OLYMPIAD SOLUTIONS

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OC201. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \in \mathbb{Q}$ and

$$f(x + f(y)^2) = f(x + y)^2.$$

Originally problem 2 from the third round algebra of the 2013 Iran National Mathematical Olympiad.

We received one correct solution. We present the solution by Oliver Geupel.

The two constant functions $f(x) = 0$ and $f(x) = 1$ have the required property, and we show that there are no other solutions.

Let us refer to the functional equation as $F(x, y)$. Let f be a solution and let $f(0) = q = a/b$ where a, b are integers and $b \geq 1$. From $F(x+y-q^2, 0)$ we obtain $f(x + y) = f(x + y - q^2 + f(0)^2) = f(x + y - q^2)^2$. Specialising $x = 0$, we also have $f(y) = f(y - q^2)^2$. Hence, using $F(x, y - q^2)$, we see that

$$f(x + f(y)) = f(x + f(y - q^2)^2) = f(x + y - q^2)^2 = f(x + y). \qquad (1)$$

Setting $y = 0$ in (1), we obtain $f(x + q) = f(x)$ and therefore

$$f(x + nq) = f(x) \qquad (2)$$

for every integer n. A further consequence of (1) is

$$f(f(x)) = f(x).$$

By $F(x, 0)$, we have $f(x + q^2) = f(x)^2$. Mathematical induction yields

$$f(x + nq^2) = f(x)^{2^n} \qquad (n \in \mathbb{N}). \qquad (3)$$

From $F(x - q^2, 0)$, we see that $f(x) = f(x - q^2 + f(0)^2) = f(x - q^2)^2 \geq 0$. By (2) and (3), $f(x) = f(x + aq) = f(x + bq^2) = f(x)^{2^b}$, so that $f(x) \in \{0, 1\}$; whence also $q \in \{0, 1\}$. We consider the cases $q = 1$ and $q = 0$ in succession.

Case $q = 1$. We show that f is the constant function $f(x) = 1$. The proof is by contradiction. Suppose for some real number t it holds $f(t) = 0$. Then,

$$1 = f(0) = f(f(t)) = f(t) = 0,$$

a contradiction which shows that f is the constant function $f(x) = 1$.

Case $q = 0$. We prove that f is the constant function $f(x) = 0$. The proof is again by contradiction. Assume $f(t) = 1$ for some real number t. By (1),

$$f\left(\frac{1}{2} + f\left(\frac{1}{2}\right)\right) = f\left(\frac{1}{2} + \frac{1}{2}\right) = f(1) = f(f(t)) = f(t) = 1.$$

If $f\left(\frac{1}{2}\right)=0$, then $f\left(\frac{1}{2}+f\left(\frac{1}{2}\right)\right)=0$, which is impossible. Thus $f\left(\frac{1}{2}\right)=1$. Hence, by (1), $f\left(\frac{3}{2}\right)=f\left(\frac{1}{2}+1\right)=f\left(\frac{1}{2}+f\left(\frac{1}{2}\right)\right)=1$. Moreover, $0=f\left(\frac{1}{2}-\frac{1}{2}\right)=f\left(\frac{1}{2}+f\left(-\frac{1}{2}\right)\right)$. Note that $f\left(-\frac{1}{2}\right)$ is either 0 or 1. So $f\left(\frac{1}{2}+f\left(-\frac{1}{2}\right)\right)$ is equal to either $f\left(\frac{1}{2}\right)=1$ or $f\left(\frac{3}{2}\right)=1$. This is impossible and the proof is complete.

OC202. Let a, b be real numbers such that the equation $x^3-ax^2+bx-a=0$ has three positive real roots. Find the minimum of $\frac{2a^3-3ab+3a}{b+1}$.

Originally problem 1 from day 1 of the 2013 South East Mathematical Olympiad.

We received six correct solutions. We present the solution by Michel Bataille.

First, let S be the set of all pairs of real numbers (a, b) such that the equation $x^3-ax^2+bx-a=0$ has three positive real roots and let $R(a,b)=\frac{2a^3-3ab+3a}{b+1}$. We show that $\min\limits_{(a,b)\in S} R(a,b)=9\sqrt{3}$.

Now, if $a=3\sqrt{3}$, $b=9$, the equation becomes $(x-\sqrt{3})^3=0$ whose roots are clearly positive real numbers and it is readily checked that $R(3\sqrt{3},9)=9\sqrt{3}$. Thus, there just remains to prove that $R(a,b)\geq 9\sqrt{3}$ whenever $(a,b)\in S$.

Let $(a,b)\in S$ and let x_1, x_2, x_3 be positive roots of $x^3-ax^2+bx-a=0$. Then,

$$x_1+x_2+x_3=a,\quad x_1x_2+x_2x_3+x_3x_1=b,\quad x_1x_2x_3=a.$$

Note that the above shows that $a, b>0$ since the roots are positive real numbers. Observing that

$$(x_1+x_2+x_3)^3=x_1^3+x_2^3+x_3^3+3(x_1+x_2+x_3)(x_1x_2+x_2x_3+x_3x_1)-3x_1x_2x_3$$

we obtain $2a^3=2(x_1^3+x_2^3+x_3^3)+6ab-6a$, hence

$$2a^3-3ab+3a=2(x_1^3+x_2^3+x_3^3)+3ab-3a\geq 2\cdot 3x_1x_2x_3+3ab-3a=3ab+3a=3a(b+1)$$

(using AM-GM for the inequality). It follows that

$$R(a,b)\geq \frac{3a(b+1)}{b+1}=3a. \tag{1}$$

Now, by AM-GM again, $\left(\frac{x_1+x_2+x_3}{3}\right)^3\geq x_1x_2x_3$, hence $\frac{a^3}{27}\geq a$ and so $a\geq 3\sqrt{3}$. With (1), we deduce that $R(a,b)\geq 9\sqrt{3}$, as desired.

OC203. Find all positive integers m and n satisfying $2^n+n=m!$.

Originally problem 1 from day 2 of the 2013 Turkey Mathematical Olympiad.

We received one correct solution. We present the solution by Oliver Geupel.

A solution is

$$(m,n)=(3,2)$$

and we show that it is unique.

Suppose that (m, n) is any solution. Then there exists a nonnegative integer a and a positive odd integer b such that $n = 2^a b$. The exact power of 2 that divides $m! = 2^n + n = 2^a(2^{n-a} + b)$ is 2^a. Thus $m \leq 2a + 1$. If $a \leq 4$ then $m \leq 9$. A straightforward inspection shows that when $m \leq 9$, the only solution is $(m, n) = (3, 2)$. We now consider the case $a \geq 5$.

We prove that for every $a \geq 5$ it holds

$$2^{2^a} > (2a + 1)!.$$

The proof is by mathematical induction on a. The base case $a = 5$ is satisfied since

$$2^{32} > 2^{27} = 2^8 \cdot 2^7 \cdot 2^5 \cdot 2^3 \cdot 2^4 > 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 = 11!$$

Suppose that for some $a \geq 6$ we have $2^{2^{a-1}} > (2a - 1)!$. Then

$$2^{2^a} > (2a-1)!^2 > (2a-1)! \cdot 2(2a-2) \cdot 3(2a-1) > (2a-1)! \cdot 2a \cdot (2a+1) = (2a+1)!,$$

which completes the induction.

We conclude $2^n + n > 2^{2^a} > (2a + 1)! \geq m!$, a contradiction.

OC204. Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

Originally problem 6 from day 2 of the 2013 USA Mathematical Olympiad.

We received one correct solution. We present the solution by Titu Zvonaru.

There are only two such points, namely the intersection of the internal bisector of $\angle BAC$ with BC or its reflection with respect to the midpoint of BC.

Let $a = BC$, $b = CA$, $c = AB$ and let $\Gamma_1(O_1, R_1)$ and $\Gamma_2(O_2, R_2)$ be the circumcircles of triangles PAB and PAC respectively. Let M be the midpoint of AP (and XY) and let T_1 and T_2 be the points of tangency of the common external tangent through X with the circles Γ_1 and Γ_2 respectively.

By the power of a point with respect to a circle, we have $XT_1^2 = XA \cdot XP = XT_2^2$. Hence X is the midpoint of T_1T_2. Since the point M lies on O_1O_2 and $\angle XMO_2 = \angle O_2T_2X = \pi/2$, we obtain the following equivalences:

$$(XM)^2 + (MO_2)^2 = (O_2T_2)^2 + (T_2X)^2,$$

$$(XM)^2 = -(R_2 \cos(C))^2 + R_2^2 + \frac{(T_1T_2)^2}{4},$$

$$4(XM)^2 = 4R_2^2 \sin^2(C) + (O_1O_2^2 - (R_1 - R_2)^2),$$

$$(XY)^2 = 4R_2^2 \sin^2(C) + (R_1 \cos(B) + R_2 \cos(C))^2 - R_1^2 - R_2^2 + 2R_1R_2,$$

$$(XY)^2 = 4R_2^2 \sin^2(C) - R_1^2 \sin^2(B) - R_2^2 \sin^2(C) + 2R_1R_2 \cos(B)\cos(C) + 2R_1R_2.$$

Applying the law of sines, it follows that

$$(XY)^2 = (AP)^2 - \frac{(AP)^2}{4} - \frac{(AP)^2}{4} + \frac{(AP)^2\cos(B)\cos(C)}{2\sin(B)\sin(C)} + \frac{AP^2}{2\sin(B)\sin(C)},$$
$$(XY)^2 = \frac{(AP)^2(1+\cos(B)\cos(C)+\sin(B)\sin(C))}{2\sin(B)\sin(C)},$$
$$\frac{(AP)^2}{(XY)^2} = \frac{2\sin(B)\sin(C)}{1+\cos(B)\cos(C)+\sin(B)\sin(C)}.$$

Letting $x = BP/PC$, we get $BP = ax/(x+1)$ and $PC = a/(x+1)$. This yields

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB\cdot PC}{AB\cdot AC} = 1$$
$$\Leftrightarrow \frac{2\sin(B)\sin(C)}{1+\cos(B)\cos(C)+\sin(B)\sin(C)} + \frac{a^2x}{bc(x+1)^2} = 1$$
$$\Leftrightarrow \frac{x\sin^2(A)}{(x+1)^2\sin(B)\sin(C)} = \frac{1+\cos(B)\cos(C)-\sin(B)\sin(C)}{1+\cos(B)\cos(C)+\sin(B)\sin(C)}$$
$$\Leftrightarrow \frac{x(1-\cos^2(A))}{(x+1)^2\sin(B)\sin(C)} = \frac{1+\cos(B+C)}{1+\cos(B)\cos(C)+\sin(B)\sin(C)}$$
$$\Leftrightarrow x^2\sin(B)\sin(C) + x(-1+\cos(B-C)\cos(B+C)) + \sin(B)\sin(C) = 0.$$

Since

$$-1+\cos(B-C)\cos(B+C) = -1 + \frac{\cos(2B)+\cos(2C)}{2}$$
$$= -1 + \frac{1-2\sin^2(B)+1-2\sin^2(C)}{2},$$

we obtain the equation

$$x^2\sin(B)\sin(C) - x(\sin^2(B)+\sin^2(C)_+ \sin(B)\sin(C) = 0,$$

which is a quadratic equation with roots $x = \sin(C)/\sin(B)$ and $x = \sin(B)/\sin(C)$. Thus, the points P are the intersection of the internal bisector of $\angle BAC$ with BC or its reflection with respect to the midpoint of BC.

OC205. For each positive integer n determine the maximum number of points in space creating the set A which has the following properties:

1. the coordinates of every point from the set A are integers from the range $[0, n]$;
2. for each pair of different points $(x_1, x_2, x_3), (y_1, y_2, y_3)$ belonging to the set A at least one of the following inequalities $x_1 < y_1, x_2 < y_2, x_3 < y_3$ is satisfied and at least one of the following inequalities $x_1 > y_1, x_2 > y_2, x_3 > y_3$ is satisfied.

Originally problem 6 from day 2 of the 2013 Polish Mathematical Olympiad.

We received one correct solution. We present the solution by Oliver Geupel.

The answer is

$$a_n = \left\lfloor \frac{3(n+1)^2+1}{4} \right\rfloor .$$

We show that an a_n-element set with the desired properties is $A = \{(x,y,z) : x+y+z = \lfloor 3n/2 \rfloor\}$. In fact, if n is an even number, $n = 2m$, then members of A are points $(x, y, 3m-x-y)$ where $0 \le x \le m$ and $m - x \le y \le 2m$, as well as points $(x, y, 3m-x-y)$ where $m+1 \le x \le 2m$ and $0 \le y \le 3m-x$, the total number of elements being

$$\sum_{x=0}^{m}(m+1+x) + \sum_{x=m+1}^{2m}(3m+1-x) = 3m^2+3m+1 = a_n.$$

If n is odd, $n = 2m+1$, then members of A are points $(x, y, 3m+1-x-y)$ where $0 \le x \le m$ and $m-x \le y \le 2m+1$, and points $(x, y, 3m+1-x-y)$ where $m+1 \le x \le 2m+1$ and $0 \le y \le 3m+1-x$, with the total number of elements

$$\sum_{x=0}^{m}(m+2+x) + \sum_{x=m+1}^{2m+1}(3m+2-x) = 3(m+1)^2 = a_n.$$

It remains to show that every set A with the required properties has not more than a_n elements. Let us define subsets $B_0, \dots, B_n$ of the lattice cube $[0,n]^3$. The members of B_k are the points $(x, n-k, z)$ where $0 \le x \le k-1$ and $0 \le z \le n$, as well as the points (k, y, z) where $n-k \le y \le n$ and $0 \le z \le n$. So B_k consists of $2k+1$ classes of $n+1$ elements each, where the members of a single class vary only in the third coordinate. Let $P = (x,y,z) \in [0,n]^3$. It follows that, if $x+y < n$ then $P \in B_{n-y}$, whereas if $x+y \ge n$ then $P \in B_x$. Hence the sets $B_0, \dots, B_n$ constitute a disjoint partition of the lattice cube.

Let A be a set with the required properties. Then $A \cap B_k$ has not more than $2k+1$ elements because it cannot contain any two members from the same class by the given property 2. Also by property 2., the elements in $A \cap B_k$ have distinct z-coordinates. Thus $A \cap B_k$ has not more than $n+1$ elements. We obtain $|A| = \sum\limits_{k=0}^{n} |A \cap B_k| \le \sum\limits_{k=0}^{n} \min(2k+1, n+1)$. If n is an even number, $n = 2m$, then

$$\sum_{k=0}^{n}\min(2k+1,n+1) = \sum_{k=0}^{m}(2k+1) + \sum_{k=m+1}^{2m}(2m+1) = 3m^2+3m+1 = a_n.$$

If n is odd, say $n = 2m+1$, then

$$\sum_{k=0}^{n}\min(2k+1,n+1) = \sum_{k=0}^{m}(2k+1) + \sum_{k=m+1}^{2m+1}(2m+2) = 3(m+1)^2 = a_n.$$

Consequently, A has at least a_n elements, which completes the proof.