OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(1), p. 9–11.

OC211. Find maximum value of

$$|a^2 - bc + 1| + |b^2 - ac + 1| + |c^2 - ba + 1|$$

where a, b, c are real numbers in the interval $[-2, 2]$.

Originally problem 1 of day 2 of 2013 Kazakhstan National Olympiad Grade 11.

Editor's Note. This problem is a duplicate from OC179. The editor used the question from two different Kazakhstan Olympiad contests but did not realize that the question could be repeated across grades. Since the editor received different solutions to this problem as opposed to the first version, the editor will include one here. My apologies.

We received 3 correct submissions. We present the solution by Šefket Arslanagić.

Let $f(a, b, c) = |a^2 - bc + 1| + |b^2 - ac + 1| + |c^2 - ba + 1|$. By symmetry, we may also suppose that $a \geq b \geq c$. Notice immediately that $f(a, b, c) = f(-a, -b, -c)$. By using these two properties, we can assume without loss of generality that at least two values are positive , so $a \geq b \geq 0$.

Further, we can show that $f(a, b, |c|) \leq f(a, b, -|c|)$. This follows since

$$|a^2 - b|c| + 1| \leq a^2 + b|c| + 1 \qquad \text{and} \qquad |b^2 - a|c| + 1| \leq b^2 + a|c| + 1.$$

Hence, since we are looking for the maximum value, we may assume that $a \geq b \geq 0 \geq c$. Next, we show that $f(a, b, c) \leq f(a, b, -2)$. This inequality is equivalent to showing that

$$a^2 - bc + 1 + b^2 - ac + 1 + |c^2 - ab + 1| \leq a^2 + 2b + 1 + b^2 + 2a + 1 + |4 - ab + 1|$$

which is equivalent to

$$|c^2 - ab + 1| \leq 5 - ab + (2 + c)(a + b).$$

If $c^2 - ab + 1 \geq 0$, then the equality is exact by adding the inequalities $c^2 \leq 4$ and $0 \leq (2 + c)(a + b)$. If $c^2 - ab + 1 \leq 0$, then the above becomes via a sequence of if and only if statements

$$\begin{aligned} 2ab &\leq 6 + c^2 + (2 + c)(a + b) \\ 2ab - 2a - 2b + 2 &\leq 8 + c^2 + c(a + b) \\ 2(a - 1)(b - 1) &\leq 8 + c^2 + c(a + b). \end{aligned}$$

This last inequality is true since

$$2(a - 1)(b - 1) \leq 2 \leq 4 + 4 + c^2 + c(a + b).$$

The last inequality holds since

$$4 + c^2 + c(a + b) \geq 4|c| + c(a + b) \geq (a + b)|c| + c(a + b) \geq 0$$

using the fact that $(2 - c)^2 \geq 0$ in the first inequality. Therefore, it suffices to find the maximum of the function $f(a, b, -2)$ where $a \geq b \geq 0$. This reduces to finding the maximum of

$$f(a, b, -2) = a^2 + b^2 - ab + 2a + 2b + 7.$$

with $a \in [0, 2]$ and $b \in [0, 2]$. This maximum must occur when either $a = b$, $b = 0$ or $a = 2$ (by say Calculus). Checking each of these cases reveals that the maximum value is

$$f(2, 2, -2) = |4 + 4 + 1| + |4 + 4 + 1| + |4 - 4 + 1| = 19.$$

OC212. Let $ABCDE$ be a pentagon inscribe in a circle (O). Let $BE \cap AD = T$. Suppose the parallel line with CD which passes through T cuts AB, CE at X, Y. If ω is the circumcircle of triangle AXY then prove that ω is tangent to (O).

Originally problem 3 from level X of the 2013 Romanian National Olympiad.

We received 3 correct submissions. We present the solution by Andrea Fanchini.

We have that $\angle AEC = \angle ADC$ because both are inscribed in the same arc of circle. Then since the lines XY and CD are parallel, we have also $\angle ADC = \angle DTY$. Similarly $\angle ECD = \angle TYC = \angle EAD$ and therefore the quadrilateral $ATYE$ is cyclic.

Now if we draw a tangent AW to the circle (O), we have

$$\angle WAB = \angle AEB = \angle AET,$$

but the points A, T, Y and E are concyclic, so we have also

$$\angle AET = \angle AYT = \angle AYX.$$

Therefore AW is also tangent to circle ω completing the proof.

OC213. Suppose $p > 3$ is a prime number and

$$S = \sum_{2 \leq i < j < k \leq p-1} ijk.$$

Prove that $S + 1$ is divisible by p.

Originally problem 4 of the 2013 Indonesian Mathematical Olympiad.

We received 2 correct submissions. We present the solution by Michel Bataille.

From Fermat's Little Theorem, each element of $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ is a root of the polynomial $x^p - x$ of $\mathbb{Z}_p[x]$. Thus, $x^p - x = x(x-1)(x-2)\cdots(x-(p-1))$ in $\mathbb{Z}_p[x]$. Since $x^p - x = x(x^{p-1} - 1)$, it follows that $x^{p-1} - 1 = (x-1)(x-2)\cdots(x-(p-1))$ in $\mathbb{Z}_p[x]$.

This said, let $p(x)$ be the polynomial $(x-2)(x-3)\cdots(x-(p-1))$ of $\mathbb{Z}_p[x]$. Since $(x-1)p(x) = x^{p-1} - 1$, we see that

$$p(x) = x^{p-2} + x^{p-3} + \cdots + x + 1. \tag{1}$$

However, we also have

$$p(x) = x^{p-2} - e_1 x^{p-3} + e_2 x^{p-4} - e_3 x^{p-5} + \cdots - e_{p-2} \tag{2}$$

with, modulo p,

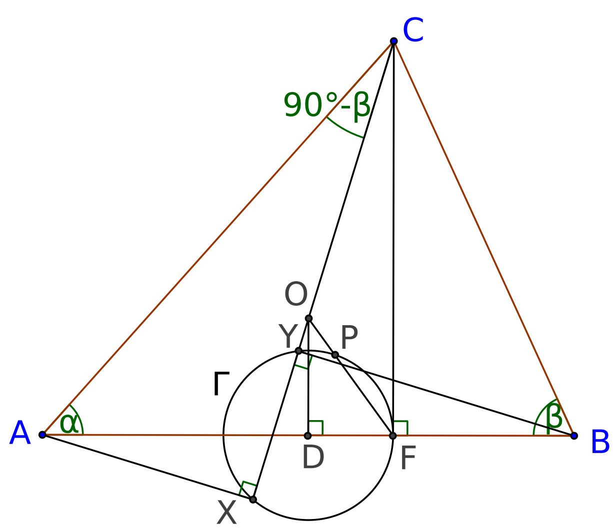
$$e_1 \equiv \sum_{i=2}^{p-1} i, \;\; e_2 \equiv \sum_{2 \le i < j \le p-1} i.j, \;\; e_3 \equiv \sum_{2 \le i < j < k \le p-1} ijk, \ldots$$

Comparing (1) and (2) yields $1 \equiv -e_3 \pmod{p}$. Since $S = e_3$, we conclude that $S + 1 \equiv 0 \pmod{p}$, that is, p divides $S + 1$.

OC214. Let ABC be an acute-angled triangle with $AC \neq BC$, and let O be the circumcentre and F the foot of the altitude through C. Furthermore, let X and Y be the feet of the perpendiculars dropped from A and B respectively to (the extension of) CO. The line FO intersects the circumcircle of FXY a second time at P. Prove that $OP < OF$.

Originally problem 6 of the 2013 South Africa National Olympiad.

We received 2 correct submissions. We present the solution by Oliver Geupel.



Let us denote $\alpha = \angle BAC$ and $\beta = \angle CBA$.

In the isosceles triangle AOC we have $\angle ACO = 90^\circ - \angle COA/2 = 90^\circ - \beta$. As a consequence, in the right triangle AXC we have $CX = AC\sin\beta$. Similarly $CY = BC\sin\alpha$. The altitude from point C in the triangle ABC has length $CF = AC\sin\alpha = BC\sin\beta$. We obtain

$$CF^2 = CX \cdot CY,$$

where the right-hand side is the power of point C with respect to the circumcircle Γ of triangle FXY. Hence the line CF is a tangent to the circle Γ.

Therefore the centre of the circle Γ lies on the line AB. Also the centre of the circle Γ lies on the perpendicular bisector p of the segment XY. Since $AC \neq BC$, the line p is not parallel to the line AB so that the lines p and AB are concurrent. The midpoint D of side AB lies on the line p because it is equidistant from the lines AX and BY. Thus, the point D is the centre of the circle Γ.

In the right triangle DFO we have $OD < OF$. The power of point O with respect to the circle Γ is

$$OF \cdot OP = OD^2 - DF^2.$$

Consequently,

$$OP = \frac{OD^2 - DF^2}{OF} < \frac{OF^2}{OF} = OF.$$

OC215. Let $n > 1$ be an integer. The first n primes are $p_1 = 2, p_2 = 3, \ldots, p_n$. Set $A = p_1^{p_1} p_2^{p_2} ... p_n^{p_n}$. Find all positive integers x, such that $\frac{A}{x}$ is even, and $\frac{A}{x}$ has exactly x divisors.

Originally problem 6 from day 2 of the 2013 South East Mathematical Olympiad.

We present the solution by Konstantine Zelator. There were no other submissions.

First, we claim that the only solution is $x = p_1 p_2 ... p_n$. Note that this is a solution. It suffices to show it is the only one. Since x is a divisor of A, we have that

$$x = \prod_{i=1}^{n} p_i^{e_i} \qquad \text{where } 0 \leq e_i \leq p_i$$

Now, e_1 cannot be 2 since then A/x is odd so $e_1 = 0$ or $e_1 = 1$. Assume towards a contradiction that $e_1 = 0$. Then

$$\frac{A}{x} = 4 \cdot \prod_{i=2}^{n} p_i^{p_i - e_i}$$

and the number of divisors this number has (which by the problem statement is equal to x) is

$$\prod_{i=2}^{n} p_i^{e_i} = x = 3\prod_{i=2}^{n}(p_i - e_i + 1)$$

Now, if $e_2 = 0$, then the left hand side above is not divisible by 3 but the right hand side is, a contradiction. If $e_2 = 1$, then the left hand side above is divisible by 1 and the right hand side is divisible by at least 3^2 coming from $3(4 - e_1) = 3^2$, another contradiction. If $e_2 = 2$, then the right hand side is even from the $(4 - e_1) = 2$ term but the left hand side is odd, again a contradiction. Lastly, if $e_3 = 3$, then note that $n \geq 3$ must be true by inspection. The remaining terms on the right must all be odd numbers since the left hand side is odd and thus, since each p_i is an odd prime for $i \geq 3$, we have that $(p_i - e_i + 1)$ is odd and hence each e_i is odd. Thus $e_i \geq 1$ for all $i \geq 3$. Hence, since $p_i^{e_i} \geq p_i \geq (p_i - e_i + 1)$, we see that

$$x = \prod_{i=2}^{n} p_i^{e_i} = 3^3 \prod_{i=3}^{n} p_i^{e_i} > 3 \cdot (4-3) \cdot \prod_{i=3}^{n} (p_i - e_i + 1) = x$$

which is a contradiction. Thus $e_1 \neq 0$ and hence $e_1 = 1$. Then as before,

$$\frac{A}{x} = 2 \cdot \prod_{i=2}^{n} p_i^{p_i - e_i}$$

and the number of divisors this number has (which by the problem statement is equal to x) is

$$2 \cdot \prod_{i=2}^{n} p_i^{e_i} = x = 2 \prod_{i=2}^{n} (p_i - e_i + 1).$$

Simplifying gives

$$\prod_{i=2}^{n} p_i^{e_i} = \prod_{i=2}^{n} (p_i - e_i + 1).$$

As before, each of the e_1 terms must be odd otherwise the right hand side is even. Hence $e_i \geq 1$ for all $i \geq 2$. As before, $p_i^{e_i} \geq p_i \geq (p_i - e_i + 1)$ with equality holding if and only if $p_i^{e_i} + e_i = p_i + 1$ and thus, since $e_i \geq 1$, equality holds if and only if $e_i = 1$ for all $i \geq 2$. Thus, $x = p_1 p_2 ... p_n$ completing the proof.