THE CONTEST CORNER

No. 43

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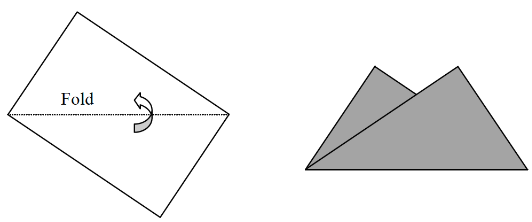
The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **January 1, 2017**, *although late solutions will also be considered until a solution is published.*

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

CC211. A rectangular sheet of paper whose dimensions are 12×18 is folded along a diagonal, which creates the M-shaped region drawn at the right. Find the area of the shaded region.



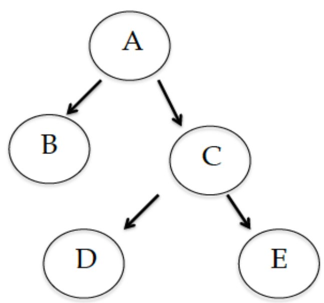
CC212. A cube that is one inch wide has had its eight corners shaved off. The cube's vertices have been replaced by eight congruent equilateral triangles, and the square faces have been replaced by six congruent octagons. If the combined area of the eight triangles equals the area of one of the octagons, what is that area? (Each octagonal face has two different edge lengths that occur in alternating order.)

CC213. A pyramid is built from solid unit cubes that are stacked in square layers. The top layer has $1 \times 1 = 1$ cube, the second $3 \times 3 = 9$ cubes and the layer below that has $5 \times 5 = 25$ cubes, and so on, with each layer having two more cubes

on a side than the layer above it. The pyramid has a total of 12 layers. Find the exposed surface area of this solid pyramid, including the bottom.

CC214. The points $(2, 5)$ and $(6, 5)$ are two of the vertices of a regular hexagon of side length two on a coordinate plane. There is a line L that goes through the point $(0, 0)$ and cuts the hexagon into two pieces of equal area. What is the slope of line L?

CC215. Each circle in this tree diagram is to be assigned a value, chosen from a set S, in such a way that along every pathway down the tree the assigned values never increase. That is, $A \geq B, A \geq C, C \geq D, C \geq E$ and $A, B, C, D, E \in S$. (It is permissible for a value in S to appear more than once.) How many ways can the tree be so numbered using only values chosen from the set $S = \{1, \ldots, 6\}$?



(Optional extension: Generalize to a case with $S = \{1, 2, 3, , \ldots, n\}$ by finding an explicit algebraic expression for the number of ways the tree can be numbered.)

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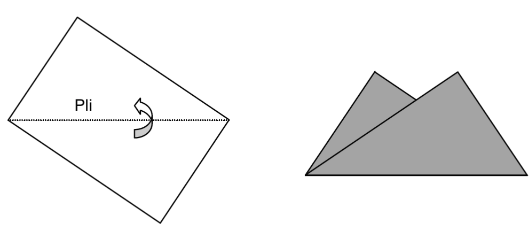
CC211. Une feuille de papier rectangulaire de taille 12×18 est pliée le long de la diagonale, formant ainsi une région en forme de M, telle qu'illustrée. Déterminer la surface de la région ombragée.



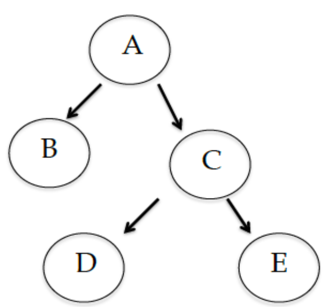
CC212. On a retranché les huit coins d'un cube dont les côtés mesurent chacun un pouce. Les sommets ont ainsi été remplacés par huit triangles équilatéraux congrus, et les faces carrées ont été remplacées par six octogones congrus. Si la

surface totale des huit triangles égale la surface d'un des octogones, quelle est cette surface ? (Chaque face octogonale comporte deux longueurs différentes de côté, en alternance.)

CC213. Une pyramide est construite à partir de cubes de taille unitaire, empilés en tranches carrées. La tranche supérieure comporte $1 \times 1 = 1$ cube, la seconde en a $3 \times 3 = 9$, celle en bas de ça en a $5 \times 5 = 25$, et ainsi de suite, chaque tranche en ayant deux de plus sur chaque côté par rapport à la tranche supérieure. La pyramide a 12 tranches au total. Déterminer la surface externe de cette pyramide, incluant le fond.

CC214. Dans le plan, les points $(2, 5)$ et $(6, 5)$ sont deux sommets d'un hexagone régulier de côté deux. Une certaine ligne L, passant par le point $(0, 0)$, coupe l'hexagone en deux parties de même surface. Quelle est la pente de la ligne L?

CC215. À chaque cercle dans l'arbre indiqué ci-bas on assigne une valeur, choisie dans un ensemble S, de façon à ce que dans chaque chemin vers le bas dans l'arbre les valeurs assignées n'augmentent jamais. C'est-à-dire $A \geq B, A \geq C, C \geq D, C \geq E$ où $A, B, C, D, E \in S$ (Il est permis qu'une valeur dans S apparaisse plus qu'une fois.) De combien de manières peut-on assigner des valeurs à l'arbre si $S = \{1, \ldots, 6\}$?



(Au choix: Généraliser au cas où $S = \{1, 2, 3, , \ldots, n\}$ à l'aide d'une expression algébrique explicite pour le nombre d'assignations.)

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(3), p. 96–97.

CC161. A number n written in base b reads 211, but it becomes 110 when written in base $b+2$. Find n and b in base 10.

A reformulation of #4 of the Santa Clara University High School Mathematics 2001 Contest.

We received ten solutions, of which eight were complete and correct. All eight solutions were nearly identical so we present a composite solution here.

We have that $n = 211$ in base b. This requires $b > 2$ and means

$$n = (211)_b = 2 \cdot b^2 + 1 \cdot b + 1,$$

while $n = 110$ in base $b + 2$ gives us

$$n = (110)_{b+2} = 1 \cdot (b+2)^2 + 1 \cdot (b+2) + 0.$$

Equating these two expressions gives

$$\begin{aligned} 2b^2 + b + 1 &= b^2 + 4 + 4b + b + 2 \\ b^2 - 4b - 5 &= 0 \\ b &= -1, 5 \end{aligned}$$

We discard the negative solution both because of the restriction on b and the fact that a base cannot be negative. Using $b = 5$ we can calculate

$$n = (211)_5 = (110)_7 = 56.$$

Therefore $n = 56$ and $b = 5$ in base 10.

CC162. What is the probability that 99 divides a randomly chosen 4-digit palindrome?

A reformulation of #3 from the team section of the 2010 Raytheon MATHCOUNTS State Competition.

We received eight submissions of which four were correct and complete. We present the solution by Titu Zvonaru.

A 4-digit palindrome is a number of the form $\overline{abba}$, with $a = 1, 2, \ldots, 9$ and $b = 0, 1, \ldots, 9$, hence there are 90 numbers which are 4-digit palindrome. Since $\overline{abba} = 1001a + 100b = 11(91a + 10b)$, we deduce that all 4-digit palindromes are divisible by 11. The number $\overline{abba}$ is divisible by 9 if and only if $a+b$ is divisible by 9. If $a+b=9$ we have the possibilities $1881, 2772, 3663, 4554, 5445, 6336, 7227, 8118, 9009$; if $a+b=18$, then there is only the number 9999. The searched probability is $10/90 = 1/9$.

Editor's Comments. Some solvers counted also the case when $a=b=0$ in the solution, but there is a flaw. Indeed, they have counted a total of 90 palindrome numbers (9 possibilities for the nonzero digit a and 10 possibilities for the digit b), but then they counted the case when $a=b=0$, a contradiction. The solution could have been consistent if they also counted the degenerate case when $a=0$ in the total number of palindromes, giving $10 \cdot 10 = 100$ palindrome numbers. In this case we also have $a=0, b=0$ and $a=0, b=9$, giving the probability $12/100 = 3/25$. The only consistent (but not correct in the strict sense) solutions are the ones given by Kathleen E. Lewis and Hannes Geupel.

CC163. If x is randomly chosen in $[-100, 100]$, what is the probability that $g[f(x)]$ is negative given that $f(x) = x^2 + 3x - 7$ and $g(x) = x^2 - 2x - 99$?

A reformulation of #8 of the 2014 University of North Colorado Math Contest.

We received eight submissions of which seven were correct and complete. We present the solution by Titu Zvonaru.

Since $g(x) = (x+9)(x-11)$, we have $g(x) < 0 \iff x \in (-9, 11)$. It follows that

$$\begin{aligned} g(f(x)) < 0 &\iff -9 < f(x) < 11 \\ &\iff -9 < x^2 + 3x - 7 < 11 \\ &\iff (x+6)(x-3) < 0 \text{ and } (x+1)(x+2) > 0. \end{aligned}$$

Thus, $g(f(x)) < 0 \iff x \in (-6, -2) \cup (-1, 3)$, and the probability is given by the total length of the combined intervals divided by the total length of the domain of x, which is

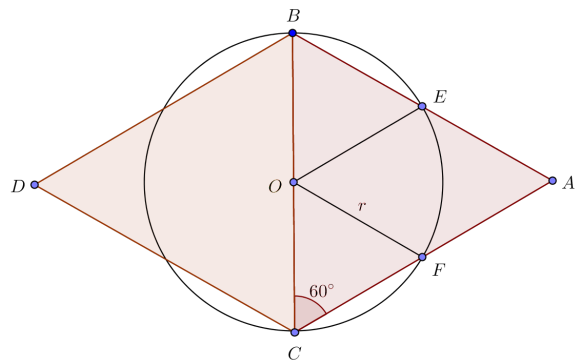
$$\frac{(-2+6)+(3+1)}{100+100} = \frac{1}{25}.$$

CC164. Build two equilateral triangles on the diameter of a circle with radius 5. What is the total area of the circle outside the equilateral triangles? (See the diagram below.)

Proposed by the editor.

We received eleven submissions of which ten were correct. We present the solution by Fernando Ballesta Yagüe, slightly modified by the editor.

Denote the center of the circle by O, the two equilateral triangles by ABC and DBC (with BC being the diameter of the circle), and the intersections of AB and AC with the circle by E and F respectively. Use r for the radius of the circle, and recall that $r = 5$.



As $\triangle ABC$ is equilateral, $\angle BCA = 60°$. Further, $OC = OF = r$, so it follows that $\triangle OCF$ is also equilateral, and $\angle FOC = 60°$.

Hence the area of the circular segment between the chord CF and the circle is equal to the area of a circular sector with central angle $60°$ minus the area of the equilateral $\triangle FOC$, that is

$$\frac{\pi \cdot r^2}{6} - \frac{1}{2} \cdot r^2 \cdot \sin(60°) = \frac{25\pi}{6} - \frac{25\sqrt{3}}{4}.$$

We can reason the same way with $\triangle OBE$, and also with the matching construction on $\triangle DBC$. It follows that the area contained inside the circle but outside the triangles consists of four congruent circular segments, and the total area is

$$4 \cdot \left(\frac{25\pi}{6} - \frac{25\sqrt{3}}{4}\right) = \frac{50\pi}{3} - 25\sqrt{3}.$$

CC165. Georges pays \$50 on each of four gas refills but the prices per litre were \$1.32, \$1.25, \$1.11 and \$1.18 as the price was fluctuating a lot in that time period. What is the average price per litre?

Proposed by the editor.

We received five correct solutions and one incorrect solution. We present the solution of Henry Ricardo.

The quantities of gas purchased were $\frac{\$50}{\$1.32/L}$, $\frac{\$50}{\$1.25/L}$, $\frac{\$50}{\$1.11/L}$, and $\frac{\$50}{\$1.18/L}$.

$$\begin{aligned}(\text{Average price per litre}) &= \frac{(\text{Total cost of gas})}{(\text{Total quantity of gas purchased})} \\ &= \frac{\$200}{\frac{\$50}{\$1.32/L} + \frac{\$50}{\$1.25/L} + \frac{\$50}{\$1.11/L} + \frac{\$50}{\$1.18/L}} \\ &\approx \$1.21/L\end{aligned}$$