Approaching the Extremum

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The concept of the derivative, which is introduced in high school, provides an easy way to find the maximum and minimum values of a given one-variable function $y = f(x)$. However, some problems require finding maximum and minimum values of a multivariate function. The methods involving a derivative in this case are much more complicated and are not studied in high school; therefore, one needs to develop elementary approaches to such problems. This article describes one such method devised in 1885 by German mathematician, Rudolf Sturm.

Let us begin by studying two problems.

Problem 1. Which convex n-gon inscribed in a unit circle has the largest area?

Problem 2. Find n numbers such that their sum is equal to 1 and the sum of their squares is as small as possible.

In Figure 1a), we have a triangle ABC inscribed in a circle with $|AB| > |BC|$. What will happen to the area of this triangle if, without changing the position of points A and C, we move the point B along the arc AC so that the lengths of sides AB and BC get closer to each other (that is, the difference $|AB| - |BC|$ gets smaller)? Let B' be the point on the arc AC on the same side of AC as B such that $|AB'| = |CB|$. To bring the lengths of AB and BC closer together, we can replace B by any point on the arc BB'. Then the altitude BH of ABC increases and hence the area of ABC increases.

This observation shows that a non-regular n-gon cannot be a solution to Problem 1. Indeed, if AB and BC are non-equal neighbouring sides of such an n-gon (see Figure 1b)), then, by replacing the point B by any point on the arc BB', we increase the area of the triangle ABC and hence the area of the n-gon.

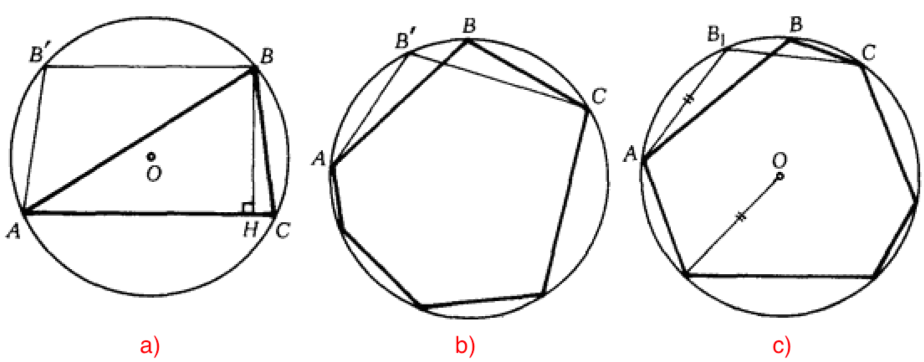

Figure 1: Illustration for Problem 1.

Moving onto Problem 2, let us see what happens to the sum of the squares of two numbers if we bring these numbers closer together without changing their sum.

Let a and b be the two given numbers with $a < b$ and let $0 < \epsilon < b - a$. Then

$$(a+\epsilon)^2 + (b-\epsilon)^2 = a^2 + b^2 - 2\epsilon(b-a-\epsilon) < a^2 + b^2.$$

We see that the sum of their squares got smaller. This observation shows that a set of n numbers with the sum of 1 which contains two unequal numbers cannot be a solution to Problem 2. Indeed, by bringing two numbers closer together, while keeping their sum constant, we reduce the sum of their squares.

One might think that we have solved Problems 1 and 2 : the solution to Problem 1 is a regular n-gon inscribed in a unit circle and the solution to Problem 2 is a set of numbers $\left\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right\}$. However, all we have proved is that no non-regular n-gon can be a solution to Problem 1 and that no set of n numbers not equal to $\left\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right\}$ can be a solution to Problem 2. But there is one more possibility : no n-gon is a solution to Problem 1 and no set of n numbers is a solution to Problem 2. Proving existence of a mathematical object with given properties is often no small feat. But solving any given problem is usually easier if you know a potential answer, which is the case with Problems 1 and 2.

Solution to Problem 1. Let us prove that the area of any n-gon inscribed in a given circle is no greater than the area of the regular n-gon inscribed in the same circle. This would mean that the regular n-gon is indeed the solution to Problem 1, whereas before we have only proved that a non-regular n-gon is not the solution to Problem 1.

Note that if we replace point B of the inscribed n-gon with B' (see Figure 1b)), we interchange the larger and smaller sides without changing the area. Repeating this operation, we can interchange the sides of a given non-regular n-gon to obtain an n-gon of the same area whose smallest and largest sides share a vertex.

Let P be the inscribed non-regular n-gon. The smallest side of this n-gon subtends an arc with the central angle less than $\frac{360^\circ}{n}$, while its largest side subtends an arc with the central angle greater than $\frac{360^\circ}{n}$. Switch the sides of the n-gon so that the smallest and the largest sides are next to each other. In the hexagon of Figure 1c), these are the sides AB and BC. On the arc AC, measure out the arc AB_1, whose central angle equals $\frac{360^\circ}{n}$. Replacing B with B_1, we will get an n-gon P_1 whose area is larger than the area of P. Moreover, the n-gon P_1 has at least one side subtending an arc with central angle equal to $\frac{360^\circ}{n}$ and, hence, has one side equal to the side of the regular n-gon. If the n-gon P_1 is not regular, we repeat the process to get an n-gon P_2 of larger area with at least one more side equal to the side of a regular n-gon. After a finite number of steps, we will arrive at the regular n-gon whose area is greater than the area of P. $\square$

Solution to Problem 2. Let the sum of numbers $a_1, a_2, \ldots, a_n$ equal 1. If not all of these numbers are equal, then the smallest one is less than $\frac{1}{n}$ and the largest one is greater than $\frac{1}{n}$. Suppose without loss of generality that $a_1 < \frac{1}{n}$ and $a_2 > \frac{1}{n}$. Replacing a_1 with $\frac{1}{n}$ and a_2 with $a_1 + a_2 - \frac{1}{n}$, we bring a_1 and a_2 closer together thereby increasing the sum of their squares without changing the total sum. As such, we get a new set of numbers that sum to 1 while the sum of their squares is less than that of the original set. Moreover, this new set has at least one number equal to $\frac{1}{n}$. If not all numbers in the new set equal $\frac{1}{n}$, we repeat the process. After a finite number of repetitions, we will arrive at the set $\left\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right\}$ with the sum of squares less than the sum of squares of the original set. Therefore, set $\left\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right\}$ is the solution to Problem 2. □

If we take some constant S, then the above argument shows that among all sets of numbers $a_1, a_2, \ldots, a_n$ that sum to S, the set with the smallest sum of squares is $\left\{\frac{S}{n}, \frac{S}{n}, \ldots, \frac{S}{n}\right\}$. Therefore, given an arbitrary set of numbers $a_1, a_2, \ldots, a_n$, the following inequality holds :

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq n\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2,$$

and therefore

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2.$$

Hence, the arithmetic mean of the squares of a set of numbers is no less than the square of the arithmetic mean of these numbers themselves. Further, equality holds only if $a_1 = a_2 = \ldots = a_n$ (left as an exercise to the reader).

Of course you notice that even though Problems 1 and 2 seemed unrelated, they were solved using essentially the same method. Both problems asked to find an extreme value of a multivariate function. The solution was constructed by repeatedly massaging only two variables at a time. In Problem 1, we examined how the area of an n-gon changed when working with two neighbouring sides ; in Problem 2, we examined how the sum of the squares of n numbers changed when working with two of these numbers. This way of finding the extremum of a multivariate function is called *Sturm's method* and will be used throughout the remainder article.

Let us start with three Lemmas.

First, consider the product of two numbers a and b ($b > a > 0$) when these numbers get closer together while their sum stays constant. Let $0 < \epsilon < b - a$. Then

$$(a + \epsilon)(b - \epsilon) = ab + \epsilon(b - a - \epsilon) > ab.$$

Therefore, we have

Lemma 1. The product of two numbers with a constant sum increases as their difference decreases.

It is slightly more complicated to figure out what happens in a similar situation with the sum of powers of two numbers $a^k + b^k$ for a natural number $k \geq 2$. Let $\epsilon > 0$ and let us compare $a^k + b^k$ and $(a + \epsilon)^k + (b - \epsilon)^k$. An easy way to do so is to study the function $f(\epsilon) = (a + \epsilon)^k + (b - \epsilon)^k$ using its derivative. Since $f'(\epsilon) = k((a + \epsilon)^{k-1} - (b - \epsilon)^{k-1})$, we have that $f'(\epsilon)$ is negative for $a + \epsilon < b - \epsilon$; that is, for $\epsilon < \frac{b-a}{2}$. Therefore, the function f is decreasing on the interval $[0, \frac{b-a}{2})$. Furthermore, $f(0) = a^k + b^k$ and hence for $0 < \epsilon < \frac{b-a}{2}$ we have

$$a^k + b^k > (a + \epsilon)^k + (b - \epsilon)^k.$$

This inequality holds also for $\epsilon \in (0, b-a)$ since the graph of the function f is symmetrical about the line $\epsilon = \frac{b-a}{2}$. So we have

Lemma 2. The sum of the k-th powers of two numbers ($k \geq 2$) with a constant sum increases as their difference decreases.

Now let us investigate the case where the numbers a and b get closer together while their product is kept constant. What happens to the sum $a+b$ in this case? Suppose $a < b$. Compare $a+b$ and $\lambda a + \frac{b}{\lambda}$ for some $\lambda > 1$ (note that the product of $a+b$ and $\lambda a + \frac{b}{\lambda}$ and is equal to ab) :

$$(a+b) - \left(\lambda a + \frac{b}{\lambda}\right) = (\lambda - 1)\left(\frac{b}{\lambda} - a\right).$$

Therefore, for $\lambda \in (1, \frac{b}{a})$, we have $a + b > \lambda a + \frac{b}{\lambda}$. This gives us

Lemma 3. The sum of two positive numbers with a constant product decreases as their difference decreases.

Lemma 1 provides an easy proof of the inequality which compares the arithmetic mean A with the geometric mean G : if $a_1, a_2, \ldots, a_n$ are positive numbers that are not all equal, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} > \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Indeed, if the numbers are not all equal, then the smallest is less than A and the largest one is greater than A. Suppose that $a_1 < A$ and $a_2 > A$. By replacing a_1 with A and a_2 with $a_1 + a_2 - A$, we bring them closer together while keeping their sum unchanged. At the same time, the arithmetic mean A stays the same, while the geometric mean G get larger (see Lemma 1). If the numbers of the new set are still not all equal to the arithmetic mean, we repeat the process. Since every time we increase the geometric mean and keep the arithmetic mean constant, in the final set of numbers the arithmetic mean equals the final geometric mean, which exceeds the original geometric mean, as claimed.

Exercise 1. Using the proof technique of Lemma 1, find 25 numbers $n_1, n_2, \ldots, n_{25}$ that sum to 1981 so that the product $n_1! n_2! \ldots n_{25}!$ is minimized.

Exercise 2. Using Lemma 2, prove that for positive numbers $a_1, a_2, \ldots, a_n$ and a natural number k, we have

$$\frac{a_1^k + a_2^k + \cdots + a_n^k}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^k.$$

Exercise 3. Using Lemma 3, prove *Huygens' inequality* :

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \geq (1 + \sqrt[n]{x_1 x_2 \cdots x_n})^n.$$

In all the above problems, a function's extremum was achieved when all the variables were equal. This is why while looking for these extrema, we were using the method of bringing the variable values closer together. In the following exercise, try using a similar approach, but instead of bringing the variables closer together, consider what happens to the function when the difference between variables increases.

Exercise 4. Let $x_1, x_2, \ldots, x_n \in [a, b]$, where $0 < a < b$. Prove that

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \le \frac{(a+b)^2}{4ab}n^2.$$

Now try your hand at these further exercises.

Exercise 5. Which n-gon inscribed in a unit semi-circle (so that one of its sides coincides with the diameter) has the maximum area?

Exercise 6. Prove that the equilateral triangle has the smallest perimeter among all triangles with a given area.

Exercise 7. Investigate what happens to the product $(1 + \frac{1}{a})(1 + \frac{1}{b})$ when the positive numbers a and b get closer together while their sum is kept constant. Show that if the sum of positive numbers $x_1, x_2, \ldots, x_n$ is equal to 1, then

$$\left(1 + \frac{1}{x_1}\right)\left(1 + \frac{1}{x_2}\right)\cdots\left(1 + \frac{1}{x_n}\right) \ge (n+1)^n.$$

Exercise 8. Investigate what happens to the sum $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2$ when the positive numbers a and b get closer together while their sum is kept constant. Show that if the sum of positive numbers $x_1, x_2, \ldots, x_n$ is equal to 1, then

$$\left(x_1 + \frac{1}{x_1}\right)^2 + \left(x_2 + \frac{1}{x_2}\right)^2 + \cdots + \left(x_n + \frac{1}{x_n}\right)^2 \ge \frac{(n^2+1)^2}{n}.$$

Exercise 9. Investigate what happens to the fraction $\frac{(1-a)(1-b)}{ab}$ when the positive numbers a and b, $a + b = 1$, get closer together. Show that if the sum of positive numbers $x_1, x_2, \ldots, x_n$ is equal to 1, then

$$\frac{(1-x_1)(1-x_2)\cdots(1-x_n)}{x_1 x_2 \cdots x_n} \ge (n-1)^n.$$

Exercise 10. Investigate what happens to the sum $\frac{1}{1+a} + \frac{1}{1+b}$ when the product of positive numbers a and b stays constant. Then prove the following inequalities.

a) If the numbers $x_1, x_2, \ldots, x_n$ are all greater than 1, then

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} \ge \frac{n}{1 + \sqrt[n]{x_1 x_2 \cdots x_n}}.$$

b) If the positive numbers $x_1, x_2, \dots, x_n$ are all less than 1, then

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} \le \frac{n}{1+\sqrt[n]{x_1 x_2 \cdots x_n}}.$$

Exercise 11. Let $\alpha, \beta, \gamma, \delta \in (0, \frac{\pi}{2})$ and suppose that $\alpha + \beta + \gamma + \delta = \pi$. Show that $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta \ge 4$.

Exercise 12. Let $\alpha, \beta, \gamma, \delta$ be positive numbers such that $\alpha+\beta+\gamma+\delta = \pi$. Show that $\sin\alpha \sin\beta \sin\gamma \sin\delta \le \frac{1}{4}$.

Exercise 13.

a) Let a, b, c, d be positive numbers. Prove that

$$\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \ge \sqrt[3]{\frac{abc+abd+acd+bcd}{4}}.$$

b) Let $x_1, x_2, \dots, x_n$ be positive numbers and let k be a natural number with $k \ge 2$. Prove that

$$\sqrt[k]{\frac{x_1^k + x_2^k + \cdots + x_n^k}{n}} \ge \sqrt[n-1]{\frac{x_1 x_2 \cdots x_n \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}{n}}.$$

Exercise 14. Given two positive numbers a and b, $a < b$, find the n numbers $x_1, x_2, \dots, x_n \in (a, b)$ so that the fraction

$$\frac{(a+x_1)(x_1+x_2)(x_2+x_3)\cdots(x_{n-1}+x_n)(x_n+b)}{x_1 x_2 \cdots x_n}$$

is minimized.

Exercise 15. The sum of some set of nonnegative numbers is equal to 3 and the sum of their squares is bigger than 1. Prove that you can pick three of these numbers such that their sum exceeds 1.

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