A Surprising Result in Cake-Sharing

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Amy and Peter are sharing a cake. Amy will cut it into two pieces. Peter then cuts one of the pieces into two. This is followed by a second cut by Amy and a second cut by Peter, so that there will be five pieces, of sizes $0 \le a_1 \le a_2 \le a_3 \le a_4 \le a_5$, with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$. Peter will get the three pieces of sizes $a_1,\ a_3$ and a_5, while Amy will get the remaining two pieces.

What is the maximum amount of the cake Peter can get?

The answer is surprising!

We first digress and consider the companion problem where Amy will get the three pieces of sizes $a_1,\ a_3$ and a_5, while Peter will get the remaining two pieces. What is the maximum amount of the cake Amy can get?

First, we prove that Peter can always get $\frac{2}{5}$ of the cake. Suppose Amy cuts the cake into two pieces of sizes x and $1 - x$, where $0 \le x \le \frac{1}{2}$. There are three cases.

Case 1. $\frac{2}{5} \le x \le \frac{1}{2}$.

Peter will cut $1 - x$ into x and $1 - 2x$. Now the three pieces are of sizes $1 - 2x < x = x$. If Amy does not cut either x, neither will Peter. Peter will then be sure of getting x plus a second piece, and $x \ge \frac{2}{5}$. If Amy cuts one of the x's, Peter will cut the other x in the same proportions. Peter will get two pieces which add up to $x \ge \frac{2}{5}$.

Case 2. $\frac{1}{5} \le x < \frac{2}{5}$.

Peter will cut x into $x - \frac{1}{5}$ and $\frac{1}{5}$. Now the three pieces are of sizes $x - \frac{1}{5} < \frac{1}{5} < 1 - x$. If Amy does not cut $1 - x$, Peter will cut it in halves. The second smallest piece cannot be less than $\frac{1}{2}(x - \frac{1}{5})$, so Peter will get at least $\frac{1-x}{2} + \frac{1}{2}(x - \frac{1}{5}) = \frac{2}{5}$. Suppose Amy cuts $1 - x$ into y and $1 - x - y$, where $0 \le y \le \frac{1-x}{2}$. Then Peter will cut $1 - x - y$ into $\frac{2}{5} - y$ and $\frac{3}{5} - x$. Now $y + (\frac{2}{5} - y) = \frac{2}{5} = (x - \frac{1}{5}) + (\frac{3}{5} - x)$. Thus Peter will get two pieces which add up to $\frac{2}{5}$.

Case 3. $0 \le x < \frac{1}{5}$.

Peter will cut $1 - x$ into $\frac{1}{5}$ and $\frac{4}{5} - x$. The situation is the same as in Case 2.

We now prove that Amy can always get $\frac{3}{5}$ of the cake. She can start by cutting the cake into two pieces of sizes $\frac{2}{5}$ and $\frac{3}{5}$. There are two cases.

Case 1. Peter cuts $\frac{2}{5}$ into x and $\frac{2}{5} - x$, where $0 \le x \le \frac{1}{5}$.

Amy will cut $\frac{3}{5}$ into x and $\frac{3}{5} - x$. Now the four pieces are of sizes $x = x \le \frac{2}{5} - x < \frac{3}{5} - x$. No matter what Peter does, the size of the second largest piece is at most $\frac{2}{5} - x$ and the size of the fourth largest piece is at most x. Hence Peter gets at most $(\frac{2}{5} - x) + x = \frac{2}{5}$.

Case 2. Peter cuts $\frac{3}{5}$ into x and $\frac{3}{5}-x$, where $0 \le x \le \frac{3}{10}$.

If $0 \le x \le \frac{1}{5}$, Amy will cut $\frac{2}{5}$ into x and $\frac{2}{5}-x$, and the situation is exactly the same as in Case 1. Hence we may assume that $\frac{1}{5} < x \le \frac{3}{10}$. Amy will cut $\frac{3}{5}-x$ into $\frac{1}{5}$ and $\frac{2}{5}-x$. Now the four pieces are of sizes $\frac{2}{5}-x < \frac{1}{5} < x < \frac{2}{5}$. There are four subcases.

- **Subcase 2(a).** Peter cuts $\frac{2}{5}$ into y and $\frac{2}{5}-y$, where $0 \le y \le \frac{1}{5}$.

  Since $y + (\frac{2}{5}-y) = \frac{2}{5} = x + (\frac{2}{5}-x)$, Peter will get two pieces which add up to $\frac{2}{5}$.

- **Subcase 2(b).** Peter cuts x.

  If $\frac{1}{5}$ remains the third largest piece, Amy will get at least $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$. If it becomes the second largest piece, Peter gets at most $\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$.

- **Subcase 2(c).** Peter cuts $\frac{1}{5}$ into y and $\frac{1}{5}-y$, where $0 \le y \le \frac{1}{10}$.

  Since $\frac{2}{5}-x \ge y$, the second smallest piece is at most $\frac{2}{5}-x$. Hence Peter gets at most $(\frac{2}{5}-x)+x=\frac{2}{5}$.

- **Subcase 2(d).** Peter cuts $\frac{2}{5}-x$.

  Amy will get at least $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$.

We now return to the original problem with the surprising answer. We first show that Peter can always get $\frac{35}{53}$ of the cake. Suppose Amy cuts the cake into two pieces of sizes a and b, where $a \ge b \ge 0$ and $a+b=1$. There are four cases.

Case A. $\frac{34}{53} \le a \le 1$ so that $0 \le b \le \frac{19}{53}$.

Peter will cut b into $\frac{b}{2}$ and $\frac{b}{2}$. If Amy does not cut a, Peter will just cut off $\frac{1}{53}$ from another piece, and will get three pieces with total size at least $a+\frac{1}{53} \ge \frac{35}{53}$. Suppose Amy cuts a into $c \le d$. Peter will cut d into $\frac{d}{2}$ and $\frac{d}{2}$, and will get three pieces with total size at least $c+\frac{d}{2}+\frac{b}{2} \ge \frac{a}{4}+\frac{1}{2} \ge \frac{35}{53}$.

Case B. $\frac{33}{53} \le a \le \frac{34}{53}$ so that $\frac{19}{53} \le b \le \frac{20}{53}$.

Peter will cut b into $\frac{18}{53}$ and $b-\frac{18}{53}$. If Amy does not cut a, Peter will just cut off $\frac{2}{53}$ from another piece, and will get three pieces with total size at least $a+\frac{2}{53} \ge \frac{35}{53}$. Suppose Amy cuts a into $c \ge d \ge 0$. We consider four subcases.

- **Subcase B1.** $\frac{18}{53} \ge c \ge \frac{17}{53} \ge d \ge b-\frac{18}{53}$.

  Peter will cut c into d and $c-d$, and will get three pieces with total size at least

  $$\frac{18}{53}+d+\min\left\{c-d, b-\frac{18}{53}\right\}.$$

  In the former instance, it is at least $\frac{18}{53}+c \ge \frac{35}{53}$. In the latter instance, it is at least $d+b=1-c \ge \frac{35}{53}$.

- **Subcase B2.** $\frac{17}{53} \geq c \geq d \geq b - \frac{18}{53}$.

  Peter will cut $b - \frac{18}{53}$ into $\frac{b}{2} - \frac{9}{53}$ and $\frac{b}{2} - \frac{9}{53}$, and will get three pieces with total size at least $\frac{18}{35} + d + \frac{b}{2} - \frac{9}{35} = \frac{62}{53} - \frac{b}{2} - c \geq \frac{35}{53}$.

- **Subcase B3.** $c \geq \frac{18}{53} \geq d \geq b - \frac{18}{53}$.

  Peter will cut $\frac{18}{53}$ into d and $\frac{18}{53} - d$, and will get three pieces with total size at least
  $$c + d + \min\left\{\frac{18}{35} - d, b - \frac{18}{35}\right\}.$$
  In the former instance, it is at least $c + \frac{18}{53} \geq \frac{36}{53}$. In the latter instance, it is at least $a + b - \frac{18}{53} = \frac{35}{53}$.

- **Subcase B4.** $c \geq \frac{18}{53} \geq b - \frac{18}{53} \geq d$.

  Peter will cut $\frac{18}{53}$ into $\frac{9}{53}$ and $\frac{9}{53}$. Since $d \leq b - \frac{18}{53} \leq \frac{2}{53}$, $c \geq \frac{31}{53}$. Hence he will get three pieces with total size at least $c + \frac{9}{53} \geq \frac{40}{53}$.

Case C. $\frac{27}{53} \leq a \leq \frac{33}{53}$ so that $\frac{20}{53} \leq b \leq \frac{26}{53}$.

Peter will cut a into $\frac{27}{53}$ and $a - \frac{27}{53}$. If Amy does not cut $\frac{27}{53}$, Peter will just cut off $\frac{8}{53}$ from another piece. If it is the second largest, then Amy gets two pieces with total size at most $\frac{16}{53}$. Otherwise, Peter will get three pieces with total size at least $\frac{27}{53} + \frac{8}{53} = \frac{35}{53}$. Suppose Amy cuts $\frac{27}{53}$ into $c \geq d$. There are four subcases.

- **Subcase C1.** $\frac{27}{106} \leq c \leq \frac{15}{53}$ so that $\frac{12}{53} \leq d \leq \frac{27}{106}$.

  Peter will cut $a - \frac{27}{53} = \frac{26}{53} - b$ into $\frac{13}{53} - \frac{b}{2}$ and $\frac{13}{53} - \frac{b}{2}$. He will get three pieces with total size $b + d + (\frac{13}{53} - \frac{b}{2}) \geq \frac{35}{53}$.

- **Subcase C2.** $\frac{15}{53} \leq c \leq \frac{18}{53}$ so that $\frac{9}{53} \leq d \leq \frac{12}{53}$.

  Peter will cut c into d and $c - d$. He will get three pieces with total size $b + d + \min\{c - d, a - \frac{27}{53}\}$. In the former instance, it is at least $b + c \geq \frac{35}{53}$. In the latter instance, it is at least $d + 1 - \frac{27}{53} \geq \frac{35}{53}$.

- **Subcase C3.** $\frac{18}{53} \leq c \leq \frac{24}{53}$ so that $\frac{3}{53} \leq d \leq \frac{9}{53}$.

  Peter will cut c into $\frac{c}{2}$ and $\frac{c}{2}$. Amy gets two pieces with total size at most $\frac{c}{2} + \max\{d, a - \frac{27}{53}\}$. In the former instance, it is at most $\frac{27}{53} - c \leq \frac{18}{35}$. In the latter instance, it is at most $\frac{12}{53} + \frac{6}{53} = \frac{18}{53}$.

- **Subcase C4.** $\frac{24}{53} \leq c \leq \frac{27}{53}$ so that $0 \leq d \leq \frac{3}{53}$.

  Peter will cut b into $\frac{b}{2}$ and $\frac{b}{2}$. He will get three pieces with total size at least $c + \frac{n}{2} + \min\{d, a - \frac{27}{53}\}$. In the former instance, it is at least $\frac{27}{53} + \frac{b}{2} \geq \frac{37}{53}$. In the latter instance, it is at least $c + 1 - \frac{27}{53} - \frac{b}{2} \geq \frac{37}{53}$.

Case D. $\frac{1}{2} \leq a \leq \frac{27}{53}$ so that $\frac{26}{53} \leq b \leq \frac{1}{2}$.

Peter will pass. Whichever piece Amy now cuts, Peter will cut off from the larger of the two new pieces a piece equal to $\frac{1}{3}$ of the piece Amy cuts. This piece will be the third largest, and Peter will get three pieces with total size at least $b + \frac{a}{3} = 1 - \frac{2a}{3} \geq \frac{35}{53}$.

The following two preliminary results will be useful.

Lemma 1. Suppose before Peter's final cut, the four pieces have sizes w, x, y and z in non-ascending order. If $x \leq 2y$, then Amy can get two pieces with total size at least x.

Proof. If Peter cuts either of the smallest two pieces, the second largest piece will have size x. There is nothing further to prove.

Hence we assume that Peter must cut one of the largest two pieces, into two pieces both with size smaller than x. Because $x \leq 2y$, at least one of the new pieces has size less than y. If the original piece with size y is still the third largest, then the largest has size at most w and the smallest has size at most z. Hence Peter gets three pieces with total size at most $w+y+z$, so that Amy will get two pieces with total size at least $1 - w - y - z = x$. On the other hand, if the piece with size y is now the second largest, then the size of each of the two new pieces lies between y and $x - y$. Thus, the second smallest piece has size at least $x - y$, and Amy's two pieces will have total size at least $y + (x - y) = x$. □

Lemma 2. Suppose before Peter's final cut, the four pieces have sizes w, x, y and z in non-ascending order. If $x \geq 2z$, then Amy can get two pieces with total size at least $\min\{y + z, x + \frac{z}{2}\}$.

Proof. If Peter cuts either of the smallest two pieces, the second smallest piece will have size at least $\frac{z}{2}$ while the second largest piece will have size x. Hence, Amy will get two pieces with total size at least $x + \frac{z}{2}$.

If Peter cuts either of the largest two pieces into two pieces both with size smaller than x, not both can have size smaller than z since $x \geq 2z$. Hence, the second smallest piece has size at least z while the second largest piece has size at least y. Hence, Amy will get two pieces with total size at least $y + z$. □

We now prove that Amy can always get $\frac{18}{53}$ of the cake. She will cut 1 into $\frac{20}{53}$ and $\frac{33}{53}$. There are two cases.

Case A. Peter cuts $\frac{20}{33}$ into $a \geq b \geq 0$.

There are three subcases.

- **Subcase A1.** $\frac{18}{53} \leq a \leq \frac{20}{53}$ so that $0 \leq b \leq \frac{2}{53}$.

  Amy will cut $\frac{33}{53}$ into $\frac{18}{53}$ and $\frac{15}{53}$. In Lemma 1, $w = a$, $x = \frac{18}{53}$, $y = \frac{15}{53}$ and $z = b$, with $x \leq 2y$. Hence she will get two pieces with total size at least $\frac{18}{53}$.

- **Subcase A2.** $\frac{17}{53} \leq a \leq \frac{18}{53}$ so that $\frac{2}{53} \leq b \leq \frac{3}{53}$.

  Amy will cut $\frac{33}{53}$ into $\frac{17}{53}$ and $\frac{16}{53}$. In Lemma 2, $w = a$, $x = \frac{17}{53}$, $y = \frac{16}{53}$ and $z = b$, with $x \geq 2z$. Now $y + z = \frac{16}{53} + b \geq \frac{18}{53}$ while $x + \frac{z}{2} = \frac{17}{53} + \frac{b}{2} \geq \frac{18}{53}$. Either way, she will get two pieces with total size $\frac{18}{53}$.

- **Subcase A3.** $\frac{10}{53} \le a \le \frac{17}{53}$ so that $\frac{3}{53} \le b \le \frac{10}{53}$.

  Amy will still cut $\frac{33}{53}$ into $\frac{17}{53}$ and $\frac{16}{53}$. The total size of the smallest four pieces is $1 - \frac{17}{53} = \frac{36}{53}$. She will be guaranteed to get at least half of that, which is $\frac{18}{53}$.

Case B. Peter cuts $\frac{33}{53}$ into $a \ge b \ge 0$.

There are seven subcases.

- **Subcase B1.** $\frac{27}{53} \le a \le \frac{33}{53}$, so that $0 \le b \le \frac{6}{53}$.

  Amy will cut a into $\frac{18}{53}$ and $a - \frac{18}{53}$. In Lemma 1, $w = \frac{20}{53}$, $x = \frac{18}{53}$, $y = a - \frac{18}{53}$ and $z = b$, with $x \le 2y$. Hence she will get two pieces with total size at least $\frac{18}{53}$.

- **Subcase B2.** $\frac{51}{106} \le a \le \frac{27}{53}$, so that $\frac{6}{53} \le b \le \frac{15}{106}$.

  Amy will cut a into $\frac{15}{53}$ and $a - \frac{15}{53}$. In Lemma 2, $w = \frac{20}{53}, x = \frac{15}{53}$, $y = a - \frac{15}{53}$ and $z = b$, with $x \ge 2z$. Now $y + z = a + b - \frac{15}{53} = \frac{33}{53} - \frac{15}{53} = \frac{18}{53}$ while $x + \frac{z}{2} = \frac{15}{53} + \frac{b}{2} \ge \frac{18}{53}$. Either way, she will get two pieces with total size at least $\frac{18}{53}$.

- **Subcase B3.** $\frac{25}{53} \le a \le \frac{51}{106}$, so that $\frac{15}{106} \le b \le \frac{8}{53}$.

  Amy will cut a into $a - b - \frac{3}{53}$ and $b + \frac{3}{53}$. If Peter cuts either $b + \frac{3}{53}$ or b, the second smallest piece is at least $\frac{b}{2}$, and Amy will get two pieces with total size at least $a - b - \frac{3}{53} + \frac{b}{2} = a + b - \frac{3}{53} - \frac{3b}{2} \ge \frac{18}{53}$. Suppose Peter cuts either $\frac{20}{53}$ or $a - b - \frac{3}{53}$. If both new pieces are less than b, then Amy will get two pieces with total size at least $b + \frac{3}{53} + \frac{1}{2}(a - b - \frac{3}{53}) = \frac{18}{53}$. If at least one of the new pieces is greater than b, then the second largest piece is at least $b + \frac{3}{53}$ so that she will get two pieces with total size at least $b + \frac{3}{53} + b \ge \frac{18}{53}$.

- **Subcase B4.** $\frac{23}{53} \le a \le \frac{25}{53}$, so that $\frac{8}{53} \le b \le \frac{10}{53}$.

  Amy will pass. If Peter then cuts b, Amy will get at least $\frac{20}{53}$. Hence Peter must cut a or $\frac{20}{53}$. After Peter's final cut, if b is still the third largest, then he gets three pieces with total size at most $a + b + 0 = \frac{33}{53}$, so that Amy will get two pieces with total size at least $\frac{18}{53}$. If b becomes the second smallest, then the second largest is at least $\frac{10}{53}$, and she will get two pieces with total size at least $\frac{10}{53} + b \ge \frac{18}{53}$.

- **Subcase B5.** $\frac{20}{53} \le a \le \frac{23}{53}$, so that $\frac{10}{53} \le b \le \frac{13}{53}$.

  Amy will also pass. In Lemma 1, $w = a$, $x = \frac{20}{53}$, $y = b$ and $z = 0$, with $x \le 2y$. Hence she will get two pieces with total size at least $\frac{20}{53}$.

- **Subcase B6.** $\frac{18}{53} \le a \le \frac{20}{53}$, so that $\frac{13}{53} \le b \le \frac{15}{53}$.

  Amy will still pass. In Lemma 1, $w = \frac{20}{53}$, $x = a$, $y = b$ amd $z = 0$, with $x \le 2y$. Hence she will get two pieces with total size at least $\frac{18}{53}$.

- **Subcase B7.** $\frac{33}{106} \le a \le \frac{18}{53}$, so that $\frac{15}{53} \le b \le \frac{33}{106}$.

  Amy will cut $\frac{20}{53}$ into $\frac{14}{53}$ and $\frac{6}{53}$. In Lemma 2, $w = a$, $x = b, y = \frac{14}{53}$ and $z = \frac{6}{53}$ with $x \ge 2z$. Now $y + z = \frac{20}{53}$ while $x + \frac{z}{2} = b + \frac{3}{53} \ge \frac{18}{53}$. Either way, she will get two pieces with total size at least $\frac{18}{53}$.

Therefore, Amy can always get $\frac{18}{53}$ of the cake.

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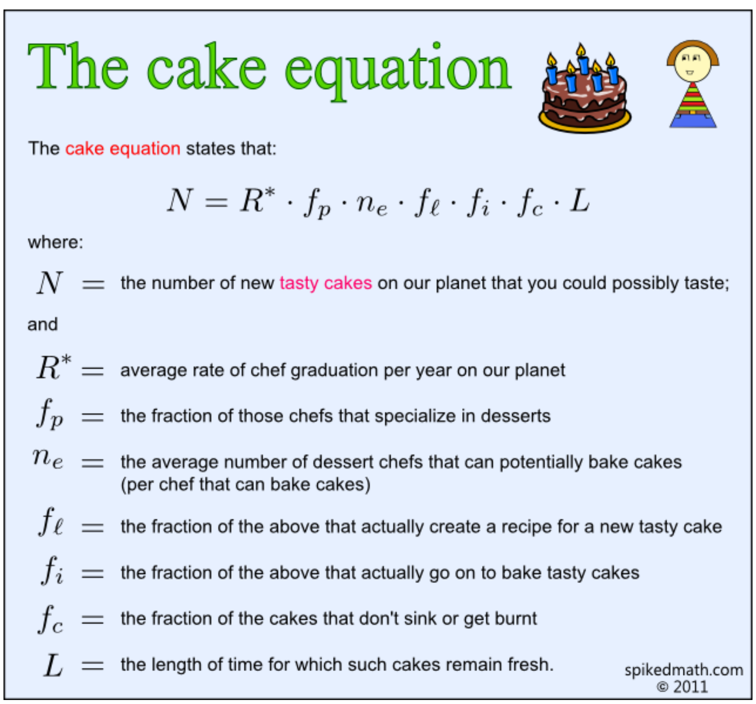
We dedicate this article to the memory of Robert Barrington Leigh (1986 – 2006), who passed away tragically 10 years ago. Robert was an extremely bright young man who won numerous mathematical and other scientific contests ; he represented Canada at the International Mathematical Olympiad in 2002 and 2003. At the time of his passing, he was enrolled in his final year at the University of Toronto in Specialist Programs both in Mathematics and in Physics and was awarded a posthumous undergraduate degree in June 2007.

This article, found as an unfinished manuscript after Robert's passing, was recently finished by his co-authors.

With apologies to Frank Drake, we present



Useful reference : `https://en.wikipedia.org/wiki/Drake_equation`