THE CONTEST CORNER

No. 46

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1 mars 2017**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

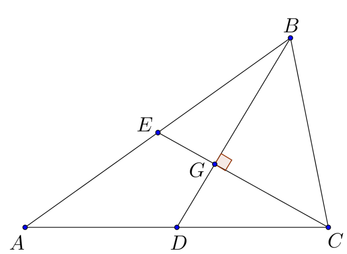
CC226. Dans le tableau qui suit, on a inscrit tous les produits de deux entiers positifs distincts de 1 à 100:

$$\begin{array}{ccccc} 1\cdot 2, & 1\cdot 3 & \ldots & 1\cdot 99, & 1\cdot 100 \\ & 2\cdot 3 & \ldots & 2\cdot 99, & 2\cdot 100 \\ & & \ddots & \vdots & \vdots \\ & & & & 99\cdot 100 \end{array}$$

Déterminer la somme de tous ces produits.

CC227. Supposons que $\{a_1, a_2, \ldots\}$ est une suite géométrique de nombres réels. La somme des n premiers termes est dénotée S_n. Si $S_{10} = 10$ et $S_{30} = 70$, déterminer la valeur de S_{40}.

CC228. Dans le triangle ABC, on sait que $AB = 2\sqrt{13}$ et $AC = \sqrt{73}$, puis que E et D sont les mi points de AB et AC respectivement. De plus, BD est perpendiculaire à CE. Déterminer la longueur de BC.



CC229. Un magasin a en vente des objets aux prix de 10, 25, 50 et 70 sous. Si Sandrine achète 40 objets et dépense sept dollars, quel est le plus grand nombre

possible d'objets à 50 sous dont elle aurait pu faire l'achat ?

CC230. Deux amis se sont mis d'accord pour se rencontrer à la bibliothèque entre 13h00 et 14h00. Ils ont décidé d'attendre 20 minutes l'un pour l'autre. Quelle est la probabilité qu'ils se rencontreront si leurs arrivées sont aléatoires durant l'heure en question et si leurs moments d'arrivée sont indépendants ?

. . . .

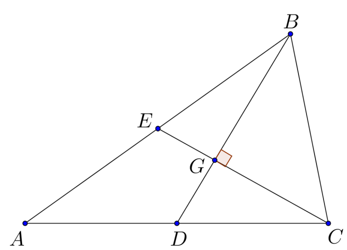
CC226. In the table below we write all the different products of two distinct counting numbers between 1 and 100:

$$\begin{array}{ccccc} 1\cdot 2, & 1\cdot 3 & \dots & 1\cdot 99, & 1\cdot 100 \\ & 2\cdot 3 & \dots & 2\cdot 99, & 2\cdot 100 \\ & & \ddots & \vdots & \vdots \\ & & & & 99\cdot 100 \end{array}$$

Find the sum of all of these products.

CC227. Suppose $\{a_1, a_2, \ldots\}$ is a geometric sequence of real numbers. The sum of the first n terms is S_n. If $S_{10} = 10$ and $S_{30} = 70$, determine the value of S_{40}.

CC228. In the triangle ABC, $AB = 2\sqrt{13}, AC = \sqrt{73}$, E and D are the midpoints of AB and AC, respectively. Furthermore, BD is perpendicular to CE. Find the length of BC.



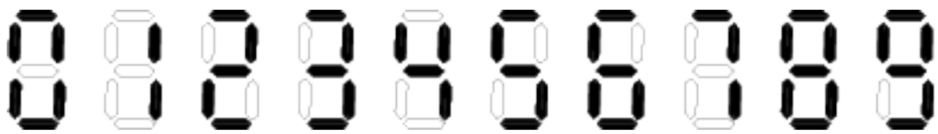
CC229. A store has objects that cost either 10, 25, 50, or 70 cents. If Sharon buys 40 objects and spends seven dollars, what is the largest quantity of the 50 cent items that could have been purchased?

CC230. Two friends agree to meet at the library between 1:00 P.M. and 2:00 P.M. Each agrees to wait 20 minutes for the other. What is the probability that they will meet if their arrivals occur at random during the hour and if the arrival times are independent?

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(6), p. 234–235.

CC176. A digital clock shows 4 digits using the following patterns:



Mathew plays the following game: "We subtract the number of lighted segments from the number which is shown. We repeat the operation on the second number and so on ..." For example, since 1234 uses 16 segments, the second number would be $1234 - 16 = 1218$. After performing this operation two times, Mathew gets 2015. What was his starting number?

Originally Problem 14 of the Championnat International des Jeux Mathématiques et Logiques 2014-15.

We received four correct submissions. We present a solution based on the submission by Alyssa Barnett.

We need to perform the inverse operation twice to get the original number.

The largest possible number of segments is 7 per digit; the smallest is 2 per digit. Given a 4 digit number, the maximum number that may be added to 2015 is 28, so the largest the first number (going backwards) can be is $2015 + 28 = 2043$. This tells us that the first two digits of our first number will definitely be 2 0. Since the number of lighted segments for the digits 2 0 is 11, and the last two digits will have at minimum 4 segments and at maximum 14 segments, the full 4-digit number must have 15 to 25 segments. Adding 15 and 25 to 2015, we find that the first number is no less than 2030 and no greater than 2040. Performing the inverse operation on $2030, 2033, \ldots, 2040$ reveals that 2038 yields the desired result: $2038 - 23 = 2015$.

Using the same process, we find that the largest possible second (i.e., original) number is $2038 + 28 = 2066$, which again has 2 and 0 as its first digits and therefore 15 to 25 segments. The original number must be no greater than 2053 and no less than 2063. Performing the inverse operation on $2053, 2054, \ldots, 2063$ reveals that 2057 yields the desired result: $2057 - 19 = 2038$.

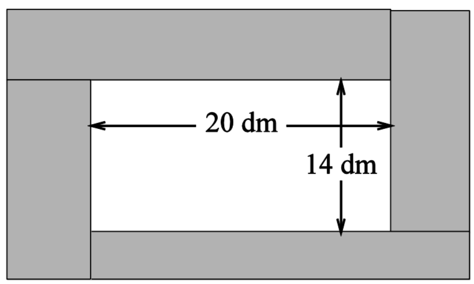
Mathew began with the number 2057.

CC177. An adventurer, born between 1901 and 1955, writes his memoirs when he is between 30 and 60 years of age. He wrote, "On this day celebrating my birthday, an extraordinary fact is made known to me: the weekday is exactly the same as the one I was born on." What was the age of the adventurer when he wrote that sentence?

Originally Problem 13 of the 2015-16 quarter finals of Le Championnat International des Jeux Mathématiques et Logiques.

We received no solutions to this problem.

CC178. A Modern Art painter Rec Tangle has painted the work of art represented here:



The white rectangle in the middle has length 20 dm and width 14 dm. The 4 grey rectangles all have equal areas and their dimensions, in decimeters, are non-zero integers. What is the minimal possible area of each of the grey rectangles? (The drawing is not to scale and a rectangle might be a square.)

Originally Problem 16 of the semi-final of the 2013-14 Championnat International des Jeux Mathématiques et Logiques.

We received two correct solutions and three incorrect solutions. We present the solution by Šefket Arslanagić.

Let t be the height of the top left rectangle, z the width of the top right rectangle, x the width of the bottom left rectangle and y the height of the bottom right rectangle. Then

$$x(14+y) = y(20+z) = z(14+t) = t(20+x) = F,$$

where F is the area of the 4 grey rectangles. We need to find the minimum value of F when x, y, z, and t are positive integers. Suppose the minimum value of F occurs for x_1, y_1, z_1, t_1 with $x_1 \neq z_1$. We may assume that $x_1 > z_1$ and because $x_1(14+y_1) = z_1(14+t_1)$ we must have $t_1 > y_1$. Taking $x = z = z_1, y = t = y_1$ would also give a valid solution with smaller area. This is a contradiction, so the minimum value of F occurs when $x = z$.

This yields

$$x(14+y) = y(20+z) = y(20+x),$$

which gives us $7x = 10y$. In order to minimize, we take $x = 10, y = 7$ which gives area $x(14 + y) = 210$.

CC179. Matthew creates a sequence of numbers starting from the number 7. Every number in his sequence is the sum of the digits of the square of the previous number, plus 1. For example, the second number in his sequence is 14 because $7^2 = 49$ and $4 + 9 + 1 = 14$. What is the 1000th number of Matthew's sequence?

Originally Problem 10 of the 2013-14 quarter final of Le Championnat International des Jeux Mathématiques et Logiques.

We received eight submissions, of which seven were correct. We present a representative solution.

Calculation of the first few terms reveals the sequence

$$7, 14, 17, 20, 5, 8, 11, 5, 8, 11, \dots,$$

which is easily identified as a 3-cycle from the 5th term onwards. We can represent the n-th term for $n \geq 5$ as

$$a_n = \begin{cases} 5, & \text{if } n = 2 \mod 3, \\ 8, & \text{if } n = 0 \mod 3, \\ 11, & \text{if } n = 1 \mod 3. \end{cases} \qquad n \geq 5$$

Since 1000 is 1 (mod 3), we find $a_{1000} = 11$.

CC180. The pages of a book are numbered $1, 2, 3, \dots$ A digit that appears in the number of the last page appears 20 times in the set of page numbers of the book. If the book had thirteen pages less, then the same digit would have been used 14 times in total. How many pages does the book have?

Originally Problem 8 of the 2013-14 semi final of Le Championnat International des Jeux Mathématiques et Logiques.

We received 3 correct solutions, and no incorrect solutions. We present the solution of Titu Zvonaru.

The book has 98 pages. It results from the following table:

	# of occurrences of digits									
# of pages	1	2	3	4	5	6	7	8	9	0
110	33	21	21	21	21	21	21	21	21	21
109	31	21	21	21	21	21	21	21	21	20
108	30	21	21	21	21	21	21	21	20	19
107	29	21	21	21	21	21	21	20	20	18
106	28	21	21	21	21	21	20	20	20	17
105	27	21	21	21	21	20	20	20	20	16
104	26	21	21	21	20	20	20	20	20	15
103	25	21	21	20	20	20	20	20	20	14
102	24	21	20	20	20	20	20	20	20	13
101	23	20	20	20	20	20	20	20	20	12
100	21	20	20	20	20	20	20	20	20	11
99	20	20	20	20	20	20	20	20	20	9
98	20	20	20	20	20	20	20	**20**	18	9
97	20	20	20	20	20	20	20	19	17	9
96	20	20	20	20	20	20	19	19	16	9
95	20	20	20	20	20	19	19	19	15	9
94	20	20	20	20	19	19	19	19	14	9
93	20	20	20	19	19	19	19	19	13	9
92	20	20	19	19	19	19	19	19	12	9
91	20	19	19	19	19	19	19	19	11	9
90	19	19	19	19	19	19	19	19	10	9
89	19	19	19	19	19	19	19	19	9	8
88	19	19	19	19	19	19	19	18	8	8
87	19	19	19	19	19	19	19	16	8	8
86	19	19	19	19	19	19	18	15	8	8
85	19	19	19	19	19	18	18	**14**	8	8