SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(1), p. 27–30.

4001. *Proposed by Cristinel Mortici and Leonard Giugiuc.*

Let $a, b, c, d \in \mathbb{R}$ with $d > 2$ such that

$$(2d+1)\cdot\frac{a}{6}+\frac{b}{2}+\frac{c}{d+1}=0.$$

Prove that there exists $t \in (0, d)$ such that $at^2 + bt + c = 0$.

We received four correct solutions and one solution that was almost complete. The first solution is due to Digby Smith and the second consists of ingredients of others.

Solution 1, by Digby Smith.

The result actually holds when $d > 1$. Let

$$2v = d + 1 \quad \text{and} \quad 3w = 2d + 1.$$

Then $0 < v < w < d$ with $6(w - v) = d - 1$. The given condition can be rewritten as

$$0 = avw + bv + c = v(aw + b) + c.$$

Let $f(t) = at^2 + bt + c$. Then

$$f(w) = aw^2 + bw + c = w(aw + b) + c = -\frac{wc}{v} + c = -\frac{c(w - v)}{v} = -\frac{c(d - 1)}{6v}.$$

If $c \neq 0$, then it follows that $f(0) = c$ and $f(w)$ have opposite signs, so that $f(t)$ has a real root in the interval $(0, w) \subseteq (0, d)$.

If $c = 0$, then $f(w) = w(aw + b) = 0$ since $v(aw + b) = 0$.

Solution 2.

Use the notation of Solution 1. Again $0 < v < w < d$. Furthermore, when $a = 0$, $f(v) = 0$. Otherwise, we may assume that $a > 0$, in which case $f(v) < avw + bv + c = 0$.

When $c = 0$, then $f(w) = 0$. When $c > 0$, then $f(0) > 0$ and $f(t)$ has a root in $(0, v)$. Finally, when $c < 0$, then $avw + bv = -c > 0$ and

$$f(w) = aw^2 + bw - avw - bv = (w - v)(aw + b) > 0$$

and $f(t)$ has a root in (v, w).

4002. *Proposed by Henry Aniobi.*

Let f be a convex function on an interval I. Let $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$ be numbers such that $x_i + y_j$ is always in I for all $1 \leq i, j \leq n$. Let $z_1, z_2, \ldots, z_n$ be an arbitrary permutation of $y_1, y_2, \ldots, y_n$. Show that

$$\begin{aligned} f(x_1 + y_1) + \ldots + f(x_n + y_n) &\geq f(x_1 + z_1) + \ldots + f(x_n + z_n) \\ &\geq f(x_1 + y_n) + f(x_2 + y_{n-1}) + \ldots + f(x_n + y_1); \end{aligned}$$

We received five submissions of which four were correct and complete. We present the solution by Joseph DiMuro.

We can prove the above statement by proving the following simpler statement:

Claim. Let $x_1 < x_2$ and $y_1 < y_2$ be numbers such that $x_i + y_j$ is always in I. Then

$$f(x_1 + y_1) + f(x_2 + y_2) \geq f(x_1 + y_2) + f(x_2 + y_1).$$

The reason why this suffices: if we choose a permutation $z_1, z_2, \ldots, z_n$ such that $z_i > z_j$ for some $i < j$, then we will have

$$f(x_i + z_i) + f(x_j + z_j) \leq f(x_i + z_j) + f(x_j + z_i).$$

We would then be able to interchange z_i and z_j without decreasing the overall sum. Thus, a permutation $z_1, z_2, \ldots, z_n$ that gives us the largest overall sum is one where $z_i \leq z_j$ whenever $i < j$; that is, $z_i = y_i$ for all i. Similarly, a permutation $z_1, z_2, \ldots, z_n$ that gives us the smallest overall sum is one where $z_i \geq z_j$ whenever $i < j$; that is, $z_i = y_{n-i+1}$ for all i.

Proof of claim. By the definition of convexity, for all $a, b \in I$ and for all $t \in [0, 1]$, we have

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b).$$

Let $a = x_1 + y_1$ and $b = x_2 + y_2$. Then $a < x_1 + y_2 < b$, so for some $t \in [0, 1]$, we have $x_1 + y_2 = ta + (1-t)b$. From that, we have:

$$\begin{aligned} x_2 + y_1 &= (x_1 + y_1 + x_2 + y_2) - (x_1 + y_2) \\ &= (a + b) - (ta + (1-t)b) \\ &= (1-t)a + tb. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x_1 + y_2) + f(x_2 + y_1) &= f(ta + (1-t)b) + f((1-t)a + tb) \\ &\leq (tf(a) + (1-t)f(b)) + ((1-t)f(a) + tf(b)) \\ &= f(a) + f(b) \\ &= f(x_1 + y_1) + f(x_2 + y_2), \end{aligned}$$

completing the proof of the claim.

4003. *Proposed by Martin Lukarevski.*

Show that for any triangle ABC, the following inequality holds

$$\sin A\sin B\sin C\left(\frac{1}{\sin A+\sin B}+\frac{1}{\sin B+\sin C}+\frac{1}{\sin C+\sin A}\right) \leq \frac{3}{4}(\cos A+\cos B+\cos C).$$

We received 13 correct solutions. We present the solution by John G. Heuver, modified slightly by the editor.

Let r, R and s denote the inradius, the circumradius and the semiperimeter of $\triangle ABC$, respectively. The following identities and inequalities are well known:

$$\sum \sin^2 A = \frac{s^2-4Rr-r^2}{2R^2}, \tag{1}$$

$$\sum \sin A\sin B = \frac{s^2+4Rr+r^2}{4R^2}, \tag{2}$$

$$\sum \cos A = \frac{R+r}{R}, \tag{3}$$

$$R \geq 2r \quad \text{Euler's inequality} \tag{4}$$

$$s^2 \leq 4R^2+4Rr+3r^2 \quad \text{Gerretsen's inequality,} \tag{5}$$

(where all the summations are taken over all angles of $\triangle ABC$).

Let L denote the left-hand side of the given inequality. By the AM-GM Inequality, we have $\sin A+\sin B \geq 2\sqrt{\sin A\sin B}$. Hence, by Cauchy-Schwarz Inequality we have

$$L \leq \frac{1}{2}\sum(\sin A)\sqrt{\sin B\sin C} \leq \frac{1}{2}\sqrt{\sum \sin^2 A}\sqrt{\sum \sin A\sin B}. \tag{6}$$

By (1) and (5), we have

$$\sum \sin^2 A \leq \frac{4R^2+2r^2}{2R^2}. \tag{7}$$

By (2) and (5), we have

$$\sum \sin A\sin B \leq \frac{4R^2+8Rr+4r^2}{4R^2}. \tag{8}$$

Using (6), (7) and (8) followed by (3) and (4), we then have

$$\begin{aligned} L \leq \frac{1}{2}\sqrt{\frac{4R^2+2r^2}{2R^2}}\cdot\sqrt{\frac{4R^2+8Rr+4r^2}{4R^2}} &= \frac{1}{2}\sqrt{2+\left(\frac{r}{R}\right)^2}\cdot\frac{R+r}{R} \\ &\leq \frac{1}{2}\sqrt{2+\left(\frac{1}{2}\right)^2}\cdot\sum\cos A \\ &= \frac{3}{4}\sum\cos A, \end{aligned}$$

which completes the proof.

Editor's comment. Digby Smith remembered that the following problem proposed by Jack Garfunkel and George Tsintsifas appeared in the August–September 1982 issue (Vol. 8, no. 7, p. 210) of ***Crux*** and a solution given by Vedula N. Murty appeared in the November 1983 issue (Vol. 9, no. 9, p. 282):

$$\frac{4}{9}\sum \sin B \sin C \le \prod \cos\frac{B-C}{2} \le \frac{2}{3}\sum \cos A.$$

Smith gave a proof by first showing that $2L \le \sum \sin B \sin C$, which together with the above inequality yields the result.

4004. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers such that $x + y + z = 2$. Prove that

$$\frac{x^5}{yz(x^2+y^2)} + \frac{y^5}{zx(y^2+z^2)} + \frac{z^5}{xy(z^2+x^2)} \ge 1.$$

We received 16 correct submissions. We present 3 solutions.

Solution 1, by Arkady Alt.

Since by Cauchy's Inequality

$$\sum_{cyc} \frac{x^5}{yz\left(x^2+y^2\right)} = \sum_{cyc} \frac{x^6}{xyz\left(x^2+y^2\right)} \ge \frac{\left(x^3+y^3+z^3\right)^2}{\sum\limits_{cyc} xyz\left(x^2+y^2\right)},$$

it suffices to prove the inequality

$$\frac{\left(x^3+y^3+z^3\right)^2}{\sum\limits_{cyc} xyz\left(x^2+y^2\right)} \ge 1.$$

We have the following equivalences:

$$\begin{aligned}\frac{\left(x^3+y^3+z^3\right)^2}{\sum\limits_{cyc} xyz\left(x^2+y^2\right)} \ge 1 \quad &\iff \quad \left(x^3+y^3+z^3\right)^2 \ge 2xyz\left(x^2+y^2+z^2\right)\\ &\iff \quad \left(x^3+y^3+z^3\right)^2 \ge xyz\left(x+y+z\right)\left(x^2+y^2+z^2\right),\end{aligned}$$

where the latter inequality holds because by AM-GM Inequality

$$x^3+y^3+z^3 \ge 3xyz$$

and by Chebyshev's Inequality

$$x^3+y^3+z^3 \ge \frac{(x+y+z)\left(x^2+y^2+z^2\right)}{3}.$$

Solution 2, by Michel Bataille.

Let $a = \frac{x}{2},\ b = \frac{y}{2}$ and $c = \frac{z}{2}$. With these notations, we are required to prove

$$\frac{a^6}{a^2+b^2} + \frac{b^6}{b^2+c^2} + \frac{c^6}{c^2+a^2} \ge \frac{abc}{2} \tag{1}$$

under the conditions $a, b, c > 0$ and $a + b + c = 1$.

The Cauchy-Schwarz inequality gives

$$\left(\frac{a^6}{a^2+b^2} + \frac{b^6}{b^2+c^2} + \frac{c^6}{c^2+a^2}\right)((a^2+b^2)+(b^2+c^2)+(c^2+a^2)) \ge (a^3+b^3+c^3)^2.$$

Hence, (1) will follow if we prove

$$\frac{(a^3+b^3+c^3)^2}{a^2+b^2+c^2} \ge abc.$$

Since $abc \le \frac{a^3+b^3+c^3}{3}$, it is sufficient to show that

$$3(a^3+b^3+c^3) \ge a^2+b^2+c^2.$$

Now, the latter follows from

$$\begin{aligned}
3(a^3+b^3+c^3) &\ge 2(a^3+b^3+c^3) + 3abc \\
&= a^3+b^3+c^3 + (a^3+b^3+c^3+3abc) \\
&\ge a^3+b^3+c^3 + ab^2 + a^2b + bc^2 + b^2c + ca^2 + c^2a \quad \text{(Schur's ineq.)} \\
&= (a^2+b^2+c^2)(a+b+c) = a^2+b^2+c^2 \quad (\text{since } a+b+c=1)
\end{aligned}$$

so we are done.

Solution 3, by Oliver Geupel.

By hypothesis $x + y + z = 2$ and by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\left(\frac{x^5}{yz(x^2+y^2)} + \frac{y^5}{zx(y^2+z^2)} + \frac{z^5}{xy(z^2+x^2)}\right)(x+y+z)xyz(x^2+y^2+z^2) \\
&= \left(\sum_{\text{cyc}} \frac{x^5}{yz(x^2+y^2)}\right)\left(\sum_{\text{cyc}} xyz(x^2+y^2)\right) \ge (x^3+y^3+z^3)^2.
\end{aligned}$$

By the power mean inequality, it holds

$$\left(\frac{x^3+y^3+z^3}{3}\right)^{1/3} \ge \left(\frac{x^2+y^2+z^2}{3}\right)^{1/2} \ge \frac{x+y+z}{3} \ge (xyz)^{1/3}.$$

Putting together we obtain

$$\begin{aligned}
&\frac{x^5}{yz(x^2+y^2)}+\frac{y^5}{zx(y^2+z^2)}+\frac{z^5}{xy(z^2+x^2)}\\
&\qquad\geq \frac{(x^3+y^3+z^3)^2}{(x+y+z)xyz(x^2+y^2+z^2)}\\
&\qquad= \frac{(x^3+y^3+z^3)^{1/3}}{x+y+z}\cdot\frac{x^3+y^3+z^3}{xyz}\cdot\frac{(x^3+y^3+z^3)^{2/3}}{x^2+y^2+z^2}\\
&\qquad\geq 3^{-2/3}\cdot 3\cdot 3^{-1/3}=1.
\end{aligned}$$

Hence the result. By the equality condition of the power mean inequality, the equality holds if and only if $x=y=z=2/3$.

4005. *Proposed by Michel Bataille.*

Let a,b,c be the sides of a triangle with area F. Suppose that some positive real numbers x,y,z satisfy the equations

$$x+y+z=4 \quad\text{and}$$

$$2xb^2c^2+2yc^2a^2+2za^2b^2-\left(\frac{4-yz}{x}\,a^4+\frac{4-zx}{y}\,b^4+\frac{4-xy}{z}\,c^4\right)=16F^2.$$

Show that the triangle is acute and find x,y,z.

We present the proposer's solution — no others were submitted.

The second equation gives

$$\begin{aligned}
&(xyz)(16F^2)\\
&=xyz(2xb^2c^2+2yc^2a^2+2za^2b^2)-yz(4-yz)a^4-zx(4-zx)b^4-xy(4-xy)c^4\\
&=(a^2yz+b^2zx+c^2xy)^2-(4a^4yz+4b^4zx+4c^4xy)\\
&=\left(\frac{x}{2}(b^2z+c^2y)+\frac{y}{2}(c^2x+a^2z)+\frac{z}{2}(b^2x+a^2y)\right)^2-(4a^4yz+4b^4zx+4c^4xy).
\end{aligned}$$

Since $t\mapsto t^2$ is a convex function and $x+y+z=4$, Jensen's inequality yields

$$\begin{aligned}
&\frac{x}{4}(b^2z+c^2y)^2+\frac{y}{4}(c^2x+a^2z)^2+\frac{z}{4}(b^2x+a^2y)^2\\
&\qquad\geq\left(\frac{x}{4}(b^2z+c^2y)+\frac{y}{4}(c^2x+a^2z)+\frac{z}{4}(b^2x+a^2y)\right)^2 \qquad (1)
\end{aligned}$$

and it follows that

$$\begin{aligned}
&(xyz)(16F^2)\\
&\leq x(b^2z+c^2y)^2+y(c^2x+a^2z)^2+z(b^2x+a^2y)^2-(4a^4yz+4b^4zx+4c^4xy)\\
&=a^4yz(y+z-4)+b^4zx(z+x-4)+c^4xy(x+y-4)+xyz(2b^2c^2+2c^2a^2+2a^2b^2)\\
&=xyz(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)\\
&=(xyz)(16F^2).
\end{aligned}$$

Thus, equality must hold in (1) and because $t \mapsto t^2$ is a strictly convex function, this calls for
$$b^2z + c^2y = c^2x + a^2z = b^2x + a^2y.$$
Setting these three expressions equal to λ and solving for x, y, z yields
$$\begin{aligned} x &= \lambda\frac{b^2+c^2-a^2}{2b^2c^2} = \frac{\lambda\cos A}{bc},\\ y &= \lambda\frac{c^2+a^2-b^2}{2c^2a^2} = \frac{\lambda\cos B}{ca},\\ z &= \lambda\frac{a^2+b^2-c^2}{2a^2b^2} = \frac{\lambda\cos C}{ab}. \end{aligned}$$
(As usual, A, B, C denote the angles of the triangle opposite sides a, b, c, respectively.) Since at most one of A, B, C is not acute and x, y, z are positive, we conclude that $\cos A, \cos B, \cos C$, and λ are positive. Thus, the triangle is acute.

In addition, we have
$$4 = \frac{\lambda\cos A}{bc} + \frac{\lambda\cos B}{ca} + \frac{\lambda\cos C}{ab}.$$
Since $a\cos A + b\cos B + c\cos C = \dfrac{2F}{R}$ and $4RF = abc$ (where R is the circumradius of the triangle), we readily find $\lambda = \dfrac{a^2b^2c^2}{2F^2}$ and obtain
$$x = \frac{a^2(b^2+c^2-a^2)}{4F^2}, \quad y = \frac{b^2(c^2+a^2-b^2)}{4F^2}, \quad z = \frac{c^2(a^2+b^2-c^2)}{4F^2}.$$
Note that conversely, if given an acute triangle, then these numbers x, y, z are positive and satisfy the two equations: $x + y + z = 4$ is readily checked; also we have $b^2z + c^2y = c^2x + a^2z = b^2x + a^2y = \lambda$, hence the calculations made at the beginning (with equality in (1)) show that the second equation holds as well.

4006. *Proposed by Dragolijub Milošević.*

Let x, y, z be positive real numbers such that $xyz = 1$. Prove that
$$\frac{2}{xy + yz + zx} - \frac{1}{x + y + z} \le \frac{1}{3}.$$

We received 15 correct solutions from 14 submitters. Ten of these solutions were along the lines of the solution presented below, with variations in how they justified the ancillary inequalities and how straightforwardly they handled the algebra. In addition, there was a MAPLE-based solution, which seemed heavy-handed for this problem. There were four other solutions that were defective in some way. We present the solution by Henry Ricardo.

Let $p = x+y+z$, $q = xy+yz+zx$ and $r = xyz = 1$. Observe that $q^2 \geq 3rp = 3p$, since, by the arithmetic-geometric means inequality,

$$\begin{aligned} q^2 &= \frac{1}{2}(x^2y^2+y^2z^2)+\frac{1}{2}(y^2z^2+z^2x^2)+\frac{1}{2}(z^2x^2+x^2y^2)+2xyz(x+y+z) \\ &\geq xy^2z+yz^2x+zx^2y+2xyz(x+y+z) \\ &= 3xyz(x+y+z) = 3rp = 3p. \end{aligned}$$

The difference between the two sides of the inequality is one-third of

$$1-\frac{6}{q}+\frac{3}{p} \geq 1-\frac{6}{q}+\frac{9}{q^2} = \left(1-\frac{3}{q}\right)^2 \geq 0,$$

and the result follows with equality if and only if $x = y = z = 1$.

Editor's comment. Oliver Geupel notes that this problem is equivalent to a problem proposed by Vasile Cîrtoaje and Mircea Lascu for the Junior TST 2003 Romania. It is also Problem 72 in Chapter 20 of *Inequalities, Theorems, Techniques and Selected Problems* by Zdravko Cvetkovski (Springer, 2012).

4007. *Proposed by Mihaela Berindeanu.*

Show that for any numbers $a, b, c > 0$ such that $a^2+b^2+c^2 = 12$, we have

$$(a^3+4a+8)(b^3+4b+8)(c^3+4c+8) \leq 24^3.$$

We received nine submission of which eight were correct and complete. We present two solutions.

Solution 1, by Ángel Plaza.

By taking logarithms, the proposed inequality may be written as

$$\frac{\ln\left(a^3+4a+8\right)+\ln\left(b^3+4b+8\right)+\ln\left(c^3+4c+8\right)}{3} \leq \ln 24.$$

Changing variables $a^2 = x$, $b^2 = y$, $c^2 = z$ the problem becomes:

For any $x, y, z > 0$ such that $x+y+z = 12$, prove that

$$\frac{\ln\left(x^{3/2}+4x^{1/2}+8\right)+\ln\left(y^{3/2}+4y^{1/2}+8\right)+\ln\left(z^{3/2}+4z^{1/2}+8\right)}{3} \leq \ln 24.$$

Let us consider function $f(x) = \ln\left(x^{3/2}+4x^{1/2}+8\right)$ for $x > 0$. Then

$$f''(x) = \frac{-8x^{3/2}-3x^{5/2}+12x-16\sqrt{x}-16}{2x^{3/2}\left(x^{3/2}+4\sqrt{x}+8\right)^2}$$

and since $f''(x) < 0$ for $x > 0$, the function f is concave. By Jensen's inequality

$$\frac{f(x)+f(y)+f(z)}{3} \leq f\left(\frac{x+y+z}{3}\right) = f(12/3) = f(4) = \ln 24.$$

Solution 2, by the proposer.

Observe that $(a-2)^4 \geq 0$ implies that $a^4 - 8a^3 + 24a^2 - 32a + 16 \geq 0$, that is

$$a^4 + 24a^2 + 80 \geq 8a^3 + 32a + 64,$$

which gives

$$(a^2+4)(a^2+20) \geq 8(a^3+4a+8)$$

and hence

$$a^3 + 4a + 8 \leq \frac{(a^2+4)(a^2+20)}{8}.$$

So,

$$\begin{aligned}(a^3+4a+8)&(b^3+4b+8)(c^3+4c+8)\\ &\leq \frac{(a^2+4)(a^2+20)}{8}\cdot\frac{(b^2+4)(b^2+20)}{8}\cdot\frac{(c^2+4)(c^2+20)}{8},\end{aligned}$$

but we know that $\sqrt[3]{xyz} \leq \dfrac{x+y+z}{3}$, therefore

$$(a^2+4)(b^2+4)(c^2+4) \leq \left(\frac{a^2+b^2+c^2+12}{3}\right)^3 = 8^3$$

and

$$(a^2+20)(b^2+20)(c^2+20) \leq \left(\frac{a^2+b^2+c^2+60}{3}\right)^3 = 24^3.$$

Finally,

$$\left(a^3+4a+8\right)\left(b^3+4b+8\right)\left(c^3+4c+8\right) \leq \frac{8^3\cdot 24^3}{8^3} = 24^3.$$

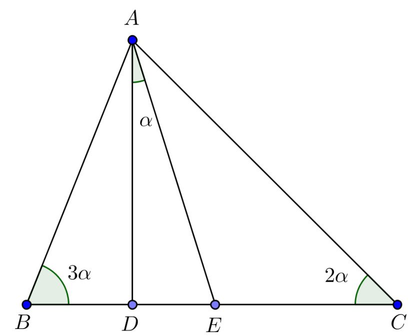
Editor's Comments. Ángel Plaza sent two solutions: the second solution consists in taking the logarithms of the given inequality, setting $a^2 = x, b^2 = y, c^2 = z$, considering the concave function (on $x \in (0,12)$) $f(x) = \ln(x^{3/2} + 4x^{1/2} + 8)$ and using Jensen's Inequality. A very similar approach was also used by Šefket Arslanagić.

4008. *Proposed by Mehmet Şahin.*

Let ABC be a triangle with $\angle ACB = 2\alpha$, $\angle ABC = 3\alpha$, AD is an altitude and AE is a median such that $\angle DAE = \alpha$. If $|BC| = a, |CA| = b, |AB| = c$, prove that

$$\frac{a}{b} = 1 + \sqrt{2\left(\frac{c}{b}\right)^2 - 1}.$$

We received 15 correct solutions and one incorrect submission. We present the solution given by Titu Zvonaru, modified slightly by the editor.



We have $AD = c\sin 3\alpha, BD = c\cos 3\alpha$, so $DE = \dfrac{a}{2} - \cos 3\alpha$. By the law of sines, we have $\dfrac{a}{\sin(180^\circ - 5\alpha)} = \dfrac{c}{\sin 2\alpha}$, so $a = \dfrac{c\sin 5\alpha}{\sin 2\alpha}$. Then

$$\begin{aligned}\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{DE}{AD} = \frac{\frac{a}{2} - \cos 3\alpha}{c\sin 3\alpha} &= \frac{\sin 5\alpha - 2\sin 2\alpha\cos 3\alpha}{2\sin 2\alpha\sin 3\alpha} \\ &= \frac{\sin 5\alpha - (\sin 5\alpha + \sin(-\alpha))}{2\sin 2\alpha\sin 3\alpha} \\ &= \frac{\sin\alpha}{2\sin 2\alpha\sin 3\alpha},\end{aligned}$$

so that

$$\cos\alpha = 2\sin 2\alpha\sin 3\alpha = \cos\alpha - \cos 5\alpha,$$

which implies that

$$\cos 5\alpha = 0, \quad \text{so} \quad 5\alpha = 90^\circ, \quad \text{or} \quad \alpha = 18^\circ.$$

Hence, $\angle BAC = 180^\circ - 5\alpha = 90^\circ$, $\angle ABC = 3\alpha = 54^\circ$ and $\angle ACB = 2\alpha = 36^\circ$. Since $\cos 36^\circ = \dfrac{1+\sqrt{5}}{4}$, we have

$$b = a\cos 2\alpha = \left(\frac{1+\sqrt{5}}{4}\right)a,$$

so

$$c = \sqrt{a^2 - b^2} = \sqrt{a^2 - \frac{3+\sqrt{5}}{8}a^2} = a\sqrt{\frac{5-\sqrt{5}}{8}}.$$

Now, $\dfrac{a}{b} = \dfrac{4}{1+\sqrt{5}} = \sqrt{5} - 1$ and

$$2\left(\frac{c}{b}\right)^2 - 1 = 2\left(\frac{5-\sqrt{5}}{8}\right)\left(\frac{4}{1+\sqrt{5}}\right)^2 - 1 = \frac{14-6\sqrt{5}}{6+2\sqrt{5}} = \frac{7-3\sqrt{5}}{3+\sqrt{5}}.$$

Therefore, we have the following equivalences:

$$\begin{aligned}
&\frac{a}{b} = 1 + \sqrt{2\left(\frac{c}{b}\right)^2 - 1} \\
\iff \quad & \sqrt{5} - 1 = 1 + \sqrt{\frac{7 - 3\sqrt{5}}{3 + \sqrt{5}}} \\
\iff \quad & (\sqrt{5} - 2)^2 = \frac{7 - 3\sqrt{5}}{3 + \sqrt{5}} \\
\iff \quad & (9 - 4\sqrt{5})(3 + \sqrt{5}) = 7 - 3\sqrt{5} \\
\iff \quad & 7 - 3\sqrt{5} = 7 - 3\sqrt{5},
\end{aligned}$$

which is true and our proof is complete.

4009. *Proposed by George Apostolopoulos.*

Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC. Prove that

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \le \frac{R}{2r^2},$$

where r and R are inradius and circumradius of ABC, respectively.

We received eleven solutions, of which ten were correct. We present two solutions.

Solution 1, by Arkady Alt.

Let F, s and h_a, h_b, h_c be the area, semiperimeter, and altitudes of the triangle. Since $m_x \ge h_x, x \in \{a, b, c\}$ and

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{a}{2F} + \frac{b}{2F} + \frac{c}{2F} = \frac{s}{2F} = \frac{1}{r}$$

then

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \le \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} \le \frac{R}{2r^2}$$

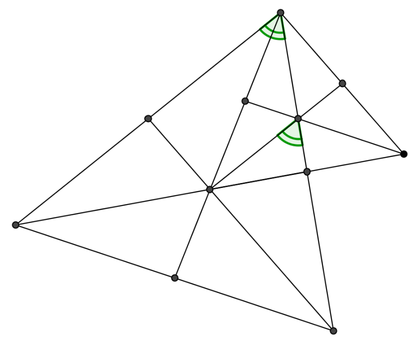
because

$$\frac{1}{r} \le \frac{R}{2r^2} \iff 2r \le R,$$

by Euler's Inequality.

Solution 2, by Edmund Swylan.

We take it as known that the triangle with side lengths $2m_a$, $2m_b$, $2m_c$ has medians of lengths $\frac{3}{2}a$, $\frac{3}{2}b$, $\frac{3}{2}c$. (See the drawing below.)



Let the area of $\triangle ABC$ be F. The area of the big triangle is then $3F$. Let the altitudes of the big triangle be H_a, H_b, H_c.

We have that $\dfrac{6F}{2m_x} = H_x$ and $H_x \leq \dfrac{3}{2}x$, for each $x \in \{a, b, c\}$. Therefore,

$$3F(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}) \leq \frac{3}{2}(a+b+c);$$

equality occurs if and only if the big triangle, and consequently $\triangle ABC$ too, is equilateral. Finally,

$$\frac{3}{2}(a+b+c) = 3F\frac{1}{r} \leq 3F\frac{1}{r}\frac{R}{2r} = 3F\frac{R}{2r^2};$$

equality occurs if and only if $\triangle ABC$ is equilateral.

4010. *Proposed by Ovidiu Furdui.*

Let $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ be a continuous function. Calculate

$$\lim_{n\to\infty} n \int_0^{\frac{\pi}{2}} \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)^{2n} f(x)\mathrm{d}x.$$

There were eight submitted solutions for this problem, all of which were correct. We present two solutions.

Solution 1, by the group of M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, expanded slightly by the editor.

The value of the required limit is $\dfrac{1}{4}\left(f(0) + f\left(\dfrac{\pi}{2}\right)\right)$. Indeed, if we denote by L the limit, then from the identity

$$\frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} = \tan\left(\frac{\pi}{4} - x\right),$$

we have

$$\begin{aligned} L &= \lim_{n\to\infty} n\int_0^{\pi/2} \left(\tan\left(\frac{\pi}{4}-x\right)\right)^{2n} f(x)\ dx \\ &= \lim_{n\to\infty} n\int_{-\pi/4}^{\pi/4} (\tan(s))^{2n} f\left(\frac{\pi}{4}-s\right)\ ds \\ &= \lim_{n\to\infty} n\int_0^{\pi/4} (\tan(s))^{2n} \left(f\left(\frac{\pi}{4}-s\right)+f\left(\frac{\pi}{4}+s\right)\right)\ ds \\ &= \lim_{n\to\infty} \int_0^1 \frac{r^{1+1/n}}{1+r^{2/n}} \left(f\left(\frac{\pi}{4}-\arctan(r^{1/n})\right)+f\left(\frac{\pi}{4}+\arctan(r^{1/n})\right)\right)\ dr, \end{aligned}$$

where we have used symmetry, and in the last step we have used the change of variable $r = (\tan(s))^n$.

Since f is a continuous function, $\exists M$ such that $|f(x)| \le M$, for $x \in [0, \frac{\pi}{2}]$, and

$$\left|\frac{r^{1+1/n}}{1+r^{2/n}} \left(f\left(\frac{\pi}{4}-\arctan(r^{1/n})\right)+f\left(\frac{\pi}{4}+\arctan(r^{1/n})\right)\right)\right| \le M$$

for all $r \in [0, 1]$, using the bound for f and that the fraction in r is bounded above by $r/(1 + r^2)$ (which is bounded by $1/2$, by looking at $(r - 1)^2 \ge 0$). In this way, we can apply the dominated convergence theorem to obtain

$$\begin{aligned} L &= \int_0^1 \lim_{n\to\infty} \frac{r^{1+1/n}}{1+r^{2/n}} \left(f\left(\frac{\pi}{4}-\arctan(r^{1/n})\right)+f\left(\frac{\pi}{4}+\arctan(r^{1/n})\right)\right)\ dr \\ &= \frac{1}{2}\left(f(0)+f\left(\frac{\pi}{2}\right)\right)\int_0^1 r\ dr = \frac{1}{4}\left(f(0)+f\left(\frac{\pi}{2}\right)\right). \end{aligned}$$

Solution 2, by Michel Bataille.

We show that the required limit is $\frac{f(0)+f(\pi/2)}{4}$. Let

$$I_n = \int_0^{\frac{\pi}{2}} \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)^{2n} f(x)\, dx = \int_0^{\frac{\pi}{2}} \left(\tan\left(\frac{\pi}{4}-x\right)\right)^{2n} f(x)\, dx.$$

The change of variables $x = \frac{\pi}{4} - \tan^{-1}(y)$ yields

$$I_n = \int_{-1}^1 \frac{y^{2n}}{1+y^2} f\left(\frac{\pi}{4}-\tan^{-1}(y)\right)\ dy.$$

But we have

$$\int_{-1}^0 \frac{y^{2n}}{1+y^2} f\left(\frac{\pi}{4}-\tan^{-1}(y)\right)\ dy = \int_0^1 \frac{u^{2n}}{1+u^2} f\left(\frac{\pi}{4}+\tan^{-1}(u)\right)\ du$$

so that

$$I_n = \int_0^1 y^{2n} g(y)\, dy,$$

where $g(y) = \left(f\left(\frac{\pi}{4} - \tan^{-1}(y)\right) + f\left(\frac{\pi}{4} + \tan^{-1}(y)\right)\right) \cdot \frac{1}{y^2+1}$. It is known that if g is continuous on $[0,1]$, then $\lim\limits_{n\to\infty} n\int_0^1 x^n g(x)\,dx = g(1)$ [for completeness, a quick proof is given at the end]. From this result, it follows that

$$\lim_{n\to\infty} (2n)\cdot I_n = g(1) = \frac{f(0)+f(\pi/2)}{2}$$

and so

$$\lim_{n\to\infty} n\cdot I_n = \frac{f(0)+f(\pi/2)}{4},$$

as claimed.

For the proof of the property used above, let $\epsilon > 0$. Using the continuity of g, we choose $\delta \in (0,1)$ such that $|g(x) - g(1)| \le \epsilon$ whenever $x \in [\delta, 1]$. Then we have

$$\begin{aligned}\left|(n+1)\int_0^1 x^n\cdot g(x)\,dx - g(1)\right| &= \left|(n+1)\int_0^1 x^n\cdot(g(x)-g(1))\,dx\right|\\ &\le (n+1)\int_0^\delta x^n|g(x)-g(1)|\,dx + (n+1)\int_\delta^1 x^n|g(x)-g(1)|\,dx\\ &\le M\cdot\delta^{n+1}+\epsilon\end{aligned}$$

where M denotes the maximum of the continuous function $x \mapsto |g(x) - g(1)|$ on $[0,1]$. Since $0 < \delta < 1$, we deduce $\limsup\limits_{n\to\infty} \left|(n+1)\int_0^1 x^n\cdot g(x)\,dx - g(1)\right| \le \epsilon$. Since the latter holds for any positive ϵ, we must have

$$\lim_{n\to\infty}\left((n+1)\int_0^1 x^n\cdot g(x)\,dx - g(1)\right) = 0,$$

and the result follows.

Editor's Comments. This type of problem has its roots in Fourier analysis, where we are interested in limits such as the one in the problem statement. This particular limit picks out half the arithmetic mean of the function's value at the endpoints of the interval $[0, \frac{\pi}{2}]$; more classical Fourier analysis will focus on limits which pick out the function's value at a specific point, like the 'Dirac delta' distribution. All solutions aside from Solution 1 essentially followed Bataille's Solution 2, including the proposer's; À. Plaza's solution used a limit result from the proposer's own book (O. Furdui, *Limits, Series and Fractional Part Integrals*, Springer, Second ed., 2013) to skip a substitution. The techniques in both solutions (i.e. utilizing dominated convergence in a clever way, and separating an integral up into two parts which are handled using the two different functions involved in the integrand) are common techniques in classical analysis. A. Stadler's approach (namely, using 'Big O' notation instead of more precise estimates) is equally successful and is common in analytic number theory.