SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

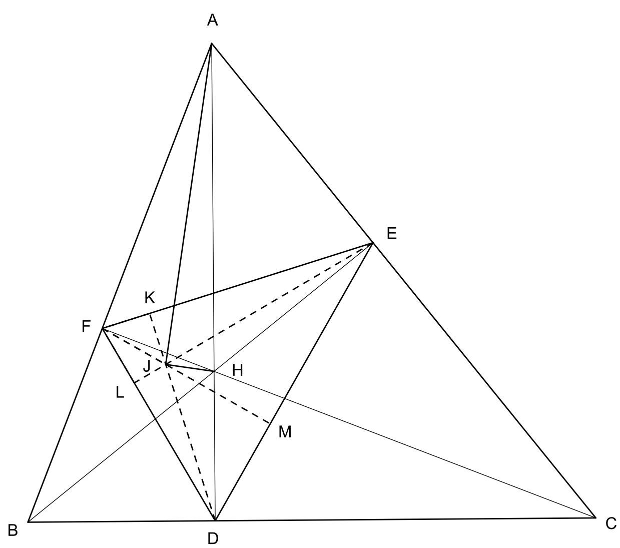
Statements of the problems in this section originally appear in 2015: 41(2), p. 71–74.

4011. *Proposed by Abdilkadir Altinas.*

In non-equilateral triangle ABC, let H be the orthocentre of ABC and J be the orthocentre of the orthic triangle DEF of ABC (that is the triangle formed by the feet of the altitudes of ABC). If $\angle BAC = 60^\circ$, show that $AJ \perp HJ$.

We received nine solutions. Eight of the solutions used angle-chasing in cyclic quadrilaterals, and one solution used barycentric coordinates. The former type of solutions were simpler, but they all missed the fact that there are subtleties if the orthocenter is not interior to the triangle.

We present the solution by Ricardo Barroso Campos slightly modified by the editor.



Since $\angle HFA = \angle HEA = 90^\circ$, quadrilateral $AFHE$ is cyclic. Hence

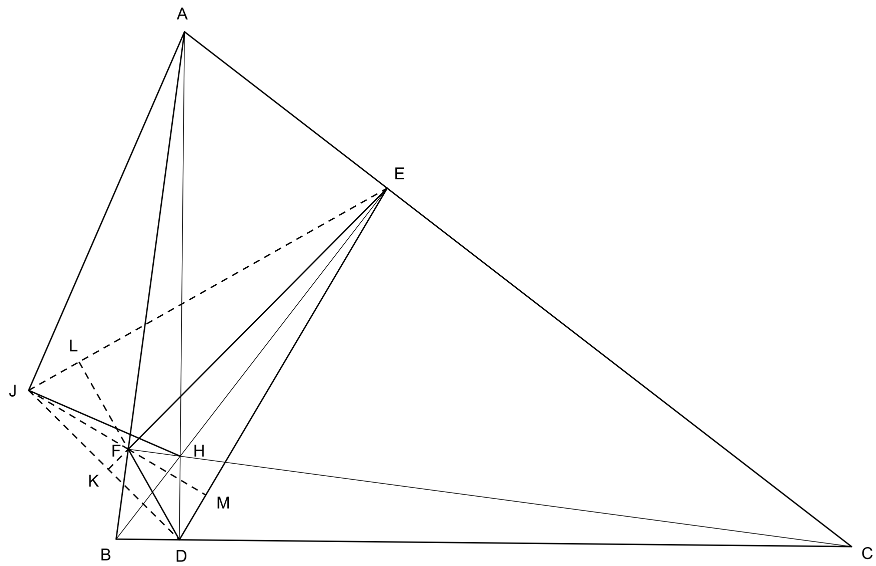
$$\angle EHF = 180^\circ - \angle CAB = 120^\circ.$$

Denote by Γ the circumcircle of $AFHE$, and note that AH is the diameter of this circle (since $\angle AFH = 90^\circ$).

In the cyclic quadrilateral $BDHF$, $\angle BDF = \angle BHF = 180^\circ - \angle EHF = 60^\circ$. Similarly, from the cyclic quadrilateral $CDHE$, we get $\angle CDE = 60^\circ$. Hence $\angle FDE = 180^\circ - (\angle BDF + \angle CDE) = 60^\circ$.

Denote the feet of the altitudes from D, E, and F by K, L and M respectively, as in the diagram. $DLJM$ is cyclic, thus $\angle LJM = 180° - \angle FDE = 120°$. Hence $\angle FJE = \angle LJM = 120°$, which implies that the quadrilateral $AFJE$ is cyclic (since $\angle FJE + \angle FAE = 180°$). It follows that J is on the circle Γ. Hence, since AH is the diameter of Γ, we get $\angle AJH = 90°$, so $AJ \perp HJ$.

Editor's Comments. The provided solution fails if one of the orthocentres is not interior to its triangle. The following diagram shows the case where the point J is not interior to DEF (for one, $\angle FJE = 60°$, not $120°$). Note however that it is not difficult to adjust the solution for these cases.



4012. *Proposed by Leonard Giugiuc.*

Let n be an integer with $n \geq 3$. Consider real numbers a_k, $1 \leq k \leq n$ such that

$$a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq 1 \geq a_n \geq 0 \quad \text{and} \quad \sum_{k=1}^{n} a_k = n.$$

Prove that

$$\frac{(n-2)(n+1)}{2} \leq \sum_{1 \leq i < j \leq n} a_i a_j \leq \frac{n(n-1)}{2}.$$

We received eight submissions of which seven were correct and complete. We present the solution by Ivan Chan Kai Chin.

Since we have

$$2 \cdot \sum_{1 \leq i \leq j \leq n} a_i a_j = (\sum_{k=1}^{n} a_k)^2 - \sum_{k=1}^{n} a_k^2 = n^2 - \sum_{k=1}^{n} a_k^2,$$

it suffices to prove that

$$n \leq \sum_{k=1}^{n} a_k^2 \leq n+2.$$

The left inequality holds by Cauchy-Schwarz, since

$$\sum_{k=1}^{n} a_k^2 \geq \frac{1}{n} \cdot (\sum_{k=1}^{n} a_k)^2 = n,$$

with equality when $a_1 = a_2 = \cdots = a_n = 1$.

For the other inequality, set $b_k = a_k - 1$ for all $1 \leq k \leq n$. Then $b_k \geq 0$ for all $1 \leq k \leq n-1$, $-1 \leq b_n \leq 0$, and $\sum_{k=1}^{n} b_i = 0$. We have

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} (1+b_k)^2 = \sum_{k=1}^{n-1} (1+b_k)^2 + (1-(b_1+b_2+\cdots+b_{n-1}))^2. \qquad (1)$$

Define the quantity $S = b_1 + b_2 + \cdots + b_{n-1} \leq 1$, and

$$f(b_1, b_2, \cdots, b_{n-1}) = \sum_{k=1}^{n-1} (1+b_k)^2.$$

For any $1 \leq i \leq j \leq n-1$,

$$f(\cdots b_i, \cdots, b_j, \cdots) \leq f(\cdots b_i + b_j, \cdots, 0, \cdots),$$

since

$$(1+b_i)^2 + (1+b_j)^2 \leq (1+b_i+b_j)^2 + 1 \iff 2b_ib_j \geq 0$$

holds for all b_i, b_j, $1 \leq i \leq j \leq n-1$. Thus we have

$$f(b_1, b_2, \cdots, b_{n-1}) \leq f(b_1 + b_2 + \cdots b_{n-1}, 0, \cdots, 0) = f(S, 0, \cdots, 0)$$

and (1) becomes

$$\begin{aligned} f(S, 0, \cdots, 0) + (1-S)^2 &= (1+S)^2 + (n-2) + (1-S)^2 \\ &= n + 2S^2 \\ &\leq n+2 \end{aligned}$$

Equality holds when $S = 1$, $b_2 = b_3 = \cdots = b_{n-1} = 0$, $b_1 = 1$ and $b_n = -1$, which corresponds to $a_1 = 2, a_2 = a_3 = \cdots = a_{n-1} = 1, a_n = 0$.

4013. *Proposed by Mehmet Şahin.*

Let a, b, c be the sides of triangle ABC, D be the foot of the altitude from A and E be the midpoint of BC. Define $\theta = \angle DAE$ and suppose that $\angle ACB = 2\theta$. Prove that the sides of the triangle satisfy

$$(a-b)^2 = 2c^2 - b^2.$$

We received 16 submissions. Among them one simply stated that the claim was incorrect and provided a counterexample, while 15 proved the claim under the additional assumption that $b > c$; moreover, 5 proved that for the claim to be correct, the assumption that $b > c$ is both necessary and sufficient, and 3 of those submissions went on to provide a complete description of triangles that satisfy the given hypotheses.

We present the solution by Joel Schlosberg, supplemented by ideas from C. R. Pranesachar.

We shall prove that if a triangle satisfies $\angle ACB = 2\angle DAE$ and, moreover, $b > c$, then $(a-b)^2 = 2c^2 - b^2$; if $b < c$ then $a = 2b$ (and the claimed equation fails to hold). Note that if $b = c$ then $A = D = E$ is the midpoint of the segment BC, and the triangle is degenerate.

Scale $\triangle ABC$ so that $b = 1$. By right-angle trigonometry, $AD = \sin 2\theta = 2\sin\theta\cos\theta$, so that

$$AD^2 = 4\sin^2\theta(1-\sin^2\theta).$$

Use signed lengths for segments on BC, with BC positive. Then

$$DE = pAD\tan\theta = 2p\sin^2\theta,$$

where $p = 1$ if B and D are on one side of E, and C is on the other (which happens if and only if $b > c$); otherwise, when E is between B and D (and, equivalently, $b < c$) then we set $p = -1$. Furthermore, we have

$$DC = \cos 2\theta = 1 - 2\sin^2\theta$$

$$EC = DC - DE = 1 - (2+2p)\sin^2\theta$$

$$a = BC = 2EC = 2 - (4+4p)\sin^2\theta$$

$$BD = BC - DC = 1 - (2+4p)\sin^2\theta.$$

By the Pythagorean theorem (using $p^2 = 1$),

$$c^2 = AD^2 + BD^2 = 1 - 8p\sin^2\theta + (16+16p)\sin^4\theta, \tag{1}$$

while (using $b = 1$)

$$\frac{(a-b)^2 + b^2}{2} = \frac{1}{2}a^2 - ab + b^2 = 1 - (4+4p)\sin^2\theta + (16+16p)\sin^4\theta. \tag{2}$$

Comparing equations (1) and (2), we see that $(a-b)^2 = 2c^2 - b^2$ iff $p = 1$. On the other hand, setting $p = -1$ and $b = 1$ in equation (2) we get $a = 2$ and deduce that $a = 2b$.

Editor's Comments. This problem should be compared with problem 4008 whose solution appeared in the previous issue. It dealt with triangles for which $\angle ACB = 2\angle DAE$ and, in addition, $\angle ABC = 3\angle DAE$. One finds that this can happen if and only if $\angle A = 90°, \angle B = 54°$, and $\angle C = 36°$; of course this implies that $b > c$ and, consequently, that $(a-b)^2 = 2c^2 - b^2$.

4014. *Proposed by Mihaela Berinedanu.*

Let n be a natural number and let x, y and z be positive real numbers such that $x + y + z + nxyz = n + 3$. Prove that

$$(1 + \frac{y}{x} + nyz)(1 + \frac{z}{y} + nzx)(1 + \frac{x}{z} + nxy) \geq (n+2)^3$$

and determine when equality holds.

We received six correct solutions. We present the solution by Dionne Bailey, Elsie Campbell and Charles Diminnie (joint).

The arithmetic-geometric means inequality yields that

$$n+3 = x+y+z+nxyz \geq (n+3)\left[x \cdot y \cdot z \cdot (xyz)^n\right]^{1/n+3} = (n+3)\left[xyz\right]^{(n+1)/(n+3)},$$

so that $xyz \leq 1$.

The inequality is equivalent to

$$(x + y + nxyz)(y + z + nxyz)(z + x + nxyz) \geq (n+2)^3(xyz)$$

or

$$(n + 3 - z)(n + 3 - x)(n + 3 - y) \geq (n+2)^3 xyz.$$

Using the arithmetic-geometric means inequality and the fact that $xyz \leq (xyz)^{2/3}$, we obtain that

$$\begin{aligned}
&(n+3-x)(n+3-y)(n+3-z) \\
&\qquad = (n+3)^3 - (n+3)^2(x+y+z) + (n+3)(xy+yz+zx) - xyz \\
&\qquad \geq (n+3)^3 - (n+3)^2[(n+3) - nxyz] + 3(n+3)(xyz)^{2/3} - xyz \\
&\qquad \geq n(n+3)^2 xyz + 3(n+3)xyz - xyz \\
&\qquad = (n^3 + 6n^2 + 9n + 3n + 9 - 1)xyz = (n+2)^3 xyz,
\end{aligned}$$

as desired, with equality if and only if $x = y = z = 1$.

4015. *Proposed by Michel Bataille.*

Find all real numbers a such that

$$a \cos x + (1-a)\cos\frac{x}{3} > \frac{\sin x}{x}$$

for every nonzero x of the interval $(-\frac{3\pi}{2}, \frac{3\pi}{2})$.

There were four submitted solutions for this problem, all of which were correct. We present the solution by Joel Schlosberg.

We will prove that the inequality

$$a\cos(x) + (1-a)\cos\left(\frac{x}{3}\right) > \frac{\sin(x)}{x} \tag{1}$$

holds for $x \in (-\frac{3\pi}{2}, \frac{3\pi}{2}) \setminus \{0\}$ if and only if $a \in (-\infty, 1/4]$.

Substituting $y = x/3$ and dividing by y^2, the above inequality is equivalent to

$$a\left(\frac{\cos(y) - \cos(3y)}{y^2}\right) < \frac{3y\cos(y) - \sin(3y)}{3y^3}, \tag{2}$$

for $|y| \in (0, \pi/2)$. By repeated applications of l'Hospital's Rule,

$$\lim_{y\to 0} \frac{\cos(y) - \cos(3y)}{y^2} = \lim_{y\to 0} \frac{-\sin(y) + 3\sin(3y)}{2y} = \lim_{y\to 0} \frac{-\cos(y) + 9\cos(3y)}{2} = 4,$$

and similarly,

$$\begin{aligned} \lim_{y\to 0} \frac{3y\cos(y) - \sin(3y)}{3y^3} &= \lim_{y\to 0} \frac{3\cos(y) - 3y\sin(y) - 3\cos(3y)}{9y^2} \\ &= \lim_{y\to 0} \frac{-6\sin(y) - 3y\cos(y) + 9\sin(3y)}{18y} \\ &= \lim_{y\to 0} \frac{-9\cos(y) + 3y\sin(y) + 27\cos(3y)}{18} = 1. \end{aligned}$$

Suppose that (1) holds for $x \in (-\frac{3\pi}{2}, \frac{3\pi}{2}) \setminus \{0\}$. Then the rewritten inequality (2) holds for non-zero y in a neighbourhood of 0. Taking $y \to 0$ on both sides yields $4a \le 1$, and so $a \in (-\infty, 1/4]$.

Since both sides of (2) are even functions of y, it is sufficient to prove (2) for $y \in (0, \pi/2)$. We let

$$f(x) = 9y\cos(y) - 4\sin(3y) + 3y\cos(3y).$$

For $y \in (0, \pi/2)$, it is well-known that $\tan(y) > y$, so

$$\begin{aligned} f'(y) &= 9(\cos(y) - y\sin(y) - \cos(3y) - y\sin(3y)) \\ &= 9(4(1-\cos^2(y))\cos(y) - 4y(1-\sin^2(y))\sin(y)) \\ &= 36(\sin^2(y)\cos(y) - y\cos^2(y)\sin(y)) \\ &= 36\cos^2(y)\sin(y)(\tan(y) - y) > 0 \end{aligned}$$

(via triple-angle formulas). Therefore, for $y \in (0, \pi/2)$, $f(y) > f(0) = 0$, which is equivalent to (2) when $a = 1/4$. Hence (1) holds for $a = 1/4$.

Suppose now that $a < 1/4$. For $y \in (0, \pi/2)$, we have

$$\cos(y) - \cos(3y) = 4(1-\cos^2(y))\cos(y) = 4\sin^2(y)\cos(y) > 0,$$

so that

$$a\left(\frac{\cos(y)-\cos(3y)}{y^2}\right) < \frac{1}{4}\left(\frac{\cos(y)-\cos(3y)}{y^2}\right) < \frac{3y\cos(y)-\sin(3y)}{3y^3},$$

and we are done.

Editor's Comments. All four solution methods involved similar elements: trigonometric identities used to rewrite the inequality, a limit (either by power series or by L'Hospital's Rule), and some calculus. Deiermann noted that if we set the right-hand side of the original inequality equal to 1 for $x = 0$, then we may allow equality at $x = 0$. Deiermann also suggested a generalization, propped up by Mathematica: if $n \geq 3$, then we have

$$a\cos(x) + (1-a)\cos\left(\frac{x}{n}\right) > \frac{\sin(x)}{x}.$$

for all non-zero $x \in (-\frac{3\pi}{2}, \frac{3\pi}{2}) \setminus \{0\}$ if and only if $a \leq \frac{n^2-3}{3(n^2-1)}$. A quick sketch of the argument by the editor seems to indicate that it is true, but the conclusion of the proof is still out of reach.

4016. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers. Find the maximal value of the expression

$$\frac{x+2y}{2x+3y+z} + \frac{y+2z}{2y+3z+x} + \frac{z+2x}{2z+3x+y}.$$

We received 21 submissions, all of which were correct. We present two solutions.

Solution 1, by Arkady Alt.

Let $S(x, y, z)$ denote the given expression. Then by using Cauchy-Schwarz Inequality we have

$$\begin{aligned} 3 - S(x,y,z) &= \sum_{cyc}\left(1 - \frac{x+2y}{2x+3y+z}\right) \\ &= \sum_{cyc}\frac{x+y+z}{2x+3y+z} \\ &= \frac{6(x+y+z)}{6}\sum_{cyc}\frac{1}{2x+3y+z} \\ &= \frac{1}{6}\sum_{cyc}(2x+3y+z)\cdot\sum_{cyc}\frac{1}{2x+3y+z} \\ &\geq \frac{1}{6}\cdot 9 = \frac{3}{2}. \end{aligned}$$

Hence, $S(x,y,z) \leq \dfrac{3}{2}$ and $S(x,x,x) = \dfrac{3}{2}$.

Solution 2, by Šefket Arslanagić.

Since the given inequality is homogeneous, we may assume that $x+y+z=1$. By the AM-HM Inequality, we have

$$\begin{aligned} S(x,y,z) &= 3-\left(\frac{1}{1+x+2y}+\frac{1}{1+y+2z}+\frac{1}{1+z+2x}\right) \\ &\leq 3-\frac{9}{(1+x+2y)+(1+y+2z)+(1+z+2x)} \\ &= 3-\frac{9}{6}=\frac{3}{2}. \end{aligned}$$

Hence, the maximum value of $S(x,y,z)$ is $\frac{3}{2}$ attained when $x=y=z$.

Editor's Comments. Kee-Wai Lau made an interesting and not-so-easy-to-see observation that

$$S(x,y,z)-\frac{3}{2}=-\frac{\sum(3x+y+2z)(x+y-2z)^2}{6\prod(2x+3y+z)}\leq 0.$$

4017. *Proposed by Michel Bataille.*

Let P be a point of the incircle γ of a triangle ABC. The perpendiculars to BC, CA and AB through P meet γ again at U, V and W, respectively. Prove that one of the numbers $PU\cdot BC, PV\cdot CA, PW\cdot AB$ is the sum of the other two.

From the 6 correct submissions we received, we present a composite of the similar solutions by Šefket Arslanagić, Ricard Peiró i Estruch, and Joel Schlosberg.

Since $PV\perp CA$ and $PW\perp AB$, $\angle VPW$ is either equal to or supplementary to $\angle BAC$, so

$$\sin\angle VPW=\sin\angle BAC=\sin A;$$

similarly,

$$\sin\angle WPU=\sin B \quad \text{and} \quad \sin\angle UPV=\sin C.$$

Moreover, because $PUVW$ is cyclic we have

$$\sin\angle VPW=\sin\angle VUW,\ \sin\angle WPU=\sin\angle WVU,\ \sin\angle UPV=\sin\angle UWV.$$

Finally, the Law of Sines applied to ΔUVW implies

$$\frac{VW}{WU}=\frac{\sin\angle VUW}{\sin\angle WVU} \quad \text{and} \quad \frac{UV}{WU}=\frac{\sin\angle UWV}{\sin\angle WVU},$$

while applied to ΔABC implies

$$\frac{\sin A}{\sin B}=\frac{BC}{CA} \quad \text{and} \quad \frac{\sin C}{\sin B}=\frac{AB}{CA}.$$

Let us suppose that the diagram has been labeled so that the quadrilateral $PUVW$ is cyclic in that order, whence PV is the diagonal, and Ptolemy's theorem says that $PV \cdot WU = PU \cdot VW + PW \cdot UV$. Putting it all together, we get

$$\begin{aligned} PV &= PU \cdot \frac{VW}{WU} + PW \cdot \frac{UV}{WU} \\ &= PU \cdot \frac{\sin \angle VUW}{\sin \angle WVU} + PW \cdot \frac{\sin \angle UWV}{\sin \angle WVU} \\ &= PU \cdot \frac{\sin \angle VPW}{\sin \angle WPU} + PW \cdot \frac{\sin \angle UPV}{\sin \angle WPU} \\ &= PU \cdot \frac{\sin A}{\sin B} + PW \cdot \frac{\sin C}{\sin B} \\ &= PU \cdot \frac{BC}{CA} + PW \cdot \frac{AB}{CA}. \end{aligned}$$

Thus, we conclude that $PV \cdot CA = PU \cdot BC + PW \cdot AB$; in other words, the product involving the diagonal of the quadrilateral equals the sum of the products involving the sides.

Additionally, the proposer observed (and proved) that the area of ΔUVW is independent of the choice of P on γ.

4018. *Proposed by Ovidiu Furdui.*

Let

$$I_n = \int_0^1 \cdots \int_0^1 \ln(x_1 x_2 \cdots x_n) \ln(1 - x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n,$$

where $n \geq 1$ is an integer. Prove that this integral converges and find its value.

We received three solutions, all of which were correct and complete. We present the solution by the proposer.

The integral equals

$$n\left(n + 1 - \zeta(2) - \zeta(3) - \cdots - \zeta(n+1)\right),$$

where ζ denotes the Riemann zeta function.

We have, based on symmetry reasons, that for all $i, j = 1, 2, \ldots, n$

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 \ln x_i \ln(1 - x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n \\ &\quad = \int_0^1 \cdots \int_0^1 \ln x_j \ln(1 - x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n, \end{aligned}$$

and this implies that

$$\begin{aligned}
I_n &= n\int_0^1 \cdots \int_0^1 \ln x_1 \ln(1 - x_1x_2\cdots x_n)dx_1dx_2\cdots dx_n\\
&= n\int_0^1 \cdots \int_0^1 -\ln x_1 \sum_{k=1}^{\infty}\frac{(x_1\cdots x_n)^k}{k}dx_1dx_2\cdots dx_n\\
&\overset{(*)}{=} n\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 \cdots \int_0^1 -\ln x_1(x_1\cdots x_n)^k dx_1dx_2\cdots dx_n\\
&= n\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 (-x_1^k\ln x_1)\mathrm{d}x_1\int_0^1 x_2^k\mathrm{d}x_2\cdots\int_0^1 x_n^k\mathrm{d}x_n\\
&= n\sum_{k=1}^{\infty}\frac{1}{k(k+1)^{n+1}}.
\end{aligned}$$

We used at step (*) Tonelli's Theorem for nonnegative functions, which allows us to interchange the integration sign and the summation sign.

Let $S_{n+1} = \sum_{k=1}^{\infty}\frac{1}{k(k+1)^{n+1}}$. Since

$$\frac{1}{k(k+1)^{n+1}} = \frac{1}{k(k+1)^n} - \frac{1}{(k+1)^{n+1}},$$

we have, by summation, that $S_{n+1} = S_n - (\zeta(n+1) - 1)$. This implies, since $S_1 = 1$, that

$$S_{n+1} = S_1 - (\zeta(2)+\zeta(3)+\cdots+\zeta(n+1)-n) = n+1-\zeta(2)-\zeta(3)-\cdots-\zeta(n+1).$$

Hence

$$I_n = n\left(n+1-\zeta(2)-\zeta(3)-\cdots-\zeta(n+1)\right),$$

and the problem is solved.

4019. *Proposed by George Apostolopoulos.*

A triangle with side lengths a, b, c has perimeter 3. Prove that

$$a^3+b^3+c^3+a^4+b^4+c^4 \geq 2(a^2b^2+b^2c^2+c^2a^2).$$

We received 21 correct solutions. We present the solution by AN-anduud Problem Solving Group.

The claimed inequality is equivalent to

$$(a^3+b^3+c^3)(a+b+c)+3(a^4+b^4+c^4) \geq 6(a^2b^2+b^2c^2+c^2a^2)$$

or

$$[(a^3b+b^3a)+(a^3c+c^3a)+(b^3c+c^3b)]+[2(a^4+b^4)+2(b^4+c^4)+2(c^4+a^4)]$$
$$\geq 6(a^2b^2+b^2c^2+c^2a^2).$$

By the AM-GM Inequality we have

$$a^3b+b^3a \geq 2a^2b^2,\ a^3c+c^3a \geq 2a^2c^2,\ b^3c+c^3b \geq 2b^2c^2$$

and

$$a^4+b^4 \geq 2a^2b^2,\ b^4+c^4 \geq 2b^2c^2,\ c^4+a^4 \geq 2a^2c^2.$$

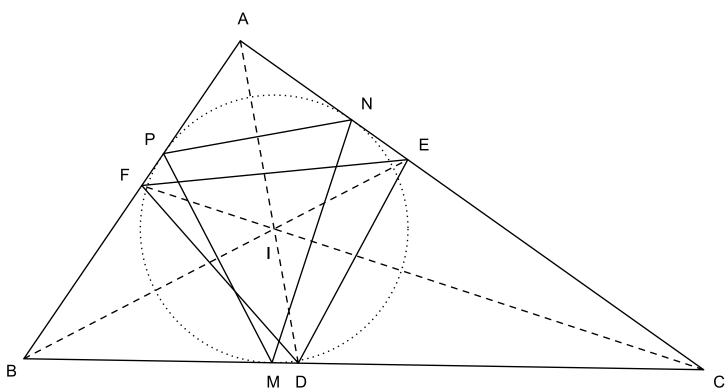
Adding the above inequalities, we obtain the desired inequality. Equality holds if and only if $a=b=c=1$.

4020. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let ABC be a triangle and let the internal bisectors from A, B and C intersect the sides BC, CA and AB in D, E and F, respectively. The incircle of ΔABC touches the sides BC, CA and AB in M, N, and P, respectively. Prove that $[MNP] \leq [DEF]$, where $[\cdot]$ denotes the area of the specified triangle.

We received eleven submissions, of which nine were correct and complete. We present the solution by Šefket Arslanagić, slightly modified by the editor.

Denote by α, β and γ the angles BAC, ABC and respectively ACB of the triangle, and let r be the radius of the incircle.



From the quadrilateral $PIMB$ note that $\angle PIM = 180^\circ - \angle ABC = 180^\circ - \beta$, whence

$$[PIM] = \frac{PI \cdot MI}{2} \sin(\angle PIM) = \frac{r^2}{2} \sin(180^\circ - \beta) = \frac{r^2}{2} \sin \beta.$$

Similarly, we calculate $[MIN]$ and $[NIP]$, and we get

$$\begin{aligned}[MNP] &= [PIM]+[MIN]+[NIP] \\ &= \frac{r^2}{2} \cdot (\sin\beta + \sin\gamma + \sin\alpha).\end{aligned} \tag{1}$$

On the other hand, we have $\angle FID = \angle AIC = 180^\circ - \frac{\alpha}{2} - \frac{\gamma}{2}$, and so

$$[FID] = \frac{ID \cdot IF}{2} \sin(\angle FID) = \frac{ID \cdot IF}{2} \sin \frac{\alpha + \gamma}{2}.$$

Similarly, calculate the area of $\triangle EIF$ and $\triangle DIE$. We have

$$\begin{aligned}[DEF] &= [DIE] + [EIF] + [FID] \\ &= \frac{ID \cdot IE}{2} \sin \frac{\alpha + \beta}{2} + \frac{IE \cdot EF}{2} \sin \frac{\beta + \gamma}{2} + \frac{ID \cdot IF}{2} \sin \frac{\alpha + \gamma}{2}.\end{aligned} \tag{2}$$

The triangles $\triangle PIF$, $\triangle MID$ and $\triangle NIE$ are all right-angled triangles, from which it follows that $IF \geq r$, $ID \geq r$ and $IE \geq r$. Hence, from the formula for $[DEF]$ above we get

$$[DEF] \geq \frac{r^2}{2} \left(\sin \frac{\alpha + \beta}{2} + \sin \frac{\beta + \gamma}{2} + \sin \frac{\alpha + \gamma}{2} \right).$$

Comparing this with the formula for $[MNP]$ in (1), in order to show that $[MNP] \leq [DEF]$ it is sufficient to show that

$$\sin \alpha + \sin \beta + \sin \gamma \leq \sin \frac{\alpha + \beta}{2} + \sin \frac{\beta + \gamma}{2} + \sin \frac{\alpha + \gamma}{2}. \tag{3}$$

However, using the sum to product trigonometric formula, we have

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \leq 2 \sin \frac{\alpha + \beta}{2},$$

where the inequality follows from the fact that $\sin \frac{\alpha+\beta}{2} \geq 0$ and $\cos \frac{\alpha-\beta}{2} \leq 1$. Similarly we have $\sin \beta + \sin \gamma \leq 2 \sin \frac{\beta+\gamma}{2}$ and $\sin \gamma + \sin \alpha \leq 2 \sin \frac{\alpha+\gamma}{2}$, which we can add to get the inequality in (3), the final step we need in order to conclude that $[MNP] \leq [DEF]$.