Conjugate numbers

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The reader probably has seen seemingly difficult geometry problems that can be solved nearly instantaneously if one point is replaced by its reflection in some line. Symmetry arguments are very useful in algebra as well.

In this article, we will consider situations where a number of the form $a + b\sqrt{d}$ appears, and the solution involves its *conjugate* $a - b\sqrt{d}$. We will see how this simple step helps to solve problems of varying difficulty in algebra and analysis, from easy estimates to contest and olympiad problems. Many of our examples can serve as a first acquaintance with deep mathematical theories that the reader can then explore further.

Pairs of conjugates appear naturally in solving a quadratic equation in which the discriminant is not a perfect square. For example, the equation $\lambda^2 - \lambda - 1 = 0$ has a pair of conjugate roots $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$. We will come back to this idea. For now, let us begin by considering the mechanism of "transferring" square roots

... from numerator to denominator and vice versa

Suppose you are solving a problem and you get $\frac{1}{3-\sqrt{7}}$, but the answer at the back of the book is $\frac{3+\sqrt{7}}{2}$. Do not rush into looking for an error : these numbers are equal since $(3+\sqrt{7})(3-\sqrt{7}) = 3^2 - 7 = 2$. Here are some characteristic examples where rationalizing the numerator proves useful.

1. Compute the sum

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$

Indeed, this sum collapses (like a telescope, if you will) if we re-write it as

$$(\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) + \cdots + (\sqrt{100}-\sqrt{99}) = -1 + 10 = 9.$$

2. Prove that for any natural numbers m and n, we have

$$\left|\frac{m}{n} - \sqrt{2}\right| \geq \frac{1}{\alpha n^2}, \tag{1}$$

where $\alpha = \sqrt{3} + \sqrt{2}$.

Indeed, we have

$$\left|\frac{m - n\sqrt{2}}{n}\right| = \frac{|m^2 - 2n^2|}{(m+n\sqrt{2})n} \geq \frac{1}{(m+n\sqrt{2})n} \tag{2}$$

since the number $|m^2 - 2n^2|$ is a non-zero integer [Ed. The fact that it is impossible to have $m^2 = 2n^2$ is one of the oldest theorem in number theory]. Towards contradiction, suppose (1) does not hold ; that is,

$$m < n\sqrt{2} + \frac{1}{\alpha n}$$

and hence

$$\begin{aligned} n(m + n\sqrt{2}) &< n\left(2n\sqrt{2} + \frac{1}{\alpha n}\right) \\ &= 2n^2\sqrt{2} + \frac{1}{\sqrt{3}+\sqrt{2}} \\ &= 2n^2\sqrt{2} + \sqrt{3} - \sqrt{2} \\ &\leq n^2(\sqrt{2}+\sqrt{3}) = \alpha n^2. \end{aligned} \tag{3}$$

However, (1) follows from (2) and (3). Therefore, out assumption is not correct and (1) holds.

Note that (1) shows that the number $\sqrt{2}$ is badly approximated by fractions with small denominators (we will see other approximations of $\sqrt{2}$ later, see example 4). Of course, (1) holds also for all $\alpha > \sqrt{3} + \sqrt{2}$, but the constant α here is not the smallest possible. Analogous inequalities, but with different values for α hold for other square roots.

3. Find the limit of the sequence defined by $a_n = (\sqrt{n^2+1} - n)n$.

We have :

$$a_n = (\sqrt{n^2+1} - n)n = \frac{n}{\sqrt{n^2+1} - n} = \frac{1}{1 + \sqrt{1 + 1/n^2}}.$$

It is now clear that the sequence a_n is decreasing and approaching $1/2$.

4. Consider two sequences $a_n = \sqrt{n+1} + \sqrt{n}$ and $b_n = \sqrt{4n+2}$. Prove that

a) $[a_n] = [b_n]$ (where $[x]$ denotes the integer part of x),

b) $0 < b_n - a_n < \frac{1}{16n\sqrt{n}}$.

Triple irrationality arises in the difference $b_n - a_n$ and we will come back to these types of irrationalities (see problem 8). For now, we will consider $a_n = \sqrt{n+1}+\sqrt{n}$ as a whole. Notice that the number $a_n^2 = 2n+1+2\sqrt{n(n+1)}$ lies between $4n+1$ and $4n+2 = b_n^2$ since $n < \sqrt{n(n+1)} < n+1$. As such, we already proved that $a_n < b_n$, which is the left inequality in part b). Furthermore, the number $b_n^2 = 4n+2$ leaves remainder of 2 when divided by 4 and hence cannot be a perfect square. Therefore, the square root of the integer $[b_n]$ is no more than $4n+1$. Part a) now follows from the inequalities $[b_n] \leq \sqrt{4n+1} < a_n < b_n$.

It remains to estimate the difference $b_n - a_n$ from above. Here, we will "transfer" square roots from numerator to denominator twice :

$$\begin{aligned}\sqrt{4n+2} - \sqrt{n} - \sqrt{n+1} &= \frac{2n+1-2\sqrt{n(n+1)}}{\sqrt{4n+2}+\sqrt{n}+\sqrt{n+1}} \\ &= \frac{1}{\sqrt{4n+2}+\sqrt{n}+\sqrt{n+1}} \cdot \frac{1}{2n+1+2\sqrt{n(n+1)}} \\ &\le \frac{1}{(2\sqrt{n}+\sqrt{n}+\sqrt{n})(2n+2n)} \\ &= \frac{1}{16n\sqrt{n}}.\end{aligned}$$

(We got lucky above since $(2n+1)^2 - 4n(n+1) = 1$.)

Replace a plus with a minus

We already mentioned the usefulness of symmetry in geometrical problems. In algebra, one type of symmetry can be achieved by replacing a plus sign with a minus sign. We will use the following fact (a and b are rational, $\sqrt{d}$ is irrational) :

$$(a+b\sqrt{d})^n = p+q\sqrt{d} \quad \Longrightarrow \quad (a-b\sqrt{d})^n = p-q\sqrt{d}. \tag{4}$$

[Ed. This is a special case of the Irrational Conjugate Theorem, which states that if $a+b\sqrt{c}$, (a, b, c rational, $\sqrt{c}$ irrational,) is a root of a polynomial with rational coefficients, then $a-b\sqrt{c}$ is also a root. We can think of this as saying that the field of rational numbers cannot "see" the difference between $a+b\sqrt{c}$ and its conjugate. However, the reader is warned that this is true only for the arithmetic operations $+, -, *, /$, and that conjugates are not interchangeable with regard to other properties (such as inequalities and convergence).]

5. Prove that the equation

$$(x+y\sqrt{5})^4 + (z+t\sqrt{5})^4 = 2+\sqrt{5}$$

does not have rational solutions for x, y, z and t.

Of course, you can find the sum on the left hand side not containing square roots (and equate it to 2) and the coefficient of $\sqrt{5}$ (and equate it to 1). But it is not clear what to do next with the awkward resulting system of equations. Instead, let us use (4) and replace the sign in front of $\sqrt{5}$:

$$(x-y\sqrt{5})^4 + (z-t\sqrt{5})^4 = 2-\sqrt{5}.$$

Now on the left we have a nonnegative number, while on the right we have a negative one.

6. Prove that there are infinitely many pairs (x, y) of natural numbers such that

$$|x^2 - 2y^2| = 1. \tag{5}$$

In other words, do the hyperbolas $2y^2 - x^2 = 1$ and $2y^2 - x^2 = -1$ go through infinitely many lattice points in the grid ?

Several such pairs are easily found by hand : $(1, 2), (3, 2), (7, 5), (17, 12), \ldots$ But how do we continue such a set ? Can we write down a general formula for these solutions ? To find answers to these questions, let us turn to the number $1 + \sqrt{2}$. The consistent pattern becomes evident in the following table :

n	$(1+\sqrt{2})^n$	x_n	y_n	$x_n^2 - 2y_n^2$	$(1-\sqrt{2})^n$
1	$1+\sqrt{2}$	1	1	$1-2=-1$	$1-\sqrt{2}$
2	$3+2\sqrt{2}$	3	2	$9-8=1$	$3-2\sqrt{2}$
3	$7+5\sqrt{2}$	7	5	$49-50=-1$	$7-5\sqrt{2}$
4	$17+12\sqrt{2}$	17	12	$289-288=1$	$17-12\sqrt{2}$
5	$41+29\sqrt{2}$	41	29	$1681-1682=-1$	$41-29\sqrt{2}$
...	...	...	...	...	...

You can see that the coefficients in $x_n + y_n\sqrt{2} = (1 + \sqrt{2})^n$ will give the desired pair. To prove this, take a closer look at the table and use (4) again :

$$x_n - y_n\sqrt{2} = (1 - \sqrt{2})^n.$$

Multiplying the last two equalities, we get :

$$x_n^2 - 2y_n^2 = (1 + \sqrt{2})^n(1 - \sqrt{2})^n = ((1 + \sqrt{2})(1 - \sqrt{2}))^n = (-1)^n$$

and so the overall expression is equal to 1 or -1. By adding and subtracting the same two equalities, we also get the expressions for x_n and y_n :

$$x_n = ((1 + \sqrt{2})^n + (1 - \sqrt{2})^n)/2,$$
$$y_n = ((1 + \sqrt{2})^n - (1 - \sqrt{2})^n)/2\sqrt{2}.$$

In this above problem about integers, can we get away with using only integers and avoiding irrational numbers $1 \pm \sqrt{2}$? Knowing the answer, we can easily express the pair (x_{n+1}, y_{n+1}) through the previous pair (x_n, y_n) : from $x_{n+1} + y_{n+1}\sqrt{2} = (x_n + y_n\sqrt{2})(1 - \sqrt{2})$, we get :

$$x_{n+1} = x_n + 2y_n, \quad y_{n+1} = x_n + y_n. \tag{6}$$

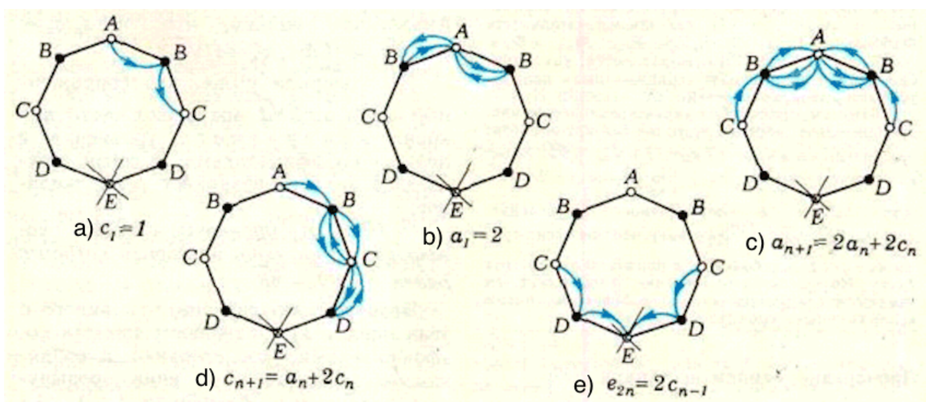
You could have arrived at this recurrence relation from seeing the first couple of solutions and checking that $|x_n^2 - 2y_n^2| = |x_{n+1}^2 - 2y_{n+1}^2|$. By adding the initial condition $x_1 = y_1 = 1$, we can prove by induction that $|x_n^2 - 2y_n^2| = 1$ for all n. Next, by expressing (x_n, y_n) through (x_{n+1}, y_{n+1}) you can prove that this sequence exhausts all solutions to (5) in natural numbers x and y. Similarly, one can solve any *Pell equation* $x^2 - dy^2 = c$, but the initial equation might have several series of solutions.

Recurrence relations like (6) arise not only in number theory, but also in areas like analysis and probability theory. Here is an example of a combinatorial problem like this (originally used in the 1979 International Math Olympiad in London) :

7. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Find the number e_m of distinct paths of exactly m jumps starting at A and ending at E.

Colour the vertices of the octagon in alternating black and white colours. It then becomes clear that $e_{2k-1} = 0$ for all k since the colour of the vertex changes with each jump. Let a_n and c_n denote the number of ways that the frog can get in $2n$ jumps to the vertex A and one of the vertices C, respectively (from the symmetry, it is clear that the frog can get to each vertex C in the same number of ways).

Consider the following graphics :



We have :

a) To get from A to C in two jumps can only be done in one way, so $c_1 = 1$.

b) You can get from A to A in two jumps in two different ways, so $a_1 = 2$.

c) You can get from C to A in two ways and from A in two ways, so $a_{n+1} = 2a_n + 2c_n$.

d) You can get to C from A in one way and from C in two ways, so $c_{n+1} = a_n + 2c_n$.

e) You can get from C to E in two ways, so $e_{2n} = 2c_{n-1}$.

You can verify that $a_1 = 2$, $c_1 = 1$ and

$$\begin{cases} a_{n+1} = 2a_n + 2c_n, \\ c_{n+1} = a_n + 2c_n. \end{cases} \tag{7}$$

Then we have $e_{2n} = 2c_{n-1}$.

How can we find the formula for a_n and c_n ? Let us re-write the recurrence (7) as

$$a_{n+1} + c_{n+1}\sqrt{2} = (a_n + c_n\sqrt{2})(2+\sqrt{2}) \tag{8}$$

and, as you might have already guessed,

$$a_{n+1} - c_{n+1}\sqrt{2} = (a_n - c_n\sqrt{2})(2+\sqrt{2}) \tag{9}$$

From here, by induction using (7), we get

$$a_n + c_n\sqrt{2} = (2+\sqrt{2})^{n-1}(a_1 + c_1\sqrt{2}) = (2+\sqrt{2})^n$$

and

$$a_n - c_n\sqrt{2} = (2-\sqrt{2})^{n-1}(a_1 - c_1\sqrt{2}) = (2-\sqrt{2})^n.$$

Therefore, $c_n = ((2+\sqrt{2})^n - (2-\sqrt{2})^n)/2\sqrt{2}$ and since $e_{2n} = 2c_{n-1}$, we finally get

$$e_{2n} = ((2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1})/\sqrt{2}$$

with $e_{2n-1} = 0$. Problem is solved. It only remains unclear how, given the original problem, one could have dreamt up the idea to use formulas containing $\pm\sqrt{2}$. (The original Olympiad problem did have the final answer in its statement.)

Actually, the conjugate numbers would have appeared automatically if we used basic linear algebra to solve (7). First, one could figure out what geometric progressions $a_n = a_0\lambda^n$ and $c_n = c_0\lambda^n$ satisfy the recurrence : the values for which such progressions exist are called *characteristic values* arising from the so-called *characteristic equation*. For (7), the characteristic equation is $\lambda^2 + 2 - 4\lambda = 0$ and its roots are exactly $2 \pm \sqrt{2}$. Knowing these roots, we can get the solutions to the recurrence as linear combination of these roots. It is therefore not surprising that many integer recurrences with characteristic equation having integer coefficients (such as (6), (7) and the Fibonacci sequence) contain conjugate irrational numbers.

Note that the greater characteristic number determines how fast the sequence is growing : for large n in example 7 we have $e_n \approx (2+\sqrt{2})^n/\sqrt{2}$ or, alternatively, $\lim_{n\to\infty}(e_{n+1}/e_n) = 2+\sqrt{2}$. For problem 6, we have an analogous observation that $\lim_{n\to\infty}(x_n/y_n) = \sqrt{2}$ shows that both summands in $x_n + y_n\sqrt{2}$ are approximately equal for large n. We will see an interesting generalization of this fact in the following problem with many conjugate irrationalities.

Change all the signs

8. Let $(1+\sqrt{2}+\sqrt{3})^n = q_n + r_n\sqrt{2} + s_n\sqrt{3} + t_n\sqrt{6}$, where q_n, r_n, s_n and t_n are all integers. Find the following limits :

$$\lim_{n\to\infty} r_n/q_n, \quad \lim_{n\to\infty} s_n/q_n, \quad \text{and} \quad \lim_{n\to\infty} t_n/q_n.$$

We can express $(q_{n+1}, r_{n+1}, s_{n+1}, t_{n+1})$ in terms of (q_n, r_n, s_n, t_n) using the fact that

$$q_{n+1} + r_{n+1}\sqrt{2} + s_{n+1}\sqrt{3} + t_{n+1}\sqrt{6} = (1+\sqrt{2}+\sqrt{3})(q_n + r_n\sqrt{2} + s_n\sqrt{3} + t_n\sqrt{6}).$$

But our previous experience tells us that easier formulas will arise not for q_n, r_n, s_n and t_n, but for some of their combinations. We already know one such combination : namely, that $q_n + r_n\sqrt{2} + s_n\sqrt{3} + t_n\sqrt{6} = (1+\sqrt{2}+\sqrt{3})^n$. It is not hard to imagine what others will be like. Together with $\lambda_1 = 1+\sqrt{2}+\sqrt{3}$, consider other conjugates $\lambda_2 = 1-\sqrt{2}+\sqrt{3}$, $\lambda_3 = 1+\sqrt{2}-\sqrt{3}$ and $\lambda_4 = 1-\sqrt{2}-\sqrt{3}$. Then

$$\begin{aligned} q_n - r_n\sqrt{2} + s_n\sqrt{3} - t_n\sqrt{6} &= \lambda_2^n, \\ q_n + r_n\sqrt{2} - s_n\sqrt{3} - t_n\sqrt{6} &= \lambda_3^n, \\ q_n - r_n\sqrt{2} - s_n\sqrt{3} + t_n\sqrt{6} &= \lambda_4^n. \end{aligned}$$

We can now express q_n, r_n, s_n, t_n through $\lambda_1, \lambda_2, \lambda_3, \lambda_4$:

$$\begin{aligned} q_n &= (\lambda_1^n + \lambda_2^n + \lambda_3^n + \lambda_4^n)/4, \\ r_n &= (\lambda_1^n - \lambda_2^n + \lambda_3^n - \lambda_4^n)/4\sqrt{2}, \\ s_n &= (\lambda_1^n + \lambda_2^n - \lambda_3^n - \lambda_4^n)/4\sqrt{3}, \\ t_n &= (\lambda_1^n - \lambda_2^n - \lambda_3^n + \lambda_4^n)/4\sqrt{6}. \end{aligned}$$

Now note that $\lambda_1 > |\lambda_2|, \lambda_1 > |\lambda_3|, \lambda_1 > |\lambda_4|$. Hence,

$$\lim_{n\to\infty} r_n/q_n = \lim_{n\to\infty} \frac{1-(\lambda_2/\lambda_1)^n + (\lambda_3/\lambda_1)^n - (\lambda_4/\lambda_1)^n}{1+(\lambda_2/\lambda_1)^n + (\lambda_3/\lambda_1)^n + (\lambda_4/\lambda_1)^n} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Analogously, we find that $\lim_{n\to\infty} s_n/q_n = \frac{1}{\sqrt{3}}$ and $\lim_{n\to\infty} t_n/q_n = \frac{1}{\sqrt{6}}$.

We mentioned earlier that conjugate numbers $a \pm b\sqrt{d}$ often arise as roots of a quadratic equation with integer coefficients. In light of the previous problem, the following one arises naturally :

9. Find a quartic equation with integer coefficients with $1+\sqrt{2}+\sqrt{3}$ as one of its roots.

In light of the previous example (and using the same notation), we can suspect that the roots of this equation will be $\lambda_1, \lambda_2, \lambda_3$ and λ_4. As such, we can write that equation as

$$(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4) = 0,$$

or, equivalently,

$$(x-1-\sqrt{2}-\sqrt{3})(x-1+\sqrt{2}-\sqrt{3})(x-1-\sqrt{2}+\sqrt{3})(x-1+\sqrt{2}+\sqrt{3}) = 0.$$

After some algebraic manipulations, we get :

$$\begin{aligned} ((x-1)^2 - 5 - 2\sqrt{6})((x-1)^2 - 5 + 2\sqrt{6}) &= 0, \\ (x^2 - 2x - 4)^2 - 24 &= 0, \\ x^4 - 4x^3 - 4x^2 - 16x - 8 &= 0. \end{aligned}$$

In fact, this is the lowest degree equation with integer coefficients with a root $\lambda_1 = 1+\sqrt{2}+\sqrt{3}$: prove this fact.

We will end this article with several exercises which continue some of the themes that we have touched upon. Some of them, however, require innovative ideas. [Ed. Note that this method does have its limitations; in particular, one should be very careful when using it with problems involving inequalities or limits.]

Exercises.

1. Which is larger : $\sqrt{1979}+\sqrt{1980}$ or $\sqrt{1978}+\sqrt{1981}$?

2. Prove that for all positive x we have

$$|\sqrt{x^2+1}-x-1/2x|<\frac{1}{8x^3}.$$

3. Sketch the graph of the function $y=\sqrt{x^2-1}$ and prove that for all $|x|\geq 1$ we have

$$0<|x|-\sqrt{x^2-1}\leq\frac{1}{|x|}.$$

4. In the formula $\sqrt{2}=1+\frac{1}{\sqrt{2}+1}$, replace the $\sqrt{2}$ in the denominator of the right-hand side with the same expression to get

$$\sqrt{2}=1+\cfrac{1}{2+\frac{1}{\sqrt{2}+1}}.$$

In this formula, again replace the lowest $\sqrt{2}$ by $1+\frac{1}{\sqrt{2}+1}$ and so on n times. If we now replace the remaining $\sqrt{2}$ by 1 or 2, we will get two rational numbers p_n and q_n. Prove that $\sqrt{2}$ is bounded by these two numbers and that $\lim_{n\to\infty}p_n=\lim_{n\to\infty}q_n=\sqrt{2}$. (Did we see these numbers in one of the previous problems?)

5. Prove that the equations $x^2-3y^2=1$ and $x^2-3y^2=2$ have infinitely many solutions in integers.

6. Prove that the function $y=\ln(\sqrt{1+x^2}+x)$ is odd and sketch its graph.

7. a) Prove that for any natural number n, we have

$$2(\sqrt{n+1}-1)<1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}-1.$$

b) Prove that the sequence

$$U_n=1+\frac{1}{\sqrt[4]{2^3}}+\frac{1}{\sqrt[4]{3^3}}+\cdots+\frac{1}{\sqrt[4]{n^3}}-4\sqrt[4]{n}$$

is decreasing and has a limit.

8. a) Prove that the sequence $\{(2+\sqrt{3})^n\}$, where $\{x\}$ denotes the fractional part of x, converges and find its limit.

b) What are the first 100 digits in the decimal expansion of $(\sqrt{50}+7)^{100}$?

9. Prove that for any natural number d, which is not a perfect square, there exists α such that for any m and n,

$$\left|\frac{m}{n}-\sqrt{d}\right| \geq \frac{1}{\alpha n^2}.$$

10. Prove that for any natural number n, the number $[(35+\sqrt{1157})^n/2^n]$ is divisible by 17 and in general for any natural numbers k and n, the number $[(2k+1+\sqrt{4k^2+1})^n/2^n]$ is divisible by k.

11. Prove that for any $p>2$, there exists β such that for any n we have

$$\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{2+\beta}}}}}_{n \text{ radicals}} = \beta^{2^n}+\beta^{-2^n}.$$

12. Prove that the sequence $b_m = 1+17m^2$ contains infinitely many perfect squares.

13. Find a quadratic equation with integer coefficients with a root $(3+\sqrt{5})/4$.

14. Find a fourth degree equation with roots $\pm\sqrt{p}\pm\sqrt{q}$ and solve it as a biquadratic equation. By comparing the solution with the given roots, derive the following well-known formulas for the double radicals :

$$\sqrt{a\pm\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} \pm \sqrt{\frac{a-\sqrt{a^2-b}}{2}},$$

where $a^2>b>0$ and $a>0$.

15. Rationalize the denominators of the following fractions :

$$\text{a) } \frac{1}{1+\sqrt{2}+\sqrt{3}}; \qquad \text{b) } \frac{1}{\sqrt{10}+\sqrt{14}+\sqrt{21}+\sqrt{15}}.$$

16. A frog can jump from any vertex to any other vertex of an equilateral triangle ABC. Find the number a_n of distinct paths of exactly n jumps starting at A and ending at A. Prove that $\lim_{n\to\infty} a_{n+1}/a_n$ exists and find it.

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