OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015 : 41(4), p. 149–150.

OC226. In a triangle ABC, let D be the point on the segment BC such that $AB+BD = AC+CD$. Suppose that the points B, C and the centroids of triangles ABD and ACD lie on a circle. Prove that $AB = AC$.

Originally problem 1 of the 2014 India National Olympiad.

We received six correct submissions. We present the solution by Titu Zvonaru (similar to Šefket Arslanagić).

As usual, let a, b, c be the sides of ΔABC and let $2s = a + b + c$. Let T be the midpoint of AD and let G_1 and G_2 be the centroids of triangles ABD and ACD respectively. Since $BD - CD = b - c$ and $BD + CD = a$, we see that $BD = s - c$ and $CD = s - b$. Now, BG_qG_2C is cyclic if and only if $TG_1 \cdot TB = TG_2 \cdot TC$. Since medians are divided by the centroid in a $2 : 1$ ratio, we see this holds if and only if $\frac{TB^2}{3} = \frac{TC^2}{3}$. This is true if and only if $4TB^2 = 4TC^2$ which, by the formula for a median's length, holds if and only if

$$2BD^2 + 2BA^2 - AD^2 = 2CD^2 + 2CA^2 - AD^2.$$

Substituting the values from before, this holds if and only if $(s - c)^2 + c^2 = (s - b)^2 + b^2$ which is equivalent to $(b - c)(b + c - a) = 0$. Since $a \neq b + c$ by the triangle inequality, we see that $b = c$ and hence $AB = AC$.

OC227. In a bag there are 1007 black and 1007 white balls, which are randomly numbered 1 to 2014. In every step we draw one ball and put it on the table ; also if we want to, we may choose two different colored balls from the table and put them in a different bag. If we do that we earn points equal to the absolute value of their differences. How many points can we guarantee to earn after 2014 steps ?

Originally problem 1 from day 1 of the 2014 Turkey Mathematical Olympiad.

No submitted solutions.

OC228. Let k be a nonzero natural number and m an odd natural number . Prove that there exist a natural number n such that the number $m^n + n^m$ has at least k distinct prime factors.

Originally problem 4 from day 1 of the 2014 Romanian Team Selection Test.

No submitted solutions.

OC229. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $x, y \in \mathbb{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = f(y)$$

Originally problem 4 from day 2 of the 2014 Iran Team Selection Test.

We received two correct submissions. We present the solution by Oliver Geupel.

The function

$$f(x) = \frac{1}{x}$$

is a solution because, for $x, y \in \mathbb{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = \frac{1}{(x+1)y} + \frac{x}{(x+1)y} = \frac{1}{y} = f(y).$$

We show that there is no other solution.

Suppose f is any solution of the problem.

To begin with, we prove that

$$f(x) \leq \frac{1}{x} \tag{1}$$

for every $x > 0$. Suppose that, contrary to our claim, $f(a) > \dfrac{1}{a}$ for some $a > 0$. Putting

$$x = \frac{1}{af(a) - 1}, \qquad y = a,$$

we obtain

$$\frac{x+1}{xf(y)} = y,$$

that is,

$$f\left(\frac{y}{f(x+1)}\right) = f(y) - f\left(\frac{x+1}{xf(y)}\right) = 0,$$

a contradiction. This proves (1) for $x > 0$.

Next we show that, for every $x \geq 1$,

$$f(x) = \frac{1}{x}. \tag{2}$$

By (1), we have for all $x, y \in \mathbb{R}^+$,

$$f(y) = f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) \leq \frac{f(x+1)}{y} + \frac{x}{x+1} \cdot f(y),$$

so that $yf(y) \leq (x+1)f(x+1)$. It follows that $xf(x)$ is identically constant for $x > 1$, say $xf(x) = c \leq 1$. Moreover, for all $x > 1$, we have

$$\frac{x}{f(x+1)} = \frac{x(x+1)}{c} > 1, \qquad \frac{x+1}{xf(x)} = \frac{x+1}{c} > 1,$$

so that

$$\frac{c^2}{x} = \frac{c^2}{(x+1)x} + \frac{c^2 x}{(x+1)x} = f\left(\frac{x}{f(x+1)}\right) + f\left(\frac{x+1}{xf(x)}\right) = f(x) = \frac{c}{x},$$

which implies $c = 1$. We obtain (2) for every $x > 1$.

By (1), we have $\frac{2}{f(1)} \geq 2$, whence

$$f(1) = f\left(\frac{1}{f(2)}\right) + f\left(\frac{2}{f(1)}\right) = \frac{1}{2} + \frac{1}{2}f(1),$$

that is, $f(1) = 1$. This proves that (2) holds for every $x \geq 1$.

Finally, let $P(n)$ denote the assertion that equation (2) is true for every $x \geq 2^{-n}$. It is enough to prove $P(n)$ for all nonnegative integers n. We do so by mathematical induction. We have already established the base case $n = 0$. For the induction step suppose $P(n)$. Let

$$\frac{1}{2^{n+1}} \leq y < \frac{1}{2^n}, \qquad x = \frac{2^n y}{1 - 2^n y}.$$

Then,

$$f(x+1) = \frac{1}{x+1} = 1 - 2^n y$$

and

$$\frac{y}{f(x+1)} = \frac{y}{1 - 2^n y} \geq \frac{1}{2^n}.$$

By (1),

$$\frac{x+1}{xf(y)} \geq \frac{x+1}{x} \cdot y = \frac{1}{2^n}.$$

By induction we deduce

$$f(y) = f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = \frac{1}{(x+1)y} + \frac{x}{x+1} \cdot f(y)$$

and conclude

$$f(y) = \frac{1}{y},$$

which proves $P(n+1)$ and the proof is complete.

OC230. Find, with justification, all positive real numbers a, b, c satisfying the system of equations :

$$a\sqrt{b} = a + c, b\sqrt{c} = b + a, c\sqrt{a} = c + b.$$

Originally problem 2 of the 2014 Singapore Senior Math Olympiad.

We received seven correct submissions. We present the solution by Albert Stadler.

We claim that $a = b = c = 4$ is the only solution in positive numbers a, b, c. It is easy to verify that this is a solution so we now show this is the only one. Let $a = u^2$, $b = v^2$ and $c = w^2$ in order to eliminate square roots. With these new variables, we need to show that $u = v = w = 2$ is the only positive solution of

$$u^2 v = u^2 + w^2 \qquad v^2 w = v^2 + u^2 \qquad w^2 u = w^2 + v^2$$

Using the AM-GM inequality, we see that

$$u^2 v = u^2 + w^2 \geq 2uw$$

implying that $uv \geq 2w$. Similarly, $vw \geq 2u$ and $wu \geq 2v$. Thus, multiplying the first two inequalities gives

$$(uv)(vw) \geq 4wu$$

implying that $v^2 \geq 4$ and hence $v \geq 2$ (recall we only want positive solutions). Similarly, $u \geq 2$ and $w \geq 2$. Then, the triple of equalities above implies that

$$v - 1 = \left(\frac{w}{u}\right)^2 \qquad w - 1 = \left(\frac{u}{v}\right)^2 \qquad u - 1 = \left(\frac{v}{w}\right)^2$$

and hence, multiplying these together yields $(u-1)(v-1)(w-1) = 1$. However, $u \geq 2$, $v \geq 2$ and $w \geq 2$. Hence, $u = v = w = 2$.