THE CONTEST CORNER

No. 42

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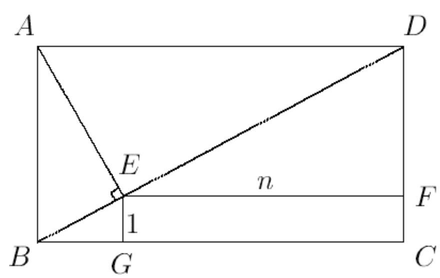
Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

CC206. Un rectangle $ABCD$ a une diagonale de longeur d. On abaisse une perpendiculaire AE à la diagonale BD. Le rectangle $EFCG$ a des côtés de longueurs n et 1. Démontrer que $d^{2/3} = n^{2/3} + 1$.



CC207. On considère les dix nombres $ar, ar^2, \ldots, ar^{10}$. Déterminer leur produit, sachant que leur somme est égale à 18 et que la somme de leurs inverses est égale à 6.

CC208.

a) Soit deux chiffres A et B. (A et B sont donc des symboles de 0 à 9 utilisés pour écrire les entiers.) Sachant que le produit des deux nombres de trois chiffres, $2A5$ et $13B$, est divisible par 36, déterminer les *quatre* couples (A, B) possibles. Justifier sa réponse.

b) Un entier n est un multiple de 7 si $n = 7k$ pour un entier quelconque k.

   i) Si a et b sont des entiers tels que $10a+b = 7m$ pour un entier quelconque m, démontrer que $a - 2b$ est un multiple de 7.

   ii) Si c et d sont des entiers tels que $5c+4d$ est un multiple de 7, démontrer que $4c - d$ est aussi un multiple de 7.

CC209.

a) Déterminer les deux valeurs de x qui vérifient $x^2 - 4x - 12 = 0$.

b) Déterminer *la* valeur de x qui vérifie $x - \sqrt{4x + 12} = 0$. Justifier sa réponse.

c) Déterminer toutes les valeurs réelles de c pour lesquelles l'équation

$$x^2 - 4x - c - \sqrt{8x^2 - 32x - 8c} = 0$$

admet exactement deux racines réelles distinctes.

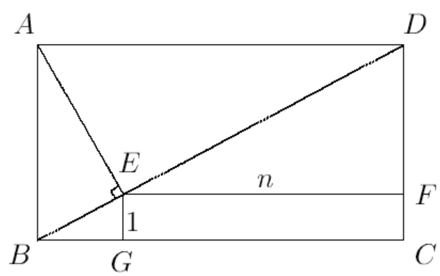
CC210. Il existe un unique triplet d'entiers strictement positifs (a, b, c) tel que $a \leq b \leq c$ et

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}.$$

Déterminer la valeur $a + b + c$.

. . . .

CC206. A rectangle $ABCD$ has diagonal of length d. The line AE is drawn perpendicular to the diagonal BD. The sides of the rectangle $EFCG$ have lengths n and 1. Prove that $d^{2/3} = n^{2/3} + 1$.



CC207. Consider the ten numbers $ar, ar^2, \ldots, ar^{10}$. If their sum is 18 and the sum of their reciprocals is 6, determine their product.

CC208.

a) Let A and B be digits (that is, A and B are integers between 0 and 9 inclusive). If the product of the three-digit integers $2A5$ and $13B$ is divisible by 36, determine with justification the *four* possible ordered pairs (A, B).

b) An integer n is said to be a multiple of 7 if $n = 7k$ for some integer k.

   i) If a and b are integers and $10a + b = 7m$ for some integer m, prove that $a - 2b$ is a multiple of 7.

ii) If c and d are integers and $5c + 4d$ is a multiple of 7, prove that $4c - d$ is also a multiple of 7.

CC209.

a) Determine the two values of x such that $x^2 - 4x - 12 = 0$.

b) Determine the *one* value of x such that $x - \sqrt{4x + 12} = 0$. Justify your answer.

c) Determine all real values of c such that

$$x^2 - 4x - c - \sqrt{8x^2 - 32x - 8c} = 0$$

has precisely two distinct real solutions for x.

CC210. There is a unique triplet of positive integers (a, b, c) such that $a \leq b \leq c$ and

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}.$$

Determine $a + b + c$.

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015 : 41(2), p. 48–50.

CC156. Describe and accurately sketch the region

$$\{(x, y, z) : \ |x| + |y| \leq 1, \ |y| + |z| \leq 1, \ |z| + |x| \leq 1\}.$$

Originally problem 2 of the 2014 Science Atlantic Math Contest.

We received two incorrect submissions and no correct solutions.

CC157. Show that if a 5×5 matrix is filled with zeros and ones, there must always be a 2×2 submatrix (that is, the intersection of the union of two rows with the union of two columns) consisting entirely of zeros or entirely of ones.

Originally problem 3 of the 2014 Science Atlantic Math Contest.

We received four correct submissions. We present the solution given by Kathleen Lewis.

Suppose that a 5×5 matrix of zeroes and ones contains no 2×2 submatrix of zeroes. We will show that it must contain a submatrix of ones. Let A be the column with the fewest ones, or one of those columns if there are several with the same smallest number of ones, and let n be the number of ones in column A.

Case 1 : $n = 0$. In this case, no other column can contain more than one zero, since otherwise that column and column A would share a 2×2 submatrix of zeroes. Then any two columns other than A must each have at least 4 ones, so they share at least three rows of ones. Thus, there are 2×2 submatrices of ones using any pair of columns besides A.

Case 2 : $n = 1$. If column A contains only one one, then it has zeroes in 4 rows. In those four rows, no other column will contain a pair of zeroes, so each of the other columns must have at least 3 ones in those four rows. Therefore any two columns (excluding A) will have ones in two of the same rows, so again we have a 2×2 submatrix of ones.

Case 3 : $n = 2$. Each of the other columns must contain at least two ones in the rows in which A has zeroes. If any column has three ones in these rows, then it will share a 2×2 submatrix of ones with any of the others. If not, each of these four column has two ones among the three rows. But there are only $\binom{3}{2} = 3$ ways to do this, so two columns must have the same arrangement. Then these two contain a 2×2 submatrix of ones.

Case 4 : $n \geq 3$. In this case, each column has at least three ones. So, given any two columns, they either share two rows with ones, or between them have at least one one in each row. The first arrangement gives us the 2×2 submatrix of ones that we are looking for. In the second arrangement, any third column must share two rows of ones with one of the original two columns, again giving us a 2×2 submatrix of ones.

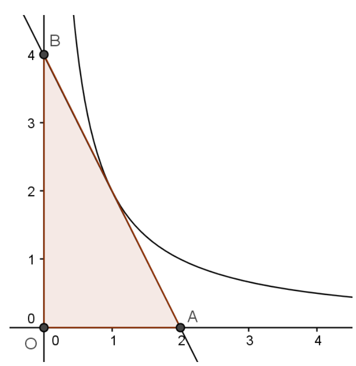
Therefore, in every case, if we don't have a 2×2 submatrix of zeroes, we must have a 2×2 submatrix of ones.

CC158. Suppose movable points A, B lie on the positive x-axis and y-axis, respectively, in such a way that $\triangle ABO$, where O is the origin, always has area 4. Find an equation for a curve in the first quadrant which is tangent to each of the line segments AB.

Originally problem 6 of the 2014 Science Atlantic Math Contest.

We received three solutions, two of which were completely correct. We present the solution by Andrea Fanchini.

We consider a generic point A that lies on the positive x-axis, at $(t, 0)$. If the triangle ABO has area 4 then the point B must be at $(0, 8/t)$. The family of lines AB thus satisfy the equation $y = -\frac{8}{t^2}(x - t)$, or $f(x, y, t) = t^2 y + 8x - 8t = 0$.



The envelope of this family of lines is defined as the set of points for which

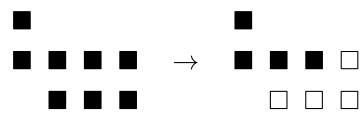
$$\begin{cases} f(x, y, t) = 0, \\ \dfrac{\partial f(x, y, t)}{\partial t} = 0, \end{cases}$$

so we have

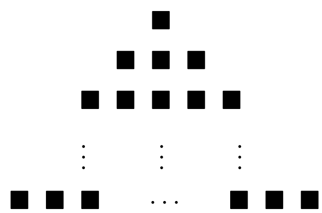
$$\begin{cases} t^2 y + 8x - 8t = 0, \\ 2yt - 8 = 0. \end{cases}$$

Solving, we obtain the equation of the envelope that is the hyperbola $xy = 2$.

CC159. The following pattern of eight square tiles can be divided into two congruent sets of four tiles as shown. (Note that one set is the mirror image of the other — this is legal.)



Find a way to divide the following pattern of 100 tiles into two congruent sets of fifty tiles, or show it cannot be done.



(19 tiles in the 10th row)

Originally problem 2 of the 2015 Science Atlantic Math Contest.

We received no submissions to this problem. We present a solution sketch by the editor.

First, prove the following two facts :

- The bottom two corners are the only two tiles with one coordinate equal and the other differing by 18.
- The only pairs in which both coordinates differ by 9 consists of the top corner and one of the bottom corners.

Now, suppose for a contradiction that a partition exists. By the pigeonhole principle, one set contains two of the three corners. By the two facts above, the other set cannot contain two tiles equivalently situated.

CC160. Find all triples of continuous functions $f, g, h : \mathbb{R} \mapsto \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$f(g(x)) = g(h(x)) = h(f(x)) = x .$$

Originally problem 3 of the 2015 Science Atlantic Math Contest.

We received one correct solution. We present the solution by Konstantine Zelator.

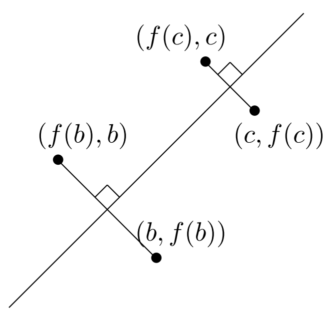
First, we observe that $f(g(x)) = x$ implies that $g(x) = f^{-1}(x)$. Furthermore, $g(h(x)) = x$ implies that $h(x) = g^{-1}(x)$. Combining these facts, $h(x) = (f^{-1})^{-1}(x) = f(x)$. So then $x = h(f(x)) = f(f(x))$. So f (and by similar reasoning, g and h) is its own inverse. Furthermore, $f = g = h$.

We claim that $y = f(x)$ intersects $y = x$ at least once, and if it intersects more than once then $f(x) = x$. If it intersects $y = x$ exactly once then it is a line of the form $f(x) = -x + k$ for some $k \in \mathbb{R}$.

Suppose that $f(x) \neq x$. Then we can find a so that $(a, f(a))$ is not on the line $y = x$. But since f is its own inverse it follows that $f(f(a)) = a$ and so $(f(a), a)$ is on $y = f(x)$. But these points are on opposite sides of $y = x$, so by the continuity of $f(x)$ it must intersect with $y = x$ at least once.

Now, suppose that $y = f(x)$ intersects $y = x$ more than once, say at points (a, a) and (b, b) with $a < b$. Since it is invertible, f must be one-to-one, and hence increasing or decreasing. In this case, f is clearly increasing. Suppose that there is some $(c, f(c))$ not on $y = x$. Then $c < f(c)$, but as we've stated before, $(f(c), c)$ is also on the curve, contradicting its increasing property. Thus no c can exist and every point must lie on $y = x$. So $f(x) = x$.

If, on the other hand, $y = f(x)$ and $y = x$ intersect only at one point, say (a, a) then by considering another point $(b, f(b))$ with $b \neq f(b)$ we can conclude that f is decreasing since $(f(b), b)$ is also on $y = f(x)$. Consider a third point, $(c, f(c))$ with $c \neq f(c)$. If $(b, f(b))$ and $(c, f(c))$ don't fall on a line perpendicular to $y = x$ then we can see, by considering the four points $(b, f(b))$, $(c, f(c))$, $(f(b), b)$, and $(f(c), c)$ that f will not be decreasing, a contradiction.



So all the points on $y = f(x)$ must lie on a line perpendicular to x. Therefore, $f(x) = -x + 2a$.

In summation, $f(x) = g(x) = h(x) = x$ or $-x + k$ for some $k \in \mathbb{R}$.