CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(1), p. 4–5.

CC151. Consider a non-zero integer n such that $n(n+2013)$ is a perfect square.

a) Show that n cannot be prime.

b) Find a value of n such that $n(n+2013)$ is a perfect square.

Originally question 1 from 2013 Pan African Mathematics Olympiad.

We received eleven correct solutions. We present two of the solutions.

Solution 1, by Andrea Fanchini.

a) We denote $m = n(n+2013)$. If m is a perfect square and n is prime then n must divide $n+2013$. By the divisibility properties n must then also be a factor of $2013 = 3 \cdot 11 \cdot 61$. Thus there are three possibilities for m:

$$m = 3(3+2013), \qquad m = 11(11+2013), \qquad m = 61(61+2013).$$

None of these numbers are square, so n cannot be prime.

b) We know that the sum of odd numbers gives a perfect square. If we set

$$n = 1 + 3 + \cdots + 2011 = 1006^2,$$

then

$$n + 2013 = 1 + 3 + \cdots + 2011 + 2013 = 1007^2$$

and $m = 1006^2 \cdot 1007^2$ is a perfect square.

Solution 2, by Albert Stadler.

Put $t = \gcd(n, 2013)$. Note that $\gcd(n, n+2013) = \gcd(n, 2013) = t$. Then

$$n(n+2013) = t^2 \cdot \frac{n}{t} \cdot \frac{n+2013}{t}$$

and $\gcd(\frac{n}{t}, \frac{n+2013}{t}) = 1$. So $n(n+2013)$ is a perfect square if and only if both $\frac{n}{t}$ and $\frac{n+2013}{t}$ are perfect squares. Set $\frac{n}{t} = a^2$ and $\frac{n+2013}{t} = b^2$. Then

$$\frac{2013}{t} = b^2 - a^2 = (b-a)(b+a).$$

Since $2013 = 3 \cdot 11 \cdot 61$, we have $t \in \{1, 3, 11, 33, 61, 183, 671, 2013\}$.

$$
\begin{aligned}
t = 2013 &: (b-a)(b+a) = 1 \text{ has no solutions in positive integers } a, b.\\
t = 671 &: (b-a)(b+a) = 3 \text{ implies } (a,b) = (1,2).\\
t = 183 &: (b-a)(b+a) = 11 \text{ implies } (a,b) = (5,6).\\
t = 61 &: (b-a)(b+a) = 33 \text{ implies } (a,b) = (16,17) \text{ or } (a,b) = (4,7).\\
t = 33 &: (b-a)(b+a) = 61 \text{ implies } (a,b) = (30,31).\\
t = 11 &: (b-a)(b+a) = 183 \text{ implies } (a,b) = (91,92) \text{ or } (a,b) = (29,32).\\
t = 3 &: (b-a)(b+a) = 671 \text{ implies } (a,b) = (335,336) \text{ or } (a,b) = (25,36).\\
t = 1 &: (b-a)(b+a) = 2013 \text{ implies } (a,b) = (1006,1007) \text{ or}\\
&\quad (a,b) = (334,337) \text{ or } (a,b) = (86,97) \text{ or } (a,b) = (14,47).
\end{aligned}
$$

With $n = a^2 t$ we obtain that $n(n+2013)$ is a perfect square if and only if n is one of 196, 671, 976, 1875, 4575, 7396, 9251, 15616, 29700, 91091, 111556, 336675, or 1012036, none of which is prime.

CC152. A square of an $n \times n$ chessboard with $n \geq 5$ is coloured in black and white in such a way that three adjacent squares in either a line, a column or a diagonal are not all the same colour. Show that for any 3×3 square inside the chessboard, two of the squares in the corners are coloured white and the two others are coloured black.

Originally question 5 from 2013 Pan African Mathematics Olympiad.

We received only one incorrect submission.

CC153. A sequence $a_0, a_1, \ldots, a_n, \ldots$ of positive integers is constructed as follows:

- if the last digit of a_n is less than or equal to 5, then this digit is deleted and a_{n+1} is the number consisting of the remaining digits; if a_{n+1} contains no digits, the process stops;
- otherwise, $a_{n+1} = 9a_n$.

Can one choose a_0 so that we can obtain an infinite sequence?

Originally question 5 from 2010 Pan African Mathematics Olympiad.

We received two correct solutions and one incomplete submission. We present the solution by Titu Zvonaru.

It is not possible to obtain an infinite sequence. If the last digit of a_n is less than or equal to 5, then it is obvious that $a_{n+1} < a_n$. If the last digit of a_n is greater than 5, then the last digit of a_{n+1} is less than 5. It results that

$$a_{n+2} = [a_{n+1}/10] = [9a_n/10] < a_n.$$

So if we had an infinite sequence (a_n) of positive integers we would find an infinite strictly decreasing subsequence, a contradiction.

CC154. The numbers $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2012}$ are written on the blackboard. Alice chooses any two numbers from the blackboard, say x and y, erases them and instead writes the number $x + y + xy$. She continues to do so until there is only one number left on the board. What are the possible values of the final number?

Originally question 4 from 2012 Pan African Mathematics Olympiad.

There was one correct solution for this problem and two incomplete submissions. We present the solution by Konstantine Zelator.

Note that $x = (x + 1) - 1$ and $xy + x + y = (x + 1)(y + 1) - 1$.

We use the following lemma:

> If $X = (a_1 + 1)\cdots(a_k + 1) - 1$ and $Y = (a_{k+1} + 1)\cdots(a_{k+m} + 1) - 1$, then $XY + X + Y = (a_1 + 1)\cdots(a_{k+m} + 1) - 1$.

By writing $XY + X + Y = (X + 1)(Y + 1) - 1$, the lemma follows immediately.

From this lemma, it follows that if the board starts with numbers $n_1, n_2, \ldots, n_t$ and the given operation is applied to the numbers in any order until a single number remains, that number will be $(n_1 + 1)(n_2 + 1)\cdots(n_t + 1) - 1$.

For the set of numbers $1, \frac{1}{2}, \ldots, \frac{1}{2012}$ the final answer will thus be

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{2013}{2012} - 1 = 2012.$$

CC155. Find all real solutions x to the equation $[x^2 - 2x] + 2[x] = [x]^2$. Here $[a]$ denotes the largest integer less than or equal to a.

Originally question 3 from 2012 Pan African Mathematics Olympiad.

There were three correct solutions for this problem and one incorrect submission. We present the solution by the Missouri State University Problem Solving Group.

The equation is true for any integer x, so we need only fnid the non-integer solutions. Suppose x is a non-integer solution and $\lfloor x \rfloor = n$. Then $x = n + \epsilon$ for some ϵ with $0 < \epsilon < 1$. We will make use of the fact that for any integer k, $\lfloor a + k \rfloor = \lfloor a \rfloor + k$. We have:

$$\begin{aligned}
\lfloor (x-1)^2 - 1 \rfloor + 2\lfloor x \rfloor &= \lfloor x \rfloor^2 \\
\lfloor (x-1)^2 \rfloor &= \lfloor x \rfloor^2 - 2\lfloor x \rfloor + 1 \\
\lfloor (n - 1 + \epsilon)^2 \rfloor &= (n-1)^2 \\
\lfloor (n-1)^2 + 2\epsilon(n-1) + \epsilon^2 \rfloor &= (n-1)^2 \\
\lfloor 2\epsilon(n-1) + \epsilon^2 \rfloor &= 0
\end{aligned}$$

Equivalently,

$$0 \le 2\epsilon(n-1) + \epsilon^2 < 1. \tag{1}$$

Since we assumed x is not an integer, $\epsilon \neq 0$. Taking both sides of inequality (1) we get:

$$1 - \frac{\epsilon}{2} \le n \le \frac{1 + 2\epsilon - \epsilon^2}{2\epsilon}.$$

Since $0 < \epsilon < 1$, we have $0 < 1 - \epsilon/2 < 1$, implying $n > 0$.

Taking the right side of inequality (1) and completing the square gives:

$$\begin{aligned} (\epsilon + (n-1))^2 &< (n-1)^2 + 1 \\ \epsilon + n - 1 &< \sqrt{(n-1)^2 + 1} \\ x = n + \epsilon &< \sqrt{(n-1)^2 + 1} + 1 \end{aligned}$$

For any positive integer n, this gives the following interval for the solution x:

$$(n, \sqrt{(n-1)^2 + 1} + 1).$$

Thus, the set of all solutions is given by

$$\bigcup_{n=1}^{\infty} \left(n, \sqrt{(n-1)^2 + 1} + 1\right) \bigcup \mathbb{Z}.$$