Limits before epsilons and deltas

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Introduction

It has been observed that when somebody, in a casual chat mentions that their job is related to teaching or research in mathematics, this statement often kills the conversation. One of the reasons is that mathematics tends to be very formal, in order to be rigorous and precise in its conclusions. While this is a norm for a mathematician, other people often cannot grasp the essence behind formal derivations and quickly lose interest with the subject. Most regretfully, this happens to students attending mathematics classes. For example, many students of calculus who have seen the formal definition of the limit in terms of epsilon and delta can start the sentence, "For any positive epsilon, there exists a delta such that ...". Unfortunately, not so many can complete the definition in a meaningful way. If people do not understand mathematics, they get bored, frustrated, and are reluctant to discuss it.

However, there are some mathematical statements that produce quite an opposite effect. Try for example, asking your friend whether the following is true or false :

$$0.999\ldots = 1. \tag{1}$$

Here the left-hand side represents an infinite decimal fraction while the right-hand side is just a natural number 1. Or you can enquire whether the equality of the following infinite nested square root and the number holds :

$$\sqrt{4\sqrt{4\sqrt{4\ldots}}} = 4. \tag{2}$$

Before reading further you are advised to respond to these questions yourself and then compare with our discussion below. When answering the second question, it is also useful to think about the following generalization : for which values of a, b and n does the following equality hold ?

$$\sqrt[n]{a\sqrt[n]{a\sqrt[n]{a\ldots}}} = b \tag{3}$$

It is not obvious that the left- and right-hand sides of the equation (1) or (2) are equal, and you might sense it from people's responses based on their intuition. The popular belief that both statements are false is contradictory to the result obtained in calculus. Well, at least people are willing to form and defend their opinion about a mathematical equation !

From a psychological point of view, the expression on the left-hand side is infinitely long, so it never stops. People tend to consider it as an infinite process of writing either 9's or $\sqrt{4}$'s. This is obviously quite different from writing a single digit on the right-hand side. However, the infinite process can be encapsulated to form an object. If we ask ourselves about the value of the infinitely long expression, we must see beyond the infinite process towards its completion. Since the time of Aristotle, philosophers distinguish the potential and actual infinity. This distinction becomes relevant to our examples. The process of writing either 9's or $\sqrt{4}$'s is potentially infinite as we can keep writing but will never have it completed. At the same time grasping the final result of this process as a totality corresponds to Aristotle's notion of the actual infinity, as it refers to the expression perceived as truly containing infinitely many 9's or $\sqrt{4}$'s.

One may ask, what all these psychological and philosophical considerations have to do with mathematical analysis ? It appears that they are extremely important for our discussion about limits. But as promised in the title, we will try to do it less formally, by means that avoid the epsilon and delta technique.

The Heine approach

Our first example is a good starting point for talking about infinite sequences. If we do not want to rely on our intuition with infinitely long decimals and trust only decimals that have finite representation, we may look at the numbers 0.9, 0.99, 0.999, etc. What we have here is a sequence of numbers $S(n)$ each of which has exactly n 9's, where $n = 1, 2, 3, \ldots$. We want to know the value of the expression with infinitely many 9's. In mathematical language, we need to find the limit of the sequence $S(n)$ when n approaches infinity.

We will discuss a method in the spirit of Heine. But before we do, we need to explain a couple of mathematical notions. First, an interval on a number line that includes all points x between numbers A and B is *open*, if it does not include the endpoints A and B. It is denoted by (A, B). Thus, x belongs to an open interval (A, B) means $A < x < B$. For example, all numbers that satisfy the inequalities $1 < x < 2$ form the open interval $(1, 2)$. Second, we say that *almost all* terms of a sequence $S(n)$, $n \geq 1$ belong to an open interval (A, B) if all but a finite number of terms of the sequence satisfy the inequality $A < a_n < B$.

German mathematician Heinrich Eduard Heine, who lived in the 19th century, proposed the following definition of the limit. His approach is equivalent to the one found in standard calculus textbooks, but it avoids epsilon-delta language.

Definition. A number S is called the *limit* of a sequence $S(n)$ as n approaches infinity if **every** open interval containing S also contains almost all terms of the sequence.

We can use this definition to show that $S = 1$. Observe that every open interval that contains 1 has the form $(1 - a, 1 + b)$ for some $a > 0$ and $b > 0$. Now, take $a > 0$ and choose a natural r such that $10^{-r} < a$. Then, $S(r) = 1 - 10^{-r} > 1 - a$. From the above relation, we can also see that as n increases, the value of $S(n)$ grows as well, particularly, $1 - a < S(r) < S(r+1) < S(r+2) < \cdots$. In addition, note that $S(n) < 1$ for all $n \geq 1$. Thus, every open interval $(1 - a, 1 + b)$ contains all terms $S(n)$ for $n \geq r$. That is, every open interval of the form $(1 - a, 1 + b)$ contains both 1 and almost all terms of the sequence, so the limit of the sequence $S(n)$ as n approaches infinity, is 1.

As an exercise, we can also show that both cases $S < 1$ and $S > 1$ are not possible according to the definition.

Indeed, suppose there exists a limit $S < 1$. Let $S < 1 - 10^{-r}$ for some natural number r. In this case the interval of the form $(0, 1 - 10^{-r})$ contains S but it contains only a finite number of terms of the sequence, namely, only terms $S(n)$, where $n < r$. This contradicts the requirement that "every open interval containing S also contains almost all terms of the sequence."

Now suppose $S > 1$. Let $S > 1 + 10^{-r}$ for some natural number r. Then the interval $(1 + 10^{-r}, 2)$ contains S but none of the terms of the sequence. This again contradicts the requirement that "every open interval containing S also contains almost all terms of the sequence."

In a sense, this approach formalizes the following logic : if you believe that $0.999\ldots$ is less than 1, "you must tell how much the deficit is, and when you do, we can show you a large enough chunk of (finitely) many 9's in this decimal which are closer to 1 than the deficit" ([3], p. 62).

Finally, note that our result is consistent with the following algebraic derivation. Suppose that $0.999\ldots$ is a real number and denote it by $S = 0.999\ldots$. Then $S - 0.9 = 0.099\cdots = S/10$. Equivalently, $0.9S = 0.9$. Solving for S, we obtain (1).

Equation (2) can be treated similarly. The sequence of which we need to find the limit is

$$Q(1) = \sqrt{4}, \quad Q(2) = \sqrt{4\sqrt{4}}, \quad Q(3) = \sqrt{4\sqrt{4\sqrt{4}}}, \ldots$$

Again, we regard the infinite nested square root as an object rather than a process and denote it by Q. If we look carefully, we observe that $\sqrt{4Q} = Q$. Squaring both sides we get $4Q = Q^2$. Since $Q(1) > 0$ and $Q(1) < Q(2) < Q(3) < \cdots$, we need $Q > 0$. Thus $Q = 4$. Observe that $Q(1) = 4^{1/2}$, $Q(2) = 4^{1/2+1/4}$, $Q(3) = 4^{1/2+1/4+1/8}$, and the term that contains exactly n nested roots can be written as $Q(n) = 4^{M(n)}$, where $M(n) = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n$, $n \geq 1$.

Note that $M(n)$ is a finite geometric series. It is well known how to compute its sum. However, even if you do not remember the general formula, here is a way to do so. Multiply $M(n)$ by $1/2$ and observe that if you subtract the product from $M(n)$, most of the terms cancel, and you obtain $M(n) - \frac{1}{2}M(n) = \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}$. Thus, $M(n) = 1 - 2^{-n}$. By using the definition proposed by Heine, we can justify that the limit of the sequence $M(n)$ as n approaches infinity is 1. Then we can conclude that the value of the infinite nested root is $Q = 4^1 = 4$.

More ways to show that the infinite sum of powers of $1/2$ is 1 are given in [2]. Some of these methods are easily adaptable to other geometric series in order to illustrate that

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}, \qquad \text{for } -1 < x < 1. \tag{4}$$

Various viewpoints

So far we have considered several methods which allowed us to establish the equality of an infinite expression and a real number. From a mathematical standpoint, we calculated either the limits of appropriate sequences or the sums of appropriate series. One philosophical idea was to treat the infinite expression as an object rather that an infinite process and manipulate this object mathematically. As an exercise you can show that (3) holds for $a = b^{n-1}$, $b \geq 0$ and integer $n \geq 2$, for example, if $a = 4$, $n = 3$ and $b = 2$, we have $\sqrt[3]{4\sqrt[3]{4\sqrt[3]{4\ldots}}} = 2$.

One remark is in order. In fact, infinite expressions could be assigned different meanings (and values) in different theoretical frameworks and such assignments should be validated in order to avoid possible inconsistencies. In this article we rely on Heine's definition when, for example, value 1 is assigned to the infinite expression $0.999\ldots$. However, seemingly reasonable mathematical derivations can bring us to the conclusion that an integer $111\ldots$ is equal to $-1/9$. Indeed, denote $T = 111\ldots$ and observe that $10T + 1 = T$, then solve for T. The same result could be *derived* from (4) by letting $x = 10$, that is, applying formula (4) beyond the interval $|x| < 1$ in which it was previously considered. Another famous assignment is Ramanujan's celebrated summation : $1 + 2 + 3 + 4 + \cdots = -1/12$. Note that even if formally established, to be validated such assignments require a different framework as we can see that they do not meet the criteria of Heine's definition of the limit. Moreover, they are inconsistent with a view that an unrestrictedly growing sequence of numbers cannot represent a finite rational number. However, a discussion of frameworks required for validation of the above assignments (e.g. p-adic analysis or complex analysis) is beyond the scope of this paper.

Telescoping

Telescoping is another useful technique for investigating infinite series. The main idea is to change the appearance of the sum and look for similar terms to cancel each other. Consider the following example. Find the numerical value of P, where

$$P = \frac{1}{10 \cdot 11} + \frac{1}{11 \cdot 12} + \frac{1}{12 \cdot 13} + \cdots .$$

Here then the sequence of which we need to find the limit is $P(1) = \frac{1}{10 \cdot 11}$, $P(2) = \frac{1}{10 \cdot 11} + \frac{1}{11 \cdot 12}$, $P(3) = \frac{1}{10 \cdot 11} + \frac{1}{11 \cdot 12} + \frac{1}{12 \cdot 13}$, ... Now, observing that

$$\frac{1}{10 \cdot 11} = \frac{1}{10} - \frac{1}{11}, \quad \frac{1}{11 \cdot 12} = \frac{1}{11} - \frac{1}{12}, \quad \frac{1}{12 \cdot 13} = \frac{1}{12} - \frac{1}{13}, \quad \ldots,$$

we get $P(n) = \frac{1}{10} - \frac{1}{10+n}$ for $n \geq 1$. Using Heine's definition we can argue that the limit of this sequence is $P = 1/10$. This is because the positive numbers $(1/11, 1/12, 1/13, \ldots)$ which are subtracted from $1/10$ get smaller and smaller as you add more summands.

One can contrast this derivation with the following example. Consider the infinite sum

$$\ln \frac{10}{11} + \ln \frac{11}{12} + \ln \frac{12}{13} + \cdots .$$

The sum $R(n)$, which contains exactly the first n terms, can be written as

$$R(n) = \ln 10 - \ln 11 + \ln 11 - \ln 12 + \cdots + \ln(9+n) - \ln(10+n) = \ln 10 - \ln(10+n),$$

$n \geq 1$. Note that, as in the previous example, cancellations also take place. However, the infinite sum is not equal to $\ln 10$ (and in fact does not have any finite value) because the positive numbers $(\ln 11, \ln 12, \ldots)$ which are subtracted from $\ln 10$ get larger and larger as you add more summands, and so the sequence $R(n)$ deviates from, rather than approaches $\ln 10$, as n approaches infinity.

Conclusion

As the above examples show, knowing simple techniques from algebra and geometry may be sufficient to obtain numerical answers. At the same time, the study of series and sequences constitutes an important branch in analysis. There are mathematical subtleties that require special attention as well as new ideas that are needed for evaluating some limits. For example, note that the formula for summation of a geometric series (4) can be viewed as a representation of function $f(x) = 1/(1-x)$ by a polynomial with infinitely many terms, called the power series. This representation is valid for $|x| < 1$. More examples of power series representing a function on the interval $|x| < 1$ can be obtained by either taking the (first, second, etc.) derivative or integrating both sides of (4) term by term. Similarly, other functions can be represented by power series, e.g. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$ for all x, and (by taking the derivative of both sides) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$ for all x. Some interesting relations can be derived from these equations if they are evaluated for particular values of x. We leave it for the reader to show that

$$\frac{2\pi^2}{3!} - \frac{4\pi^4}{5!} + \frac{6\pi^6}{7!} - \frac{8\pi^8}{9!} + \cdots = 1.$$

Exercises

1. Let $a \geq 2$ be an integer. Show that $a^{1/a} \cdot (a^2)^{1/a^2} \cdot (a^3)^{1/a^3} \cdots = a^{a/(a-1)^2}$.

2. Show that

$$\frac{2}{5} + \frac{3}{25} + \frac{4}{125} + \cdots = \frac{9}{16}.$$

3. Let F_n, $n \geq 1$ be the sequence of Fibonacci numbers, that is, the first two terms are $F_1 = F_2 = 1$, and every following term is the sum of the two previous terms in the sequence, $F_3 = 1 + 1 = 2, F_4 = 1 + 2 = 3, F_5 = 2 + 3 = 5, F_6 = 3 + 5 = 8, \ldots$. Evaluate $\dfrac{F_3}{F_2 \cdot F_4} + \dfrac{F_4}{F_3 \cdot F_5} + \dfrac{F_5}{F_4 \cdot F_6} + \cdots$

4. Is the following equality true?

$$2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + \ldots}}}} = 3$$

References

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[3] Lewinter, M., Widulski, W. (2002). *The Saga of Mathematics : A Brief History.* New Jersey : Prentice Hall.