))^2} = \frac{\alpha_1}{(x-x_1)^2} + \frac{\alpha_2}{(x-x_2)^2} + \frac{\alpha_3}{(x-x_3)^2}. \tag{8}$$

The values of the α_k $(k = 1, 2, 3)$ are easily obtained noticing that

$$\begin{aligned} \frac{1}{\alpha_1} = (x_1 - x_2)(x_1 - x_3) &= \left(\cos\frac{2\pi}{7} - \cos\frac{4\pi}{7}\right)\left(\cos\frac{2\pi}{7} - \cos\frac{6\pi}{7}\right) \\ &= 4\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin^2\left(\frac{4\pi}{7}\right) \\ &= \frac{abc^2}{4R^4}. \end{aligned}$$

Similarly,

$$\frac{1}{\alpha_2} = -4\sin^2\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} = -\frac{a^2bc}{4R^4}$$

and

$$\frac{1}{\alpha_3} = 4\sin\frac{\pi}{7}\sin^2\frac{2\pi}{7}\sin\frac{4\pi}{7} = \frac{ab^2c}{4R^4}.$$

The key idea is to specialize x in (7) and (8) and deduce formulas that can be transformed into relations between a, b, c by means of (1), (2) and (4). A few examples will clarify the method.

Making $x = 0$ in (7) and (8) first gives $8 = \sum_{k=1}^{3} \frac{\alpha_k}{x_k}$ and $-32 = \sum_{k=1}^{3} \frac{\alpha_k}{x_k^2}$ and then

$$8 = \frac{4R^4}{abc}\left(\frac{1}{cx_1} - \frac{1}{ax_2} + \frac{1}{bx_3}\right) \quad \text{and} \quad -32 = \frac{4R^4}{abc}\left(\frac{1}{cx_1^2} - \frac{1}{ax_2^2} + \frac{1}{bx_3^2}\right).$$

Finally, with the help of (4), we conclude that

$$\frac{1}{ac^3} + \frac{1}{ba^3} - \frac{1}{cb^3} = \frac{1}{R^4} \qquad \text{and} \qquad \frac{c}{ba^4} - \frac{a}{cb^4} - \frac{b}{ac^4} = \frac{2}{R^4}.$$

Because of the formulas $1 + \cos 2t = 2\cos^2 t$ and $1 - \cos 2t = 2\sin^2 t$, one can sensibly hope for some interesting results when taking $x = 1$ or -1 in (7) and (8). For example, we treat the case $x = -1$ in (7), leaving the similar case $x = 1$ to the reader. We are first led to

$$8 = \frac{4R^4}{abc}\left(\frac{1}{c(1+x_1)} - \frac{1}{a(1+x_2)} + \frac{1}{b(1+x_3)}\right)$$

with

$$1 + x_1 = 1 + \cos\frac{2\pi}{7} = 2\cos^2\frac{\pi}{7} = \frac{b^2}{2a^2}$$

and similarly

$$1 + x_2 = \frac{c^2}{2b^2}, \quad 1 + x_3 = \frac{a^2}{2c^2}.$$

This provides the relation

$$\frac{a}{b^3c^2} - \frac{b}{c^3a^2} + \frac{c}{a^3b^2} = \frac{1}{R^4}.$$

The calculations are slightly more complicated when applying (8); one can verify that the following formulas are obtained

$$\frac{1}{bc^4} + \frac{1}{ca^4} - \frac{1}{ab^4} = \frac{2abc}{7R^8}\left(= \frac{2\sqrt{7}}{7R^5}\right), \qquad \frac{a^3}{b^5c^2} - \frac{b^3}{c^5a^2} + \frac{c^3}{a^5b^2} = \frac{6}{R^4}.$$

Of course, the most courageous readers will be tempted to continue the pattern and first differentiate (8), getting

$$\frac{P''(x)}{(P(x))^2} - \frac{2(P'(x))^2}{(P(x))^3} = \sum_{k=1}^{3} \frac{-2\alpha_k}{(x-x_k)^3}.$$

Then, taking $x = 0$ leads to

$$\frac{b^3}{c^4} + \frac{c^3}{a^4} - \frac{a^3}{b^4} = \frac{5abc}{R^4}\left(= \frac{5\sqrt{7}}{R}\right).$$

Taking $x = 1$ and $x = -1$, with some additional effort, will produce two more amazing formulas:

$$\frac{1}{ca^6} - \frac{1}{ab^6} + \frac{1}{bc^6} = \frac{3abc}{7R^{10}}\left(= \frac{3\sqrt{7}}{7R^7}\right) \qquad \text{and} \qquad \frac{a^5}{b^7c^2} - \frac{b^5}{c^7a^2} + \frac{c^5}{a^7b^2} = \frac{31}{R^4}.$$

We conclude with a much easier exercise: obtain the relation

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{2}{R^2}$$

(found in [**1**]) from the decomposition into partial fractions of $\dfrac{P'(x)}{P(x)}$.

References

[**1**] L. Bankoff, J. Garfunkel, The Heptagonal Triangle, *Math. Magazine*, Jan. Feb 1973, p. 7-19.

[**2**] A. Altintas, Some Collinearities in the Heptagonal Triangle, *Forum Geometricorum*, 2016, p. 249-256.

[**3**] P. Steinbach, Golden Fields: A Case for the Heptagon, *Math. Magazine*, Feb. 1997, p. 22-31.

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