OLYMPIAD SOLUTIONS

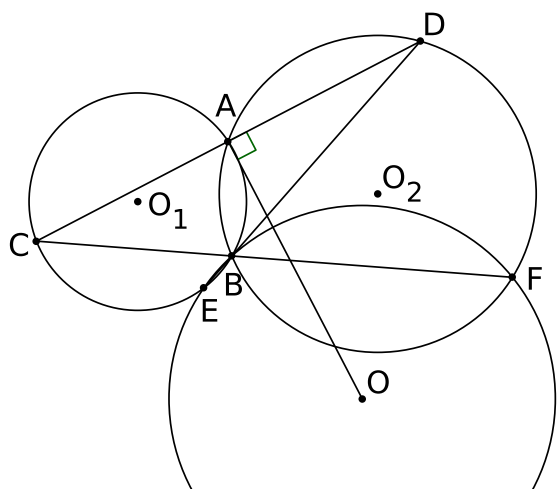
Les énoncés des problèmes dans cette section paraissent initialement dans 2014 : 40(10), p. 417–419.

OC206. Two circles K_1 and K_2 of different radii intersect at two points A and B, and let C and D be two points on K_1 and K_2, respectively, such that A is the midpoint of the segment CD. The extension of DB meets K_1 at another point E, the extension of CB meets K_2 at another point F. Let ℓ_1 and ℓ_2 be the perpendicular bisectors of CD and EF, respectively.

1. Show that ℓ_1 and ℓ_2 have a unique common point (denoted by P).
2. Prove that the lengths of CA, AP and PE are the side lengths of a right triangle.

Originally problem 1 of the 2013 China National Olympiad.

We received two correct submissions. We present the solution by Oliver Geupel.



Let K be the circumcircle of $\triangle BEF$. Let O, O_1, O_2, r, r_1, and r_2 be the centres and radii of K, K_1, and K_2, respectively. For a point X and a circle Γ, let $\mathcal{P}(X,\Gamma)$ denote the power of X with respect to Γ. We have $O \in \ell_2$ and

$$\mathcal{P}(C,K) = \mathcal{P}(C,K_2) = CA \cdot CD = DA \cdot DC = \mathcal{P}(D,K_1) = \mathcal{P}(D,K).$$

We deduce $CO = DO$; whence $O \in \ell_1$. Thus $O \in \ell_1 \cap \ell_2$.

To complete part 1, it is enough to show that $\ell_1 \neq \ell_2$. We prove it by contradiction. Suppose to the contrary that $\ell_1 = \ell_2$. Then $CD \parallel EF$, so that $\triangle BCD$ and $\triangle BFE$ are homothetic. Hence the points A, B, and the midpoint G of EF lie on a common line l. But A, G and O lie on $\ell_1 = \ell_2$. It follows $l = \ell_1$, so that $B \in \ell_1$. As a consequence CD is parallel to O_1O_2. If the distance of the lines is d, we obtain

$$r_1^2 = d^2 + \frac{AC^2}{4} = d^2 + \frac{AB^2}{4} = r_2^2,$$

which contradicts the hypothesis $r_1 \neq r_2$. Part 1 is complete.

Observe that
$$CO^2 - r^2 = \mathcal{P}(C, K) = \mathcal{P}(C, K_2) = 2CA^2.$$
In the right triangle ACO we have $AO^2 + CA^2 = CO^2 = 2CA^2 + r^2$. Consequently, $AO^2 = CA^2 + r^2 = CA^2 + OE^2$. By the converse of the Pythagorean Theorem, CA, AO, and OE are the side lengths of a right triangle. This completes part 2.

OC207. Find all injective functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy :

$$|f(x) - f(y)| \leq |x - y|$$

for any $x, y \in \mathbb{Z}$.

Originally problem X-3 of the 2013 Romanian National Olympiad.

We received four correct submissions. We present the solution by Michel Bataille.

We show that the solutions are the functions $x \mapsto x + a$ and $x \mapsto -x + a$ where a is an arbitrary integer.

Such a function is clearly a solution. Conversely, let f be any solution and let $g : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $g(x) = f(x) - f(0)$. Then g is injective, satisfies $|g(x) - g(y)| \leq |x - y|$ for any $x, y \in \mathbb{Z}$ and in addition, $g(0) = 0$. Thus, we may as well suppose that $f(0) = 0$ from the beginning and show that $f(x) = x$ for all $x \in \mathbb{Z}$ or $f(x) = -x$ for all $x \in \mathbb{Z}$.

Let f be a solution such that $f(0) = 0$. Then, $|f(x)| \leq |x|$ for any integer x and in particular $|f(1)| \leq 1$. In addition, since f is injective, we have $f(1) \neq f(0)$, that is, $f(1) \neq 0$. It follows that $f(1) = 1$ or $f(1) = -1$.

First, we suppose that $f(1) = 1$. Assume that for some positive integer n, we have $f(k) = k$ for each element k of $\{0, 1, \ldots, n\}$. Then, from $|f(n+1) - f(n)| \leq |(n+1) - n| = 1$ and $f(n+1) \neq f(n)$, we deduce that $f(n+1) - f(n) = 1$ or -1. However, $f(n+1) - f(n) = -1$ implies $f(n+1) = n - 1 = f(n-1)$, contradicting f injective. Thus, $f(n+1) = n+1$ and so $f(k) = k$ for each element k of $\{0, 1, \ldots, n+1\}$. By induction, we have proved that for any positive integer n, we have $f(k) = k$ for each element k of $\{0, 1, \ldots, n\}$ and in particular, $f(n) = n$.

The function $h : \mathbb{Z} \to \mathbb{Z}$ defined by $h(x) = -f(-x)$ is injective, satisfies $|h(x) - h(y)| \leq |x - y|$ for any $x, y \in \mathbb{Z}$ and $h(0) = 0$. Thus, $h(n) = n$ for any positive integer n, which means that $f(-n) = -n$ for any positive integer n. Gathering the results, we see that $f(x) = x$ for any integer x.

In the case when $f(1) = -1$, from what has been already obtained, the function $-f$ satisfies $(-f)(x) = x$ for any integer x, hence $f(x) = -x$ for any integer x.

OC208. Find all non-integers x such that $x + \frac{13}{x} = \lfloor x \rfloor + \frac{13}{\lfloor x \rfloor}$ where $\lfloor x \rfloor$ means the greatest integer n less than or equal to x.

Originally problem 5 of the 2013 China Northern Mathematical Olympiad.

We received seven correct submissions. We present the solution by Digby Smith.

Let $x = m + a$ with $m = \lfloor x \rfloor$ and $a \in \mathbb{R}$ between 0 and 1. Note that if $0 < x < 1$, then $m = 0$ and there is no solution (the right hand side above is undefined). Thus, suppose that $m \neq 0$. Substituting into the above equation yields

$$m + a + \frac{13}{m + a} = m + \frac{13}{m}$$

Simplifying yields

$$m(m + a) = 13$$

We proceed in cases. When $m \geq 4$, we see that $m(m + a) \geq 16$ which is a contradiction. When $m \in \{1, 2, 3\}$ then $m(m+a) < m(m+1) \leq 12$, also a contradiction. Now, if $m \leq -5$ then $m(m + a) > 20$ and once again there is no solution. For $m \in \{-1, -2, -3\}$, we see that $m(m + a) < m^2 \leq 9$, also a contradiction. Thus, this leaves only the case $m = -4$. Substituting this into the equation gives

$$(-4)(-4 + a) = 13 \qquad \Rightarrow \qquad a = 3/4$$

Hence, $x = m + a = -4 + 3/4 = -13/4$ and this is the only solution.

OC209. The sequence $a_1, a_2, \ldots, a_n$ consists of the numbers $1, 2, \ldots, n$ in some order. For which positive integers n is it possible that the $n+1$ numbers $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_n$ all have different remainders when divided by $n + 1$?

Originally problem 2 of the 2013 Canadian Mathematical Olympiad.

We present the solution by The Missouri State University Problem Solving Group. There were no other submissions.

Since for any arrangement we have $\displaystyle\sum_{i=1}^{n} a_i = \frac{n(n+1)}{2}$, if n is even, then this sum leaves a remainder of 0 when divided by $n+1$ meaning that this case is impossible in this case. Thus, suppose that n is odd. Consider the arrangement given by

$$1, n-1, 3, n-3, 5, n-5, \cdots, n.$$

This arrangement satisfies the given criteria. Indeed, observe that the sequence modulo $n + 1$ is equivalent to

$$1, -2, 3, -4, 5, -6, \cdots, n$$

and so the sequence $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_n$ modulo $n+1$ is equivalent to

$$0, 1, -1, 2, -2, 3, -3, \cdots, (n + 1)/2.$$

OC210.

Find all positive integers a such that for any positive integer $n \geq 5$ we have $2^n - n^2 \mid a^n - n^a$.

Originally problem 8 of the 2013 China Western Mathematical Olympiad.

We received no submissions to this problem.

Math Quotes

In the mathematics I can report no deficience, except that it be that men do not sufficiently understand the excellent use of the pure mathematics, in that they do remedy and cure many defects in the wit and faculties intellectual. For if the wit be too dull, they sharpen it ; if too wandering, they fix it ; if too inherent in the sense, they abstract it. So that as tennis is a game of no use in itself, but of great use in respect it maketh a quick eye and a body ready to put itself into all postures ; so in the mathematics, that use which is collateral and intervenient is no less worthy than that which is principal and intended.

Roger Bacon in John Fauvel and Jeremy Gray (eds.) "A History of Mathematics : A Reader", Sheridan House, 1987.