# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

*Statements of the problems in this section originally appear in 2015 : 41(6), p. 260–263.*

## **4051**. *Proposed by Arkady Alt.*

Let $a, b$ and $c$ be the side lengths of a triangle. Prove that

$$(a+b+c)\left(a^2b^2+b^2c^2+c^2a^2\right) \geq 3abc\left(a^2+b^2+c^2\right).$$

*We received eleven correct solutions. We present two solutions.*

*Solution 1, by Michel Bataille.*

A key to a solution is contained in solution 2 to ***Crux*** problem **3991** published in 41 (1) (December 2015).

Setting $a = \frac{y+z}{2}$, $b = \frac{z+x}{2}$, $c = \frac{x+y}{2}$ transforms the proposed inequality into

$$x^5+y^5+z^5+x^2y^2z+x^2yz^2+xy^2z^2 \geq x^3y^2+x^2y^3+y^3z^2+y^2x^3+z^3x^2+z^2x^3, \quad (1)$$

where $x$, $y$, $z$ are positive real numbers. The general Schur inequality is

$$u^r(u-v)(u-w)+v^r(v-w)(v-u)+w^r(w-u)(w-v) \geq 0$$

for $u, v, w \geq 0$ and $r$ real. We take $r = \frac{1}{2}$ and $u = x^2$, $v = y^2$, $w = z^2$ and obtain

$$x(x^2-y^2)(x^2-z^2)+y(y^2-z^2)(y^2-x^2)+z(z^2-x^2)(z^2-y^2) \geq 0.$$

Expanding and arranging directly leads to (1).

*Solution 2, by Titu Zvonaru.*

By Consequence 16.3, p. 156 from [1], all symmetric three-variable polynomials of degree less than or equal to five achieve their maximum and minimum values on $\mathbb{R}^*$ at $(a, b, c)$ if and only if $(a-b)(b-c)(c-a) = 0$ or $abc = 0$. It thus suffices to prove the given inequality for $b = c$ and $c = 0$.

If $b = c$, then we have to prove that

$$\begin{aligned}
(a+2b)(2a^2b^2+b^4) &\geq 3ab^2(a^2+2b^2)\\
2a^3b^2+ab^4+4a^2b^3+2b^5 &\geq 3a^3b^2+6ab^4\\
4a^2b^3+2b^5 &\geq a^3b^2+5ab^4\\
b^2(a-b)^2(2b-a) &\geq 0,
\end{aligned}$$

which is true by the triangle inequality.

If $c = 0$, then we have to prove that

$$(a+b)a^2b^2 \geq 0,$$

which is true.

Equality holds if and only if $a = b = c$.

[1] Z. Cvetkovski, *Inequalities - Theorems, Techniques and Selected Problems*, Springer-Verlag, 2012.

**4052**. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let $k < 0$ be a fixed real number. Let $a, b, c$ and $d$ be real numbers such that $a+b+c+d = 0$ and $ab+bc+cd+da+ac+bd = k$. Prove that $abcd \geq -k^2/12$ and determine when equality holds.

*We received six submissions, of which five were correct and complete. We present the solution by Oliver Geupel, slightly modified by the editor.*

It is sufficient to consider the case when three of the numbers $a$, $b$, $c$, $d$ have the same sign. Otherwise, $abcd \geq 0 > -k^2/12$, and we're done.

By symmetry, there is no loss of generality in assuming that it is $a$, $b$, and $c$ which have the same sign. Then $a+b+c+d = 0$ gives us $d = -(a+b+c)$, so

$$\begin{aligned} k &= ab+bc+cd+da+ac+bd \\ &= (a+b+c)d + (ab+bc+ca) \\ &= -(a+b+c)^2 + (ab+bc+ca) \\ &= -(a^2+b^2+c^2+ab+bc+ca), \end{aligned}$$

and the inequality we want to prove can be rewritten as

$$-abc(a+b+c) \geq -(a^2+b^2+c^2+ab+bc+ca)^2/12,$$

or, equivalently,

$$12abc(a+b+c) \leq (a^2+b^2+c^2+ab+bc+ca)^2. \qquad (1)$$

Let $x = |a|$, $y = |b|$, $z = |c|$, so that $x$, $y$, $z$ are nonnegative real numbers. Since $a$, $b$ and $c$ have the same sign by assumption,

$$\begin{aligned} &abc(a+b+c) = xyz(x+y+z), \text{ and} \\ &a^2+b^2+c^2+ab+bc+ca = x^2+y^2+z^2+xy+yz+zx, \end{aligned}$$

so it is sufficient to prove that (1) holds with $a$,$b$ and $c$ replaced by $x$, $y$ and $z$, respectively.

By the AM-GM Inequality (where the right hand side is treated as a sum of 8 terms, with terms repeated as indicated by the coefficients), we have

$$8x^2yz \le x^4 + 2x^3y + 2x^3z + 3y^2z^2. \tag{2}$$

The equality holds only when $x^4 = x^3y = x^3z = y^2z^2$, that is when either $x = y = z$ or $x = yz = 0$. Summing up inequality (2) and its two cyclic variants, and adding terms to both sides so we can complete the square on the right hand side, we obtain

$$12xyz(x+y+z) \le (x^2+y^2+z^2+xy+yz+zx)^2, \tag{3}$$

where the equality holds if and only if $x = y = z$. This shows that (1) holds, and thus concludes the proof that $abcd \ge -k^2/12$.

It follows from the preceding steps that the equality holds if and only if three of the four numbers $a$, $b$, $c$, and $d$ are equal. A straightforward computation (from $a+b+c+d = 0$ and $abcd = -k^2/12$) shows that the common value is $\pm\sqrt{-k/6}$, and that the fourth number has the value $\mp 3\sqrt{-k/6}$.

**4053**. *Proposed by Šefket Arslanagić.*

Prove that

$$\frac{\cos\alpha\cos\beta}{\cos\gamma} + \frac{\cos\beta\cos\gamma}{\cos\alpha} + \frac{\cos\alpha\cos\gamma}{\cos\beta} \ge \frac{3}{2},$$

where $\alpha, \beta$ and $\gamma$ are angles of an acute triangle.

*We received 13 correct solutions. We present a composite of essentially the same solution by José Luis Díaz-Barrero, Dionne Bailey, Elsie Campbell, and Charles R. Diminnie (joint), Henry Ricardo, and Lorian Saceanu.*

Since $\alpha, \beta, \gamma \in (0, \frac{\pi}{2})$ we have

$$\begin{aligned}
\frac{\cos\alpha\cos\beta}{\cos\gamma} &= \frac{\cos\alpha\cos\beta}{\cos\gamma} \cdot \frac{\tan\alpha+\tan\beta}{\tan\alpha+\tan\beta} \\
&= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\gamma} \cdot \frac{1}{\tan\alpha+\tan\beta} \\
&= \frac{\sin(\pi-\gamma)}{\cos\gamma} \cdot \frac{1}{\tan\alpha+\tan\beta} \\
&= \frac{\tan\gamma}{\tan\alpha+\tan\beta}.
\end{aligned}$$

By the well-known Nesbitt's Inequality which states that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}$$

with equality if and only if $a = b = c$, we then have

$$\sum_{\text{cyc}} \frac{\cos\alpha\cos\beta}{\cos\gamma} = \sum_{\text{cyc}} \frac{\tan\gamma}{\tan\alpha+\tan\beta} \ge \frac{3}{2}$$

with equality if and only if $\alpha = \beta = \gamma$; in other words, if the given triangle is equilateral.

*Editor's comment.* Both Ricardo and Diminnie et al pointed out that the current problem is the special case when $m = 0$ of problem #5381 in the January 2016 issue of *School, Science and Mathematics* :

If $A$, $B$, $C$ are angles of an acute triangle, then

$$\sum \left( \frac{\cos A \cos B}{\cos C} \right)^{m+1} \geq \frac{3}{2^{m+1}}$$

for all nonnegative integers $m$.

Interestingly, this general problem was proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu, two regular contributors to the ***Crux*** problem section. Ricardo actually gave a proof to the general inequality as well as a proof for the above case. The general proof uses the same argument in the featured solution above together with the power mean inequality and it appeared as solution 3 in the April 2016 issue of that journal (see `www.ssma.org/publications`).

## 4054. *Proposed by Mihaela Berindeanu.*

Find a prime $p$ such that the number

$$(p^2 - 4)^2 - 117(p^2 - 4) + 990$$

has a minimum digit sum.

*We received eight correct and complete solutions, all of which were very similar. We present the solution by Joseph DiMuro.*

Let

$$f(p) = (p^2 - 4)^2 - 117(p^2 - 4) + 990.$$

If $p \neq 3$ is a prime, then $p = 3n \pm 1$ for some integer $n$. Then

$$p^2 - 4 = 9n^2 \pm 6n - 3,$$

a multiple of 3 ; thus, $(p^2 - 4)^2$ is a multiple of 9. But 117 and 990 are also multiples of 9. So $f(p)$ is a multiple of 9, as is its digit sum.

If we had $f(p) = 0$, then by the quadratic formula we would have

$$p^2 - 4 = \frac{117 \pm \sqrt{9729}}{2},$$

which has no integer solutions, so $f(p) \neq 0$. Therefore, the digit sum of $f(p)$ must be at least 9 when $p$ is a prime other than 3.

However, the digit sum of $f(3) = 430$ is 7, so the minimum digit sum is 7, attained for $p = 3$.

**4055**. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that if $x, y > 0, x \neq y$ and $0 < a < b < \frac{1}{2} < c < d < 1$ then :

$$x\Big[\Big(\frac{y}{x}\Big)^a+\Big(\frac{y}{x}\Big)^d-\Big(\frac{y}{x}\Big)^b-\Big(\frac{y}{x}\Big)^c\Big] > y\Big[\Big(\frac{x}{y}\Big)^b+\Big(\frac{x}{y}\Big)^c-\Big(\frac{x}{y}\Big)^a-\Big(\frac{x}{y}\Big)^d\Big].$$

*We received three correct solutions and feature two of them that are similar.*

*Solution 1, by Michel Bataille.*

With $t = \frac{x}{y}$, the inequality can be re-written as

$$(t^d + t^{1-d}) - (t^c + t^{1-c}) > (t^b + t^{1-b}) - (t^a + t^{1-a}). \tag{1}$$

Let us fix $t > 0, t \neq 1$ and set $f(u) = t^u$ and $g(u) = f(u) + f(1-u)$ so that (1) is just

$$g(d) - g(c) > g(b) - g(a). \tag{2}$$

From the Mean Value Theorem, we have

$$g(b) - g(a) = (b-a)g'(\alpha), \qquad g(d) - g(c) = (d-c)g'(\beta)$$

for some $\alpha \in (a, b)$ and $\beta \in (c, d)$.

Since $g'(u) = (\ln t)(t^u - t^{1-u}) = (\ln t)(f(u) - f(1-u))$, (2) becomes

$$(d-c)(\ln t)(f(\beta) - f(1-\beta)) > (b-a)(\ln t)(f(\alpha) - f(1-\alpha)). \tag{3}$$

Applying the Mean Value Theorem again, we have $f(\beta)-f(1-\beta) = (2\beta-1)(\ln t)t^\sigma$ and $f(\alpha) - f(1-\alpha) = (2\alpha-1)(\ln t)t^\tau$ with $\sigma$ between $\beta$ and $1-\beta$ and $\tau$ between $\alpha$ and $1-\alpha$.

Substituting into (3) and because $(\ln t)^2 > 0$, we are reduced to proving

$$(d-c)(2\beta-1)t^\sigma > (b-a)(2\alpha-1)t^\tau. \tag{4}$$

Now, on the one hand $d - c > 0,\ t^\sigma > 0,\ 2\beta - 1 > 0$ (note that $\beta \in (c, d)$, hence $\beta > \frac{1}{2}$) and on the other hand, $b - a > 0,\ t^\tau > 0,\ 2\alpha - 1 < 0$ (since $\alpha \in (a, b)$). Thus,

$$(d-c)(2\beta-1)t^\sigma > 0 > (b-a)(2\alpha-1)t^\tau$$

and (4) follows.

*Solution 2, by Daniel Sitaru and Leonard Giugiuc.*

Let $f : [0,1] \to \mathbb{R}, f(\alpha) = \frac{x^{1-\alpha}y^\alpha + x^\alpha y^{1-\alpha}}{2}$, with $x, y \in (0,\infty), x \neq y$. We have :

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}} f(\alpha) = \frac{x+y}{2}, \quad \lim_{\substack{\alpha\to 1\\ \alpha<1}} f(\alpha) = \frac{x+y}{2}.$$

Since

$$f'(\alpha) = \frac{1}{2}(\ln y - \ln x)(x^{1-\alpha}y^{\alpha} - x^{\alpha}y^{1-\alpha}),$$

then

$$f'(\alpha) = 0 \quad \Rightarrow \quad x^{1-\alpha}y^{\alpha} = x^{\alpha}y^{1-\alpha} \quad \Rightarrow \quad \left(\frac{x}{y}\right)^{1-2\alpha} = 0 \Rightarrow a = \frac{1}{2}.$$

Therefore, $\min f(\alpha) = f\left(\frac{1}{2}\right) = \sqrt{xy}$ and $f(a) > f(b) > f\left(\frac{1}{2}\right)$, $\quad f\left(\frac{1}{2}\right) < f(c) < f(d)$. By adding, we get $f(a) + f(d) > f(b) + f(c)$ or

$$\begin{aligned}\frac{x^{1-a}y^{a} + x^{a}y^{1-a}}{2} + \frac{x^{1-d}y^{d} + x^{d}y^{1-d}}{2} &> \frac{x^{1-b}y^{b} + x^{b}y^{1-b}}{2} \\ &> \frac{x^{1-b}y^{b} + x^{b}y^{1-b}}{2} + \frac{x^{1-c}y^{c} + x^{c}y^{1-c}}{2}.\end{aligned}$$

This results in

$$x\left[\left(\frac{y}{x}\right)^{a} + \left(\frac{y}{x}\right)^{d} - \left(\frac{y}{x}\right)^{b} - \left(\frac{y}{x}\right)^{c}\right] > y\left[\left(\frac{x}{y}\right)^{b} + \left(\frac{x}{y}\right)^{c} - \left(\frac{x}{y}\right)^{a} - \left(\frac{x}{y}\right)^{d}\right].$$

**4056**. *Proposed by Idrissi Abdelkrim-Amine.*

Let $n$ be an integer, $n \geq 2$. Consider real numbers $a_k$, $1 \leq k \leq n$ such that $a_1 \geq 1 \geq a_2 \geq \ldots \geq a_n > 0$ and $a_1 a_2 \ldots a_n = 1$. Prove that

$$\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} \frac{1}{a_k}.$$

*We received eight solutions of which six were correct. We present the solution by Roy Barbara.*

Note first that if $0 < a, b \leq 1$, then

$$a - \frac{1}{a} + b - \frac{1}{b} \geq ab - \frac{1}{ab}. \qquad (1)$$

Indeed, multiplying by $ab$, (1) is equivalent, in succession, to

$$\begin{array}{rrcl} & a^2b - b + ab^2 - a & \geq & (ab)^2 - 1 \\ \text{or} & (a+b)(ab-1) & \geq & (ab+1)(ab-1) \\ \text{or} & (1-ab)(1-a)(1-b) & \geq & 0, \end{array}$$

which is true.

We now prove the given inequality by using induction on $n \geq 2$.

The case when $n = 2$ is trivial. Suppose the inequality holds for some $n \geq 2$, and let $a_i$, $i = 1, 2, \cdots, n+1$ satisfy $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1}$ and $a_1 a_2 \cdots a_n a_{n+1} = 1$. We need to prove that

$$\left(\sum_{k=1}^{n-1} a_k - \sum_{k=1}^{n-1} \frac{1}{a_k}\right) + \left(a_n - \frac{1}{a_n}\right) + \left(a_{n+1} - \frac{1}{a_{n+1}}\right) \geq 0. \tag{2}$$

Set $b_n = a_n a_{n+1}$.

Then clearly $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_{n-1} \geq b_n$ and $a_1 a_2 \cdots a_{n-1} b_n = 1$.

By the induction hypothesis, we have

$$\left(\sum_{k=1}^{n-1} a_k - \sum_{k=1}^{n-1} \frac{1}{a_k}\right) + \left(b_n - \frac{1}{b_n}\right) \geq 0. \tag{3}$$

Hence, to get (2), it suffices to prove that

$$\left(a_n - \frac{1}{a_n}\right) + \left(a_{n+1} - \frac{1}{a_{n+1}}\right) \geq \left(b_n - \frac{1}{b_n}\right),$$

that is,

$$\left(a_n - \frac{1}{a_n}\right) + \left(a_{n+1} - \frac{1}{a_{n+1}}\right) \geq \left(a_n a_{n+1} - \frac{1}{a_n a_{n+1}}\right),$$

which is true by (1) as $0 < a_n, a_{n+1} \leq 1$.

*Editor's comment.* The proposer of the current problem remarked that the given inequality was inspired by the following inequality due to Leonard Giugiuc : $(\sum a_k)^2 \geq n \sum \frac{1}{a_k}$, where the $a_k$'s satisfy the same conditions given in the current problem.

## 4057. *Proposed by Eeshan Banerjee.*

Let $ABC$ be a non-obtuse triangle with circumradius $R$, inradius $r$ and area $\Delta$. Prove that

$$\Delta < \left(\frac{\frac{1}{r} + 3R + 3}{7}\right)^7.$$

*We received four correct solutions. We present a composite of very similar solutions by Michel Bataille and Andrea Fanchini.*

With AM-GM, we have that

$$\frac{\frac{1}{r} + R + R + R + 1 + 1 + 1}{7} \geq \sqrt[7]{\frac{R^3}{r}},$$

so it suffices to prove that

$$\Delta < \frac{R^3}{r}.$$

From Euler's inequality $R \geq 2r$ and the inequality $R \geq \frac{2s}{3\sqrt{3}}$, we have

$$\frac{R^3}{r} \geq \frac{4r^2 \cdot 2s}{r \cdot 3\sqrt{3}} = \frac{8\sqrt{3}}{9} \cdot rs = \frac{8\sqrt{3}}{9} \cdot \Delta > \Delta,$$

completing the proof.

**4058**. *Proposed by Francisco Javier García Capitán.*

Let $ABC$ be a triangle. For any $X$ on line $BC$, let $X_b$ and $X_c$ be the circumcenters of the triangles $ABX$ and $AXC$, and let $P$ be the intersection point of $BX_c$ and $CX_b$. Prove that the locus of $P$ as $X$ varies along the line $BC$ is the conic through the centroid, orthocenter, and vertices $B$ and $C$, and whose tangents at these vertices are the corresponding symmedians. (Recall that a symmedian is the reflection of a median in the bisector of the corresponding angle.)

*We received two submissions, both of which were correct. We feature the solution by Michel Bataille with a few details added from the proposer's solution.*

As usual, set $BC = a, CA = b, AB = c$. Should $a = b$, then $CX_b$ will be the perpendicular bisector of $AB$, which immediately implies that this line is the locus of $P$; similarly, should $a = c$, then the locus of $P$ would be the perpendicular bisector of $AC$. Assume therefore that $a \neq b, c$. We shall see that under this further assumption the locus is a hyperbola. For our argument we shall use barycentric coordinates relative to $(A, B, C)$, and the following notation :

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}.$$

For later use, here are a few readily checked relations satisfied by these numbers :

$$S_B + S_C = a^2, \quad S_C + S_A = b^2, \quad S_A + S_B = c^2,$$

and

$$\begin{aligned} a^2 S_A + S_B S_C &= b^2 S_B + S_C S_A = c^2 S_C + S_A S_B \\ &= S_A S_B + S_B S_C + S_C S_A = \frac{1}{2}(a^2 S_A + b^2 S_B + c^2 S_C). \end{aligned}$$

If $(f : g : h)$ is the point at infinity of a line $\ell$, then $(gS_B - hS_C : hS_C - fS_A : fS_A - gS_B)$ is the point at infinity of the perpendiculars to $\ell$. With the help of this property, we easily obtain the equations of the perpendicular bisector $\ell_1$ of $AB$ and $\ell_2$ of $AC$,

$$\ell_1 : c^2 x - c^2 y + (a^2 - b^2) z = 0, \qquad \ell_2 : b^2 x + (a^2 - c^2) y - b^2 z = 0.$$

If $X = (0 : \beta : \gamma)$ with $\beta + \gamma = 1$, the point at infinity on $AX$ is $(-1 : \beta : \gamma)$, hence the one on the perpendiculars to $AX$ is $(\beta S_B - \gamma S_C : \gamma S_C + S_A : -S_A - \beta S_B)$. It follows that the perpendicular bisector $m$ of $AX$ is

$$x(\beta^2 S_B + \gamma^2 S_C + S_A) + y(\gamma^2 a^2 - c^2) + z(\beta^2 a^2 - b^2) = 0.$$

Note that

$$\begin{aligned}&\beta^2 S_B + \gamma^2 S_C + S_A\\&= \beta^2 S_B + (1-\beta)^2 S_C + S_A\\&= a^2\beta^2 - 2\beta S_C + b^2 = a^2\beta^2 - \beta(a^2+b^2-c^2) + b^2 = \beta c^2 + \gamma b^2 - a^2\beta\gamma.\end{aligned}$$

Let $x_1 = a^2(S_A + \beta S_B),\ y_1 = a^2\beta S_A + b^2 S_B,\ z_1 = c^2(S_C - a^2\beta)$. It is rather long but easy to check that $(x_1, y_1, z_1)$ satisfies both the equations of $\ell_1$ and $m$. Thus $X_b = (x_1 : y_1 : z_1)$. Similarly, we obtain $X_c = (x_2 : y_2 : z_2)$ with $x_2 = a^2(S_A + \gamma S_C),\ y_2 = b^2(S_B - a^2\gamma),\ z_2 = a^2\gamma S_A + c^2 S_C$.

The equations of $CX_b : xy_1 - yx_1 = 0$ and $BX_c : xz_2 - zx_2 = 0$ then provide $P = (u : v : w) = (x_1x_2 : x_2y_1 : x_1z_2)$ so that

$$v = \frac{a^2\beta S_A + b^2 S_B}{a^2(S_A + \beta S_B)} \cdot u, \quad w = \frac{a^2\gamma S_A + c^2 S_C}{a^2(S_A + \gamma S_C)} \cdot u,$$

from which we obtain

$$a^2\beta = \frac{b^2 S_B u - a^2 S_A v}{vS_B - uS_A}, \qquad a^2\gamma = \frac{c^2 S_C u - a^2 S_A w}{wS_C - uS_A}.$$

Eliminating $\beta, \gamma$ (through $\beta + \gamma = 1$) yields a necessary and sufficient condition on $u, v, w$ for $P$ to belong to the desired locus, namely

$$\begin{aligned}u^2 S_A(a^2 S_A + b^2 S_B + c^2 S_C) - uvc^2(a^2 S_A + S_B S_C) - wub^2(a^2 S_A + S_B S_C)&\\ +vwa^2(S_A S_B + S_B S_C + S_C S_A) = 0;&\end{aligned}$$

that is, $2u^2 S_A - c^2 uv - b^2 uw + a^2 vw = 0$.

Thus, the locus of $P$ is the conic $\Gamma$ with equation

$$(b^2 + c^2 - a^2)x^2 - c^2 xy + a^2 yz - b^2 zx = 0.$$

Note that because we have assumed that $a \neq b, c$, the discriminant of the conic (namely, $\frac{a^2}{4}(b^2 - a^2)(c^2 - a^2)$) is nonzero; because the coefficient of $y^2$ is zero, this nondegenerate conic must be a hyperbola, as claimed. Let $\mathcal{C}(x, y, z)$ be the left-hand side of the equation. We readily find that

$$\mathcal{C}(0,1,0) = \mathcal{C}(0,0,1) = \mathcal{C}(1,1,1) = \mathcal{C}(S_B S_C, S_C S_A, S_A S_B) = 0,$$

hence $\Gamma$ passes through $B, C, G, H$ (respectively), where $G = (1 : 1 : 1)$ and $H = (S_B S_C : S_C S_A : S_A S_B)$ denote the centroid and the orthocenter of $ABC$. In addition, the equation of the tangent to $\Gamma$ at $(x_0 : y_0 : z_0)$ is

$$2xx_0 S_A - \frac{1}{2}c^2(x_0 y + xy_0) - \frac{1}{2}b^2(x_0 z + xz_0) + \frac{1}{2}a^2(y_0 z + yz_0) = 0.$$

In particular, the tangent to $\Gamma$ at $B$ is $c^2 x - a^2 z = 0$, a line passing through the Lemoine point $K = (a^2 : b^2 : c^2)$ and through $B$. Therefore this tangent is the symmedian through vertex $B$. Similarly, the tangent to $\Gamma$ at $C$ is the symmedian through vertex $C$.

*Editor's comments.* The proposer noted the following theorem to be a consequence of his problem :

> If a conic passes through the vertices $B$ and $C$ of a triangle $ABC$, while the tangents at those points are the corresponding symmedians, then the centroid of the triangle lies on the conic if and only if the orthocenter does also.

**4059**. *Proposed by Marcel Chiriţa.*

Let $a, b \in (0, \infty)$, $a \neq b$. Determine the functions $f : \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ such that

$$f(ax) = e^x f(bx), \quad \forall x \in \mathbb{R}.$$

*We received four submissions of which three were correct and complete. We present the solution by Michel Bataille.*

We show the following :

> Let $\mathcal{P}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R} \setminus \{0\}$ that are periodic with period $\ln(b/a)$ and let $u : t \mapsto u(t) = \ln(|t|)$ for $t \neq 0$. Then the solutions are the functions $t \mapsto g(t) \cdot e^{\frac{t}{a-b}}$ where the function $g$ is defined by
>
> $$g(t) = p(u(t)) \ (t > 0), \quad g(0) = \alpha, \quad g(t) = q(u(t)) \ (t < 0)$$
>
> for some $p, q \in \mathcal{P}$ and some $\alpha \in \mathbb{R} \setminus \{0\}$.

First, a remark : if we set $g(x) = f(x) \cdot e^{-\frac{x}{a-b}}$, a simple calculation shows that solving the given equation boils down to solving the functional equation $g(ax) = g(bx)$ for functions $g : \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $f$ being then defined by $f(x) = g(x) \cdot e^{\frac{x}{a-b}}$ for $x \in \mathbb{R}$. Since $\frac{x}{a}$ takes all real values when $x$ does, substituting $\frac{x}{a}$ for $x$ even reduces the problem to seeking functions $g : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ such that

$$g(x) = g\left(x \cdot \frac{b}{a}\right) \tag{1}$$

for all $x \in \mathbb{R}$. Let $g$ be a solution. Then, for $x > 0$, we have

$$g(e^{\ln x}) = g(e^{\ln x + \ln(b/a)}),$$

that is, $p(\ln x) = p(\ln x + \ln(b/a))$ if we set $p = g \circ \exp$. Since $\ln x$ takes all real values as $x$ describes $(0, \infty)$, it follows that $p$ is periodic with period $\ln(b/a)$ and does not take the value 0, i.e. $p \in \mathcal{P}$, and that $g(x) = p(\ln x) = p(\ln(|x|)$ for positive $x$.

Similarly, for $x < 0$, we have

$$g(-e^{\ln(-x)}) = g(-e^{\ln(-x) + \ln(b/a)})$$

that is, $q(\ln(-x)) = q(\ln(-x) + \ln(b/a))$ where $q = g \circ (-\exp)$ is an element of $\mathcal{P}$, so that $g(x) = q(\ln(-x)) = q(\ln(|x|)$.

Conversely, define $g$ by

$$g(t) = p(u(t))\ (t > 0), \quad g(0) = \alpha, \quad g(t) = q(u(t))\ (t < 0)$$

where $p, q \in \mathcal{P}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then, $g$ takes only nonzero values and, if $x > 0$, $g\left(x \cdot \left(\frac{b}{a}\right)\right) = p\left(\ln\left(x \cdot \left(\frac{b}{a}\right)\right)\right) = p(\ln x + \ln(b/a)) = p(\ln x) = g(x)$ and if $x < 0$, then $g\left(x \cdot \left(\frac{b}{a}\right)\right) = q\left(\ln\left(-x \cdot \left(\frac{b}{a}\right)\right)\right) = q(\ln(-x) + \ln(b/a)) = q(\ln(-x)) = g(x)$. Thus, the equality $g\left(x \cdot \left(\frac{b}{a}\right)\right) = g(x)$ holds for all real $x$ (it is obvious for $x = 0$) and so $g$ satisfies (1) with $g(x) \neq 0$ for all real $x$. The proof is complete.

*Remark.* The solutions which are continuous at 0 are the functions $t \mapsto \alpha e^{\frac{t}{a-b}}$ where $\alpha$ is a nonzero real constant : with the above notations, $f$ is continuous at 0 if and only if $g$ is. So, we consider a solution $g$ of (1), with $g$ continuous at 0. Suppose first that $b < a$. Then,

$$g\left(x \cdot \left(\frac{b}{a}\right)^k\right) = g\left(x \cdot \left(\frac{b}{a}\right)^{k+1}\right)$$

if $k$ is a positive integer ; hence, by an immediate induction, we see that $g(x) = g\left(x \cdot \left(\frac{b}{a}\right)^n\right)$ for all positive integers $n$. Since $0 < \frac{b}{a} < 1$, we have $\lim_{n\to\infty} x \cdot \left(\frac{b}{a}\right)^n = 0$ and so $\lim_{n\to\infty} g\left(x \cdot \left(\frac{b}{a}\right)^n\right) = g(0)$. As a result, $g(x) = g(0)$ for any $x \in \mathbb{R}$. If $b > a$, the treatment is similar using the equation $g(x) = g\left(x \cdot \frac{a}{b}\right)$ which holds for all $x$ as well. Thus $g$ is a constant function. Conversely, any constant function $\mathbb{R} \to \mathbb{R} \setminus \{0\}$ is obviously a solution of (1).

*Editor's comments.* Roy Barbara proposed and solved the following generalization of the problem. Set $k = a/b$ and $t = bx$ $(t \in \mathbb{R})$. If $\varphi : \mathbb{R} \to \mathbb{R}^*$ is a function with $\varphi(0) = 1$ and $k \in (0, \infty)$, $k \neq 1$, determine all the functions $f : \mathbb{R} \to \mathbb{R}^*$ satisfying

$$f(kt) = \varphi(t) \cdot f(t) \qquad \forall t \in \mathbb{R}.$$

Unfortunately, Marcel Chiriţa passed away on 29 February 2016 and he cannot enjoy the beautiful solution given above. We will miss him and we will miss his precious contribution to the journal.

## 4060. *Proposed by Michel Bataille.*

Let

$$f(x, y) = \frac{xy(x + y)}{(1 - x - y)^3}.$$

Find the range of $f(x, y)$ when its domain is restricted to the circle $S$ that satisfies the equation $x^2 + y^2 = 1 - 2x - 2y$.

*Five correct solutions were submitted from four people. Two others made incorrect submissions. We present two solutions.*

*Solution 1, following the approach of Kee-Wai Lau.*

Suppose that $(x, y) \in S$ and let $t = x + y$. Then

$$(t+2)^2 = 2(x^2 + y^2 + 2x + 2y - 1) + 6 - (x-y)^2 = 6 - (x-y)^2 \le 6,$$

so that $-2 - \sqrt{6} \le t \le -2 + \sqrt{6} < 1$, with equality possible only if $x = y$. Since

$$2xy = (x+y)^2 + 2(x+y) - 1 = t^2 + 2t - 1,$$

we have

$$f(x, y) = \frac{t(t^2 + 2t - 1)}{2(1-t)^3}.$$

The condition for $S$ can be written as

$$(x+1)^2 + (y+1)^2 = 3,$$

so that $S$ is a circle with centre $(-1, -1)$ that intersects the line $y = x$ at

$$\left(\frac{1}{2}(-2-\sqrt{6}), \frac{1}{2}(-2-\sqrt{6})\right) \quad \text{and} \quad \left(\frac{1}{2}(-2+\sqrt{6}), \frac{1}{2}(-2+\sqrt{6})\right).$$

These points correspond to the limiting values of $t$. From the geometry, it is easily seen that as $(x, y)$ ranges over $S$, the variable $t$ assumes all values in the closed interval $[-2 - \sqrt{6}, -2 + \sqrt{6}]$. [Alternatively, using the theory of the quadratic, one can determine that the system

$$\begin{cases} (x+1) + (y+1) = t + 2, \\ (x+1)^2 + (y+1)^2 = 3, \end{cases}$$

is solvable for real values of $x$ and $y$ if and only if $(t+2)^2 \le 6$.]

Since $f(x, y) = -\sqrt{6}/18$ when $t = -2 - \sqrt{6}$ and $f(x, y) = \sqrt{6}/18$ when $t = -2 + \sqrt{6}$, and since $f$ is continuous in $t$, the range of $f$ includes the closed interval $[-\sqrt{6}/18, \sqrt{6}/18]$.

Observe that

$$\begin{aligned} f(x, y) + \frac{\sqrt{6}}{18} &= \frac{(9 - \sqrt{6})t^3 + (18 + 3\sqrt{6})t^2 - (9 + 3\sqrt{6})t + \sqrt{6}}{18(1-t)^3} \\ &= \frac{(t + 2 + \sqrt{6})[(9 - \sqrt{6})t^2 + (6 - 4\sqrt{6})t + (3 - \sqrt{6}]}{18(1-t)^3}. \end{aligned}$$

Since the quadratic factor, having zero discriminant, is the square of a linear polynomial, and since $1 - t > 0$, $f(x, y) \ge -\sqrt{6}/18$ when $t \ge -2 - \sqrt{6}$. Likewise

$$\begin{aligned} f(x, y) - \frac{\sqrt{6}}{18} &= \frac{(9 + \sqrt{6})t^3 + (18 - 3\sqrt{6})t^3 + (-9 + 3\sqrt{6})t - \sqrt{6}}{18(1-t)^3} \\ &= \frac{(t + 2 - \sqrt{6})[(9 + \sqrt{6})t^2 + (6 + 4\sqrt{6})t + (3 + \sqrt{6})]}{18(1-t)^3} \le 0, \end{aligned}$$

so that $f(x,y) \leq \sqrt{6}/18$ when $t \leq -2+\sqrt{6}$. Thus the range of $f$ is exactly $[-\sqrt{6}/18, \sqrt{6}/18]$.

*Solution 2, by the proposer.*

Let $u = x^2 + y^2$ and

$$a = \frac{2x}{u+1} \qquad b = \frac{2y}{u+1} \qquad c = \frac{u-1}{u+1}.$$

When $(x,y) \in S$, we have that

$$u+1 = 2(1-x-y) \quad \text{and} \quad u-1 = x^2+y^2-1 = -2(x+y).$$

It can be checked that

$$a+b+c = 0, \quad a^2+b^2+c^2 = 1 \quad \text{and so} \quad ab+bc+ca = -\frac{1}{2}.$$

Also

$$abc = -\frac{xy(x+y)}{(1-x-y)^3} = -f(x,y).$$

Thus, the real numbers $a, b, c$ are the roots of the polynomial

$$t^3 - \frac{1}{2}x + f(x,y).$$

Since the roots are all real, we must have $4 \cdot (1/8) \geq 27(f(x,y))^2$, so that

$$-\frac{1}{\sqrt{54}} \leq f(x,y) \leq \frac{1}{\sqrt{54}}$$

and $f(x,y)$ belongs to the closed interval $[-\sqrt{6}/18, \sqrt{6}/18]$.

Conversely, let $p \in [-\sqrt{6}/18, \sqrt{6}/18]$. We show that $p = f(x,y)$ for some $(x,y) \in S$. This is true for $p = 0$, since $f(0, -1+\sqrt{2}) = 0$. Suppose $p \neq 0$. Consider the polynomial $t^3 - \frac{1}{2}t + p$ and let $a, b, c$ be its roots. Since the discriminant condition $4 \cdot (1/8) \geq 27p^2$ holds, the roots are all real. Since $a+b+c = 0$ and $ab+bc+ca = -1/2$, we have that $a^2+b^2+c^2 = 1$. Since $abc \neq 0$, $-1 < c < 1$, so that $c = (v-1)/(v+1)$ for some $v > 0$. Now, let

$$x = \frac{a(1+v)}{2} \qquad \text{and} \qquad y = \frac{b(1+v)}{2}.$$

Then $x^2+y^2 = v = 1-2x-2y$, so that $(x,y) \in S$. Moreover

$$\begin{aligned} f(x,y) &= \frac{xy(x+y)}{(1-x-y)^3} = \frac{ab}{4}(v+1)^2 \cdot \frac{(v+1)(a+b)}{2} \cdot \frac{8}{(v+1)^3} \\ &= ab(a+b) = -abc = p. \end{aligned}$$

*Editor's Comments.* The proposer submitted a second solution that followed the same strategy as Lau, except that he analyzed the behaviour of the function $t(t^2+2t-1)(1-t)^{-3}$ by calculus. Paul Bracken used Lagrange Multipliers and located the maximum and minimum values of $f(x,y)$ as well as eight other critical points on $S$ given by the equations $(x+1)^2+(y+1)^2=3$ and $(x^2+y^2)(x+y)=xy+x+y$.

Paul Deiermann looked at the more general restriction $(x+1)^2+(y+1)^2=a$ where $0<a<9/2$, and parameterized the points of $S$ by

$$(x,y)=(-1+\sqrt{a}\cos\theta,-1+\sqrt{a}\sin\theta).$$

He found $f(x,y)$ to be equal to

$$\frac{1}{2}\cdot\frac{[q^2-2q+2-a][-2+q]}{(3-q)^3},$$

where $q=\sqrt{2a}\cos(\theta-\pi/4)$ satisfies $-\sqrt{2a}\le q\le\sqrt{2a}$. He then analyzed this function, identifying its values at the endpoints and the two critical points of the interval $[-\sqrt{2a},\sqrt{2a}]$. In the case $a=3$ of the problem, the global maximum is achieved at both an endpoint and a critical point, while the global minimum is achieved at the other endpoint and critical point. He adds that *Mathematica* graphs suggest that $a=3$ is the only value of $a$ where $f$ achieves both global extrema at an endpoint and a critical point at the same time.