

THE OLYMPIAD CORNER

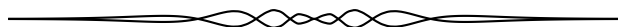
No. 340

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2016**; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.



OC266. Soit D un point sur le côté BC d'un triangle acutangle ABC . Soit O_1 et O_2 les centres respectifs des cercles circonscrits aux triangles ABD et ACD . Démontrer que la droite qui joint le centre du cercle circonscrit au triangle ABC et l'orthocentre du triangle O_1O_2D est parallèle à BC .

OC267. On a empilé des disques rouges et des disques bleus de même grandeur de manière à former une pile de forme triangulaire. Le niveau supérieur de la pile compte un disque et chaque niveau compte un disque de plus que le niveau immédiatement au-dessous. Chaque disque qui n'est pas au niveau le plus bas touche à deux disques au-dessous de lui et ce disque est bleu si les deux disques sont de la même couleur. Autrement, il est rouge.

Supposons que le niveau le plus bas compte 2048 disques dont 2014 sont rouges. Quelle est la couleur du disque au niveau supérieur ?

OC268. Soit $\mathbb{Z}_{\geq 0}$ l'ensemble des entiers supérieurs ou égaux à 0. Déterminer toutes les fonctions $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ qui vérifient la relation

$$f(f(f(n))) = f(n+1) + 1$$

pour tout $n \in \mathbb{Z}_{\geq 0}$.

OC269. Soit x, y, z les nombres réels qui satisfont à

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 8 \quad \text{et} \quad x^3 + y^3 + z^3 = 1.$$

Déterminer la valeur minimale de $x^4 + y^4 + z^4$.

OC270. Étant donné un entier pair strictement positif n , on place chacun des nombres $1, 2, \dots, n^2$ sur une des cases d'un damier $n \times n$. Soit S_1 la somme des

nombres placés sur les cases noires et S_2 la somme des nombres placés sur les cases blanches. Déterminer tous les n pour lesquels il est possible d'obtenir $\frac{S_1}{S_2} = \frac{39}{64}$.

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OC266. In an acute triangle ABC , a point D lies on the segment BC . Let O_1, O_2 denote the circumcentres of triangles ABD and ACD respectively. Prove that the line joining the circumcentre of triangle ABC and the orthocentre of triangle O_1O_2D is parallel to BC .

OC267. Blue and red circular disks of identical size are packed together to form a triangle. The top level has one disk and each level has 1 more disk than the level above it. Each disk not at the bottom level touches two disks below it and its colour is blue if these two disks are of the same colour. Otherwise its colour is red.

Suppose the bottom level has 2048 disks of which 2014 are red. What is the colour of the disk at the top?

OC268. Let $\mathbb{Z}_{\geq 0}$ be the set of all nonnegative integers. Find all the functions $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ satisfying the relation

$$f(f(f(n))) = f(n + 1) + 1$$

for all $n \in \mathbb{Z}_{\geq 0}$.

OC269. Let x, y, z be the real numbers that satisfy the following :

$$(x - y)^2 + (y - z)^2 + (z - x)^2 = 8, x^3 + y^3 + z^3 = 1.$$

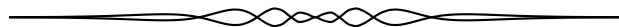
Find the minimum value of $x^4 + y^4 + z^4$.

OC270. For even positive integer n we put all numbers $1, 2, \dots, n^2$ into the squares of an $n \times n$ chessboard (each number appears once and only once). Let S_1 be the sum of the numbers put in the black squares and S_2 be the sum of the numbers put in the white squares. Find all n such that we can achieve $\frac{S_1}{S_2} = \frac{39}{64}$.



OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2014 : 40(10), p. 417–419.

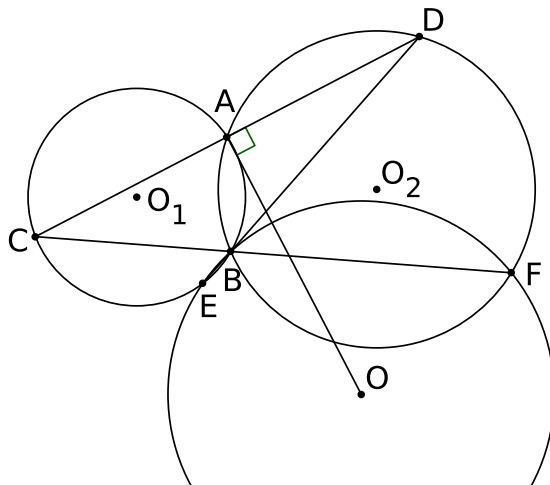


OC206. Two circles K_1 and K_2 of different radii intersect at two points A and B , and let C and D be two points on K_1 and K_2 , respectively, such that A is the midpoint of the segment CD . The extension of DB meets K_1 at another point E , the extension of CB meets K_2 at another point F . Let ℓ_1 and ℓ_2 be the perpendicular bisectors of CD and EF , respectively.

1. Show that ℓ_1 and ℓ_2 have a unique common point (denoted by P).
2. Prove that the lengths of CA , AP and PE are the side lengths of a right triangle.

Originally problem 1 of the 2013 China National Olympiad.

We received two correct submissions. We present the solution by Oliver Geupel.



Let K be the circumcircle of $\triangle BEF$. Let O , O_1 , O_2 , r , r_1 , and r_2 be the centres and radii of K , K_1 , and K_2 , respectively. For a point X and a circle Γ , let $\mathcal{P}(X, \Gamma)$ denote the power of X with respect to Γ . We have $O \in \ell_2$ and

$$\mathcal{P}(C, K) = \mathcal{P}(C, K_2) = CA \cdot CD = DA \cdot DC = \mathcal{P}(D, K_1) = \mathcal{P}(D, K).$$

We deduce $CO = DO$; whence $O \in \ell_1$. Thus $O \in \ell_1 \cap \ell_2$.

To complete part 1, it is enough to show that $\ell_1 \neq \ell_2$. We prove it by contradiction. Suppose to the contrary that $\ell_1 = \ell_2$. Then $CD \parallel EF$, so that $\triangle BCD$ and $\triangle BFE$ are homothetic. Hence the points A , B , and the midpoint G of EF lie on a common

line l . But A , G and O lie on $\ell_1 = \ell_2$. It follows $l = \ell_1$, so that $B \in \ell_1$. As a consequence CD is parallel to O_1O_2 . If the distance of the lines is d , we obtain

$$r_1^2 = d^2 + \frac{AC^2}{4} = d^2 + \frac{AB^2}{4} = r_2^2,$$

which contradicts the hypothesis $r_1 \neq r_2$. Part 1 is complete.

Observe that

$$CO^2 - r^2 = \mathcal{P}(C, K) = \mathcal{P}(C, K_2) = 2CA^2.$$

In the right triangle ACO we have $AO^2 + CA^2 = CO^2 = 2CA^2 + r^2$. Consequently, $AO^2 = CA^2 + r^2 = CA^2 + OE^2$. By the converse of the Pythagorean Theorem, CA , AO , and OE are the side lengths of a right triangle. This completes part 2.

OC207. Find all injective functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ that satisfy :

$$|f(x) - f(y)| \leq |x - y|$$

for any $x, y \in \mathbb{Z}$.

Originally problem X-3 of the 2013 Romanian National Olympiad.

We received four correct submissions. We present the solution by Michel Bataille.

We show that the solutions are the functions $x \mapsto x + a$ and $x \mapsto -x + a$ where a is an arbitrary integer.

Such a function is clearly a solution. Conversely, let f be any solution and let $g : \mathbb{Z} \rightarrow \mathbb{Z}$ be the function defined by $g(x) = f(x) - f(0)$. Then g is injective, satisfies $|g(x) - g(y)| \leq |x - y|$ for any $x, y \in \mathbb{Z}$ and in addition, $g(0) = 0$. Thus, we may as well suppose that $f(0) = 0$ from the beginning and show that $f(x) = x$ for all $x \in \mathbb{Z}$ or $f(x) = -x$ for all $x \in \mathbb{Z}$.

Let f be a solution such that $f(0) = 0$. Then, $|f(x)| \leq |x|$ for any integer x and in particular $|f(1)| \leq 1$. In addition, since f is injective, we have $f(1) \neq f(0)$, that is, $f(1) \neq 0$. It follows that $f(1) = 1$ or $f(1) = -1$.

First, we suppose that $f(1) = 1$. Assume that for some positive integer n , we have $f(k) = k$ for each element k of $\{0, 1, \dots, n\}$. Then, from $|f(n+1) - f(n)| \leq |(n+1) - n| = 1$ and $f(n+1) \neq f(n)$, we deduce that $f(n+1) - f(n) = 1$ or -1 . However, $f(n+1) - f(n) = -1$ implies $f(n+1) = n - 1 = f(n - 1)$, contradicting f injective. Thus, $f(n+1) = n + 1$ and so $f(k) = k$ for each element k of $\{0, 1, \dots, n+1\}$. By induction, we have proved that for any positive integer n , we have $f(k) = k$ for each element k of $\{0, 1, \dots, n\}$ and in particular, $f(n) = n$.

The function $h : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $h(x) = -f(-x)$ is injective, satisfies $|h(x) - h(y)| \leq |x - y|$ for any $x, y \in \mathbb{Z}$ and $h(0) = 0$. Thus, $h(n) = n$ for any positive integer n , which means that $f(-n) = -n$ for any positive integer n . Gathering the results, we see that $f(x) = x$ for any integer x .

In the case when $f(1) = -1$, from what has been already obtained, the function $-f$ satisfies $(-f)(x) = x$ for any integer x , hence $f(x) = -x$ for any integer x .

OC208. Find all non-integers x such that $x + \frac{13}{x} = [x] + \frac{13}{[x]}$ where $[x]$ means the greatest integer n less than or equal to x .

Originally problem 5 of the 2013 China Northern Mathematical Olympiad.

We received seven correct submissions. We present the solution by Digby Smith.

Let $x = m + a$ with $m = [x]$ and $a \in \mathbb{R}$ between 0 and 1. Note that if $0 < x < 1$, then $m = 0$ and there is no solution (the right hand side above is undefined). Thus, suppose that $m \neq 0$. Substituting into the above equation yields

$$m + a + \frac{13}{m + a} = m + \frac{13}{m}$$

Simplifying yields

$$m(m + a) = 13$$

We proceed in cases. When $m \geq 4$, we see that $m(m + a) \geq 16$ which is a contradiction. When $m \in \{1, 2, 3\}$ then $m(m + a) < m(m + 1) \leq 12$, also a contradiction. Now, if $m \leq -5$ then $m(m + a) > 20$ and once again there is no solution. For $m \in \{-1, -2, -3\}$, we see that $m(m + a) < m^2 \leq 9$, also a contradiction. Thus, this leaves only the case $m = -4$. Substituting this into the equation gives

$$(-4)(-4 + a) = 13 \quad \Rightarrow \quad a = 3/4$$

Hence, $x = m + a = -4 + 3/4 = -13/4$ and this is the only solution.

OC209. The sequence a_1, a_2, \dots, a_n consists of the numbers $1, 2, \dots, n$ in some order. For which positive integers n is it possible that the $n + 1$ numbers $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \dots + a_n$ all have different remainders when divided by $n + 1$?

Originally problem 2 of the 2013 Canadian Mathematical Olympiad.

We present the solution by The Missouri State University Problem Solving Group. There were no other submissions.

Since for any arrangement we have $\sum_{i=1}^n a_i = \frac{n(n+1)}{2}$, if n is even, then this sum leaves a remainder of 0 when divided by $n + 1$ meaning that this case is impossible in this case. Thus, suppose that n is odd. Consider the arrangement given by

$$1, n - 1, 3, n - 3, 5, n - 5, \dots, n.$$

This arrangement satisfies the given criteria. Indeed, observe that the sequence modulo $n + 1$ is equivalent to

$$1, -2, 3, -4, 5, -6, \dots, n$$

and so the sequence $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \dots + a_n$ modulo $n + 1$ is equivalent to

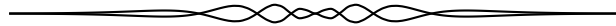
$$0, 1, -1, 2, -2, 3, -3, \dots, (n + 1)/2.$$

OC210.

Find all positive integers a such that for any positive integer $n \geq 5$ we have $2^n - n^2 \mid a^n - n^a$.

Originally problem 8 of the 2013 China Western Mathematical Olympiad.

We received no submissions to this problem.



Math Quotes

In the mathematics I can report no deficiency, except that it be that men do not sufficiently understand the excellent use of the pure mathematics, in that they do remedy and cure many defects in the wit and faculties intellectual. For if the wit be too dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it. So that as tennis is a game of no use in itself, but of great use in respect it maketh a quick eye and a body ready to put itself into all postures; so in the mathematics, that use which is collateral and intervenient is no less worthy than that which is principal and intended.

Roger Bacon in John Fauvel and Jeremy Gray (eds.) "A History of Mathematics : A Reader", Sheridan House, 1987.