FOCUS ON...

No. 20

Michel Bataille

Inequalities *via* Complex Numbers

Introduction

Consider the following famous inequality: If A, B, C, D are four points in the plane, then $AB.CD + BC.AD \geq AC.BD$ (Ptolemy's inequality). A very short proof uses complex numbers: introducing the affixes a, b, c, d of A, B, C, D, the equality $(b-a)(d-c) + (c-b)(d-a) = (c-a)(d-b)$ is readily checked. The familiar properties of the modulus of a complex number (in particular the triangle inequality) then give

$$|c-a|.|d-b| = |(b-a)(d-c) + (c-b)(d-a)| \leq |b-a|.|d-c| + |c-b|.|d-a|$$

and Ptolemy's inequality follows at once! This gem of a proof, now well-known, seems to date back to 1914 ([**1**]). In this number, we present some results in the same vein, related to more or less recent problems.

Hayashi's Inequality

Hayashi's inequality, although less known, appears in problems from time to time. To name a couple of recent examples, it is the main argument of the solutions to the *American Mathematical Monthly* problem 11536 proposed in November 2010 and to problem **OC41** [2011 : 424 ; 2012 : 361]. The inequality can be stated as follows:

If P is a point in the plane of a triangle ABC, then

$$\frac{PA \cdot PB}{CA \cdot CB} + \frac{PB \cdot PC}{AB \cdot AC} + \frac{PC \cdot PA}{BC \cdot BA} \geq 1.$$

Various identities for complex numbers can be taken as the starting point of the proof. My favourite one follows from a decomposition in partial fractions which leads to

$$\frac{1}{(p-a)(p-b)(p-c)} = \frac{1}{(b-a)(c-a)} \cdot \frac{1}{p-a} + \frac{1}{(c-b)(a-b)} \cdot \frac{1}{p-b} + \frac{1}{(a-c)(b-c)} \cdot \frac{1}{p-c}.$$

The proof then proceeds by multiplying by $(p-a)(p-b)(p-c)$ and taking moduli as in the proof of Ptolemy's inequality above.

Let us connect Hayashi's inequality to a close one which involves, besides triangle ABC, two points M, N:

$$\frac{CM \cdot CN}{CA \cdot CB} + \frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BC \cdot BA} \geq 1. \tag{1}$$

The method is similar, the inequality being deduced as above from the identity

$$\frac{(m-a)(n-a)}{(b-a)(c-a)}+\frac{(m-b)(n-b)}{(c-b)(a-b)}+\frac{(m-c)(n-c)}{(a-c)(b-c)}=1. \qquad (2)$$

However, the proof of this identity is a good opportunity to apply some results obtained in a prior Focus On (No 7). Indeed, introducing $D(z)=(z-a)(z-b)(z-c)$, the left-hand side L of (2) is

$$\frac{(m-a)(n-a)}{D'(a)}+\frac{(m-b)(n-b)}{D'(b)}+\frac{(m-c)(n-c)}{D'(c)}$$

that is,

$$mn\left(\frac{1}{D'(a)}+\frac{1}{D'(b)}+\frac{1}{D'(c)}\right)-(m+n)\left(\frac{a}{D'(a)}+\frac{b}{D'(b)}+\frac{c}{D'(c)}\right)$$
$$+\left(\frac{a^2}{D'(a)}+\frac{b^2}{D'(b)}+\frac{c^2}{D'(c)}\right)$$

and finally $L=mn\times 0-(m+n)\times 0+1=1$.

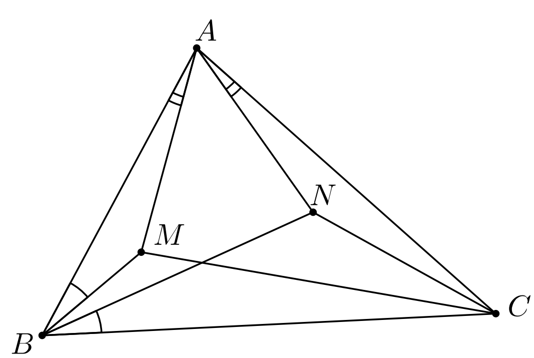
Incidentally, another interesting application of identity (2) is a variant of solution to problem **2595** ([2000 : 498 ; 2001 : 557]), which offers a case of equality in (1).

> Given that M and N are points inside the triangle ABC such that $\angle MAB=\angle NAC$ and $\angle MBA=\angle NBC$, prove that
>
> $$\frac{AM\cdot AN}{AB\cdot AC}+\frac{BM\cdot BN}{BC\cdot BA}+\frac{CM\cdot CN}{CA\cdot CB}=1.$$

Keeping the notations a,b,c,m,n for the affixes of A,B,C,M,N, the additional hypothesis on M and N leads to

$$\arg\left(\frac{(m-a)(n-a)}{(b-a)(c-a)}\right)=\arg\left(\frac{(m-b)(n-b)}{(c-b)(a-b)}\right)=0$$

(since $\angle(\overrightarrow{BM},\overrightarrow{BA})=\angle(\overrightarrow{BC},\overrightarrow{BN})$ and $\angle(\overrightarrow{AM},\overrightarrow{AB})=\angle(\overrightarrow{AC},\overrightarrow{AN})$).



As a result, both $\dfrac{(m-a)(n-a)}{(b-a)(c-a)}$ and $\dfrac{(m-b)(n-b)}{(c-b)(a-b)}$ are positive real numbers.

From (2), $\dfrac{(m-c)(n-c)}{(a-c)(b-c)}$ is a real number and because M and N are interior to the triangle, we must have $\angle(\overrightarrow{CA}, \overrightarrow{CN}) = \angle(\overrightarrow{CM}, \overrightarrow{CB})$, so that $\dfrac{(m-c)(n-c)}{(a-c)(b-c)}$ is a positive real number as well. Taking moduli in (2) then yields the desired equality. Note in passing that M, N are isogonal conjugates with respect to ABC.

More examples

Prompted by expressions evoking the modulus of a complex number, one can sometimes introduce complex numbers advantageously. Here are two examples.

We start with problem **3092**, part (a) [2005 : 544,546 ; 2006 : 526]:

> Let a, b, and c be positive real numbers such that $a+b+c = abc$. Find the minimum value of $\sqrt{1+a^2} + \sqrt{1+b^2} + \sqrt{1+c^2}$.

The statement then referred to the previous problem **2814** of which one of the featured solutions (by Guersenzvaig) used complex numbers. We can mimic the method as follows.

Since $\sqrt{1+a^2} + \sqrt{1+b^2} + \sqrt{1+c^2} = |1+ia| + |1+ib| + |1+ic| \geq |3+i(a+b+c)|$ (by the triangle inequality), we have

$$\sqrt{1+a^2} + \sqrt{1+b^2} + \sqrt{1+c^2} \geq \sqrt{9+(a+b+c)^2}.$$

Now, since $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = 1$ (from $a+b+c = abc$), the harmonic-arithmetic mean inequality gives

$$ab + bc + ca \geq \frac{3}{\frac{1}{3}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)} = 9.$$

Thus, $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca) \geq 27$ and we finally obtain

$$\sqrt{1+a^2} + \sqrt{1+b^2} + \sqrt{1+c^2} \geq 6.$$

Observing that for $a = b = c = \sqrt{3}$, we have $a+b+c = abc$ and the equality case in the above inequality, we conclude that the required minimum value is 6.

Another example is provided by problem **3686** that I proposed in 2011 [2011 : 456, 458 ; 2012 : 391].

> Let a, b, and c be real numbers such that $abc = 1$. Show that
>
> $$\left(a - \frac{1}{a} + b - \frac{1}{b} + c - \frac{1}{c}\right)^2 \leq 2\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right).$$

The problem attracted various methods and three solutions were featured. My proposed solution rested upon complex numbers: Since $abc = 1$, we have

$$\begin{aligned} 2\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) &= 2(a^2+1)(b^2+1)(c^2+1) \\ &= 2|(a+i)(b+i)(c+i)|^2 \\ &= 2|(1-a-b-c)+i(ab+bc+ca-1)|^2 \end{aligned}$$

so that

$$2\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) = 2\left[(a+b+c-1)^2+(1-(ab+bc+ca))^2\right].$$

But, $2(X^2+Y^2) \geq (X+Y)^2$ for all real numbers X, Y, hence

$$\begin{aligned} 2\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) &\geq (a+b+c-ab-bc-ca)^2 \\ &= \left(a-\frac{1}{a}+b-\frac{1}{b}+c-\frac{1}{c}\right)^2. \end{aligned}$$

As usual, we conclude this number with a couple of exercises.

Exercises

1. Prove the identity

$$vw(v-w)+wu(w-u)+uv(u-v)+(v-w)(w-u)(u-v) = 0$$

where u, v, w are complex numbers and deduce another proof of Hayashi's inequality.

2. Using complex numbers, prove the identity

$$(b^2+c^2)(c^2+a^2)(a^2+b^2) = (a^2b+b^2c+c^2a-abc)^2+(ab^2+bc^2+ca^2-abc)^2$$

for real numbers a, b, c. Deduce that if a, b, c are the side lengths of a triangle, then

$$2(b^2+c^2)(c^2+a^2)(a^2+b^2) > (a^3+b^3+c^3)^2.$$

References

[1] T. Hayashi, Two Theorems on Complex Numbers, *Tôhoku Math. Journal*, p. 75–77, (1913/1914).