Angle Bisectors in a Triangle

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In this article, we have collected some geometric facts which are directly or tangentially related to the angle bisectors in a triangle. These results vary from easy lemmas to serious theorems, but we will not classify them; rather, we will just number them. Every statement that occurs without a proof is considered as an exercise for the reader. In fact, even the presented proofs are rather concise, which allows the reader to fill in the details.

Everyone should know this

Let us first recall some standard notation : let ABC be the given triangle, S_{ABC} is its area, $|BC| = a$, $|CA| = b$, $|AB| = c$, $2p = a + b + c$, O and R are the circumcentre and the circumradius, I and r are the incentre and the inradius. Furthermore, the triangle has three excircles, each of which touches one side of a triangle and the extensions of the other two sides. Their centres and their radii will be denoted by I_a, I_b, I_c, r_a, r_b, r_c (I_a denotes the centre of the excircle touching the side BC and extensions of the sides AB and AC with r_a being its radius). Further notation will be presented as needed.

1. Suppose the internal bisector of $\angle A$ intersects the side BC at a point A_1. Then

$$\frac{|BA_1|}{|A_1C|} = \frac{|BA|}{|AC|} = \frac{c}{b}.$$

2. Suppose the external angle bisector of $\angle A$ intersects the line BC at the point A_2. Then

$$\frac{|BA_2|}{|A_2C|} = \frac{|BA|}{|AC|} = \frac{c}{b}.$$

3. $S_{ABC} = pr$.

4. $S_{ABC} = (p - a)r_a$.

5. Let M be the point of tangency of the incircle with the side AB. Then $|AM| = p - a$.

6. Let M be the point of tangency of the excircle with the centre I_a and the line AB. Then $|AM| = p$.

7. Points B and C lie on the circle with diameter II_a and the centre of that circle lies on a circumcircle (see Figure 1.)

Therefore, the centre I of the incircle has the following property : the lines AI, BI and CI (that is, the angle bisectors of the triangle) go through the centres of the circumcircles of triangles BIC, CIA and AIB, respectively. The converse is true as well, namely :

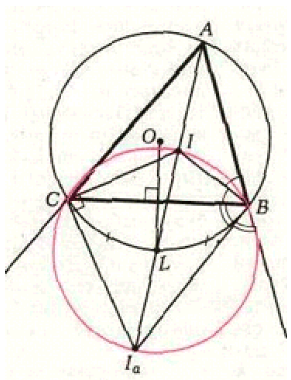

Figure 1: See Problem 7.

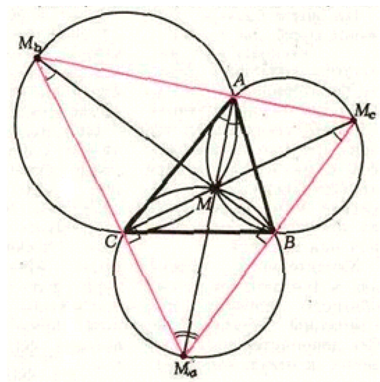

Figure 2: See Problem 8.

8. If the lines AM, BM and CM go through the centres of the circumcircles of triangles BMC, CMA and AMB, then M is the centre of the incircle of ABC.

Indeed, let M_a, M_b and M_c be the points of intersection (different from M) of the lines AM, BM and CM with the corresponding circles (see Figure 2). Then MM_a, MM_b and MM_c are diameters of those circles ; therefore, M_aA, M_bB and M_cC are altitudes of the triangle $M_aM_bM_c$. This implies that $\angle BAM = \angle BM_cM = 90^\circ - \angle BM_aC = \angle CM_bM = \angle CAM$, which means that M lies on the angle bisector of angle A and, analogously, on the angle bisectors of angles B and C.

Distances between centres of special circles

9. $|OI|^2 = R^2 - 2Rr$ (Euler's formula).

10. $|OI_a|^2 = R^2 + 2Rr_a$.

11. $|II_a|^2 = 4R(r_a - r)$.

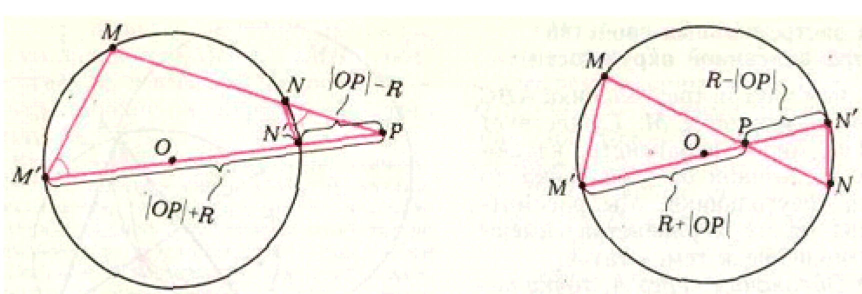

Figure 3: See Problems 9 and 10.

For the proof of **9** and **10**, recall that if M and N are points of intersection of a line passing through an arbitrary point P with the circle of radius R and centre O, then $|PM| \cdot |PN| = |R^2 - |OP|^2|$; this follows from the similarity of triangles

PMM' and PNN', where M' and N' are points of intersection of the line OP with the circle (see Figure 3). It implies that $R^2 - |OI|^2 = |IA| \cdot |IL|$, where L is the point of intersection of the angle bisector of angle A and the circumcircle (see Figure 4). But $|IA| = r/\sin(\angle A/2)$ and, by **7**, $|IL| = |LB| = 2R \times \sin(\angle A/2)$, so $R^2 - |OI|^2 = 2Rr$. Analogously,

$$|OI_a|^2 - R^2 = |I_aL| \cdot |I_aA| = 2R \times \sin(\angle A/2) \times \frac{r_a}{\sin(\angle A/2)} = 2Rr_a.$$

Finally,

$$|II_a|^2 = 2|IL| \cdot (|I_aA| - |IA|) = 4R\sin(\angle A/2) \cdot \frac{r_a - r}{\sin(\angle A/2)} = 4R(r_a - r).$$

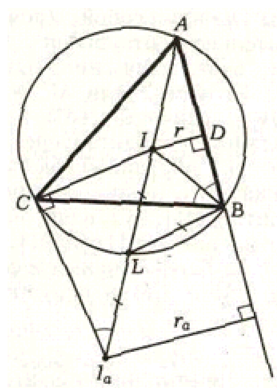

FIGURE 4: See Problems 9 and 10.

12. Consider the points symmetric to the centres of the excircles with respect to the centre of the circumcircle. These points lie on the circle of radius $2R$ with the centre I.

Two extremal properties of the centre of the incircle

Consider an arbitrary point M inside the triangle ABC. There are many inequalities concerning the distances between M and the vertices of the triangle. We will consider two such inequalities.

13. Let A_1 be the point of intersection of the line AM and the circumcircle. Then

$$\frac{|BM| \cdot |CM|}{|A_1M|} \geq 2r$$

and equality holds if M coincides with I.

Suppose that the smallest value of $f(M) = \frac{|BM| \cdot |CM|}{|A_1M|}$ is achieved when M is some point inside ABC. We will show that $M = I$. Then, since $f(I) = 2r$ (this follows, for example, from similar triangles BID and I_aIC in Figure 4), it would imply that *if* $f(M)$ *achieves its minimum inside* ABC, then $f(M) \geq 2r$. The italicized statement is far from trivial and should be carefully proven.

Construct the circumcircle of triangle AMC (see Figure 5). Consider triangles CMA_1 formed by moving the point M along the arc AC — they are all similar (why ?) and hence the ratio $|CM|/|A_1M|$ is constant for all of them. Therefore, if the minimum of $f(M)$ is achieved at M, then the line BM must go through the circumcentre of the triangle AMC (otherwise, we could reduce $|BM|$ while keeping $|CM|/|A_1M|$ constant). Now, let B_1 and C_1 be the points of intersection of lines BM and CM with the circumcircle of ABC. Then, as we saw in the proof of **9**, we have $|MA| \cdot |MA_1| = |MB| \cdot |MB_1| = |MC| \cdot |MC_1|$ and hence

$$\frac{|BM| \cdot |CM|}{|A_1M|} = \frac{|CM| \cdot |AM|}{|B_1M|} = \frac{|AM| \cdot |BM|}{|C_1M|}.$$

Therefore, lines AM and CM must also pass through the circumcentres of triangles BMC and AMB respectively. Then M is the circumcentre of ABC (by **8**).

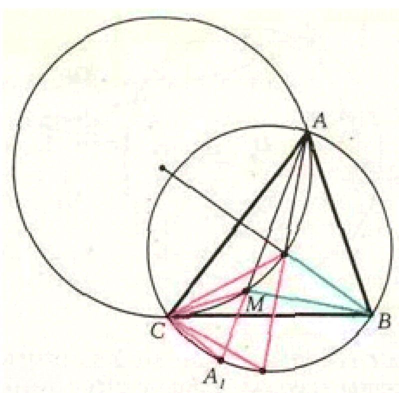

Figure 5: See Problem 13.

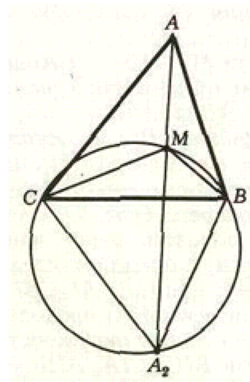

Figure 6: See Problem 14.

One must always be careful when using indirect proofs like the one above (where we did not directly prove that $f(M) \geq f(I)$ for all points M inside ABC) since a function does not always achieve its minimum and maximum.

14. Show that

$$|AM| \sin \angle BMC + |BM| \sin \angle CMA + |CM| \sin \angle AMB \leq p,$$

and equality holds if M coincides with I.

The proof of this statement will also be indirect : we will show that the point M where the left-hand side achieves its maximum (if it exists !) coincides with I.

Construct the circumcircle of triangle BMC and extend the line AM until the second point of intersection A_2 (see Figure 6). Apply Ptolemy's theorem to the quadrangle $BMCA_2$ to get :

$$|BM| \cdot |A_2C| + |CM| \cdot |A_2B| = |BC| \cdot |A_2M|.$$

Since the lengths of chords of a circle are proportional to the sines of angles subtended by those chords, we have

$$|BM| \sin \angle A_2MC + |CM| \sin \angle A_2MB = |A_2M| \sin \angle BMC$$

or

$$|BM| \sin \angle AMC + |CM| \sin \angle AMB = |A_2M| \sin \angle BMC.$$

Comparing the last equation to **14**, we see that the left side of the inequality equals $|AA_2| \sin \angle BMC$. Therefore, the line AM must go through the circumcentre of BMC since otherwise we can increase the value of the left-hand side of **14** by moving M along the arc BC. The rest of the proof is similar to that of **13**.

We leave it to the reader to prove that if $M = I$, then $|AA_2| \sin \angle BMC = p$. To see that, you can use **6** and **7** and the fact that $\angle BIC = 90^\circ + \angle A/2$.

When intuition fails

When two similar elements of a triangle are equal (such as two angles or two medians), it seems natural to expect the triangle to be isosceles. Among the problems of this type, one of the hardest to prove is the Steiner-Lehmus theorem.

15. If a triangle has two angle bisectors of equal lengths, then it is isosceles.

This problem is well-known, whereas the following amusing variation is not usually familiar even to geometry buffs.

16. Suppose a triangle ABC has angle bisectors AA_1, BB_1 and CC_1. If the triangle $A_1B_1C_1$ is isosceles, is ABC isosceles as well ?

Experiments with the graphic software suggest that the answer is no ; $\triangle A_1B_1C_1$ can be isosceles while $\triangle ABC$ is not. We do not know of any brief, elegant construction of a counterexample.

And for now, some more problems.

17. Prove that the angle bisector in a triangle bisects the angle between the circumradius and the altitude from the same vertex.

18. Let AA_1 be the angle bisector of $\angle A$ in a triangle ABC. Show that

$$|AA_1| = \sqrt{bc - |BA_1| \cdot |CA_1|} = \frac{2bc \cos(\angle A/2)}{b+c}.$$

19. Suppose a triangle ABC has angle bisectors AA_1, BB_1 and CC_1. Show that the altitudes of ABC are angle bisectors of $A_1B_1C_1$.

20. Let M and N be the projections of the point of intersection of the altitudes of ABC onto the internal and external angle bisector of angle A. Show that the line MN divides the side BC in half.

21. Let S be the sum of the areas of the three triangles whose vertices are the points where an excircle touches the sides (or their extensions) of the given triangle

ABC. Let T be the area of the triangle formed by the points where the incircle touches the sides of ABC. Prove that $S = S_{ABC} + T$.

22. Suppose a triangle ABC has angle bisectors AA_1, BB_1 and CC_1; let L and K be the points of intersection of the lines AA_1 with B_1C_1 and CC_1 with A_1B_1, respectively. Show that BB_1 bisects angle LBK.

23. Let M and N be the midpoints of the diagonals AC and BC of a cyclic quadrilateral $ABCD$. Prove that if BC bisects angle ANC, then AC bisects angle BMD.

24. In a triangle ABC, let M be the point of intersection of the angle bisector of angle B with the line passing through the midpoint of AC and the midpoint of the altitude from the vertex B. Let N be the midpoint of the angle bisector of angle B. Show that the angle bisector of angle C also bisects angle MCN.

25. Suppose a triangle ABC has angle bisectors AA_1, BB_1 and CC_1, and construct the circle O through the points A_1, B_1 and C_1. Consider the three chords of O formed by the segments of the sides of ABC lying inside O. Prove that the length of one of these chords is equal to the sum of the other two.

26. In a triangle ABC, let K and L be points on the sides AB and BC, respectively, such that $|AK| = |KL| = |LC|$. Draw the line parallel to the angle bisector of angle B through the point of intersection of the lines AL and CK. Let M be the point of intersection of this line with the line AB. Show that $|AM| = |BC|$.

27. Let $ABCD$ be a cyclic quadrilateral. Let K be the point of intersection of the extensions of the sides AB and CD; let L be the point of intersection of the extensions of the sides BC and AD. Show that the bisectors of the angles BKC and BLA are perpendicular and intersect on the line connecting the midpoints of AC and BD.

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