SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014 : 40(8), p. 345–348.

3971. *Proposed by Marcel Chiriţă.*

Let x, y, z be positive real numbers such that $4(x+y+z)^2 = 9(x^2+y^2+z^2)$. Show that

$$4-\sqrt{15} \leq \frac{x}{z} \leq 4+\sqrt{15}.$$

We received 21 submissions of which 20 were correct and complete. We present the solution by Roy Barbara.

By the hypothesis we have

$$8(xy+yz+zx) = 5(x^2+y^2+z^2). \tag{1}$$

Set $k = \frac{x}{z}$. Replacing x by kz in (1) and rearranging yields :

$$5y^2 - 8z(k+1)y + z^2(5k^2-8k+5) = 0. \tag{2}$$

Looking at (2) as a quadratic equation in y (with root y a real number), we see that its discriminant must be non-negative, that is,

$$z^2[16(k+1)^2 - 5(5k^2-8k+5)] \geq 0.$$

Hence, since $z^2 > 0$, $16(k+1)^2 - 5(5k^2-8k+5) \geq 0$, which simplifies to

$$k^2 - 8k + 1 \leq 0. \tag{3}$$

Inequality (3) shows that k must lie between the two zeros of $k^2-8k+1 = 0$, that is,

$$4-\sqrt{15} \leq \frac{x}{z} \leq 4+\sqrt{15}.$$

3972. *Proposed by Michel Bataille.*

Find all functions $f : (0,\infty) \to (0,\infty)$ such that

$$f\left(xf\left(\frac{1}{y}\right)\right) = xf\left(\frac{1}{x+y}\right)$$

for all positive real numbers x, y.

We received five solutions, of which three were correct and complete. We present two solutions.

Solution 1, by Roy Barbara.

The function $f(x) = \frac{x}{x+1}$ is a unique solution. Indeed, let f be any solution and let y be any positive real number. Set

$$x = \sqrt{\frac{y^2}{4} + f\left(\frac{1}{y}\right)^{-1}} - \frac{y}{2}. \tag{1}$$

Then, $x > 0$ and $x + yx - f\left(\frac{1}{y}\right)^{-1} = 0$, whence

$$xf\left(\frac{1}{y}\right) = \frac{1}{x+y}.$$

Hence,

$$f\left(xf\left(\frac{1}{y}\right)\right) = f\left(\frac{1}{x+y}\right).$$

Since $f\left(xf\left(\frac{1}{y}\right)\right) = xf\left(\frac{1}{x+y}\right)$ by hypothesis, we conclude that $x = 1$ and therefore by (1)

$$\sqrt{\frac{y^2}{4} + f\left(\frac{1}{y}\right)^{-1}} = \frac{y}{2} + 1.$$

Squaring yields $f\left(\frac{1}{y}\right)^{-1} = y + 1$, that is, $f\left(\frac{1}{y}\right) = \frac{1}{y+1}$. Since $y > 0$ is arbitrary, replacing y by $\frac{1}{y}$ yields $f(y) = \frac{y}{y+1}$.

Solution 2, by Mohammed Aassila.

Let

$$(x, y) = \left(\frac{-1 + \sqrt{1 + \frac{4t^2}{f(t)}}}{2t}, \frac{1}{t}\right),$$

where $t > 0$. Clearly, $x > 0$ and $y > 0$. Then we obtain

$$1 = \frac{-1 + \sqrt{1 + \frac{4t^2}{f(t)}}}{2t}.$$

Thus, $f(t) = \frac{t}{t+1}$ and it is easy to check that it satisfies the given functional equation.

Editor's Comments. We don't know anything about the injectivity of f, so it cannot be used to solve the problem! If f were injective, the problem would be trivial because if we set $x = 1$, then we get

$$f\left(f\left(\frac{1}{y}\right)\right) = f\left(\frac{1}{1+y}\right) \implies f\left(\frac{1}{y}\right) = \frac{1}{1+y},$$

which yields $f(y) = \frac{y}{y+1}$.

3973. *Proposed by Dragoljub Milošević.*

Let m_a, m_b, m_c be the medians, r_a, r_b, r_c the exradii, R the circumradius and r the inradius of a triangle. Prove that

$$\frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c} \leq \frac{2R}{r} - 1.$$

We received four correct solutions. We present the one by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, expanded slightly by the editor.

Let p_a, p_b, and p_c be the distances from the circumcentre O of the triangle to the sides BC, CA, and AB, respectively. Then it is known (see page 73 of O. Bottema et al., *Geometric inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969) that

$$m_a \leq R + p_a, \quad m_b \leq R + p_b, \quad m_c \leq R + p_c. \tag{1}$$

Now let K be the foot of the perpendicular from O to BC. Then

$$p_a = R\cos(\angle BOK) = R\cos\left(\frac{1}{2}\angle BOC\right) = R\cos A,$$

$$\text{and similarly, } \ p_b = R\cos B \text{ and } \ p_c = R\cos C. \tag{2}$$

From (1) and (2) we have

$$m_a \leq R(a + \cos A), \quad m_b \leq R(1 + \cos B), \quad m_c \leq R(1 + \cos C). \tag{3}$$

It is well known that

$$rs = (s-a)r_a = (s-b)r_b = (s-c)r_c, \tag{4}$$

where $s = \frac{1}{2}(a+b+c)$. From (3) and (4) we then obtain

$$\begin{aligned}
\sum_{cyclic} \frac{m_a}{r_a} &\leq \frac{R}{rs}\sum_{cyc}(1+\cos A)(s-a) \\
&= \frac{R}{r}\sum_{cyc}(1+\cos A) - \frac{R}{rs}\left(2s + \sum_{cyc} a\cos A\right) \\
&= \frac{R}{r}\left(3 + \sum_{cyc}\cos A\right) - \frac{2R}{r} - \frac{R}{rs}\sum_{cyc} a\cos A \\
&= \frac{R}{r} + \frac{R}{r}\sum_{cyc}\cos A - \frac{R}{rs}\sum_{cyc} a\cos A.
\end{aligned} \tag{5}$$

Now,

$$\begin{aligned}
\sum_{cyc}\cos A &= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1 + \frac{4(s-a)(s-b)(s-c)}{abc} \\
&= 1 + 4\left(\frac{F^2/s}{4RF}\right) = 1 + \frac{F}{sR} = 1 + \frac{rs}{sR} = 1 + \frac{r}{R},
\end{aligned} \tag{6}$$

where F denotes the area of the triangle. Using (2) we have

$$\sum_{cyc} a\cos A = \frac{1}{R}\sum_{cyc} ap_a = \frac{2F}{R}. \tag{7}$$

Substituting (6) and (7) into (5) we finally have

$$\sum_{cyc}\frac{m_a}{r_a} \le \frac{R}{r} + \frac{R}{r}\left(1+\frac{r}{R}\right) - \frac{R}{rs}\left(\frac{2F}{R}\right) = \frac{2R}{r} + 1 - \frac{2F}{rs} = \frac{2R}{r} - 1$$

since $F = rs$, and our proof is complete.

3974. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a+b+c=3$. Prove that

$$\sqrt{\frac{a}{b}+\frac{1}{a}} + \sqrt{\frac{b}{c}+\frac{1}{b}} + \sqrt{\frac{c}{a}+\frac{1}{c}} \ge 3\sqrt{2}.$$

We received 22 solutions and 5 incorrect submissions (mostly due to arithmetic errors). We present the solution by Mohammed Aassila.

By the power mean inequality we have

$$\frac{a+b+c}{3} \ge \left(\frac{a^{-1/4}+b^{-1/4}+c^{-1/4}}{3}\right)^{-4}.$$

Using the hypothesis $a+b+c=3$ and rearranging, we get

$$a^{-1/4}+b^{-1/4}+c^{-1/4} \ge 3.$$

Finally, applying the AM-GM inequality to each term, we have

$$\begin{aligned}\sqrt{\tfrac{a}{b}+\tfrac{1}{a}} + \sqrt{\tfrac{b}{c}+\tfrac{1}{b}} + \sqrt{\tfrac{c}{a}+\tfrac{1}{c}} &\ge \sqrt{2\sqrt{\tfrac{1}{b}}} + \sqrt{2\sqrt{\tfrac{1}{c}}} + \sqrt{2\sqrt{\tfrac{1}{a}}} \\ &= \sqrt{2}\left(a^{-1/4}+b^{-1/4}+c^{-1/4}\right) \\ &\ge 3\sqrt{2}.\end{aligned}$$

Note that equality holds if and only if $a=b=c=1$.

3975. *Proposed by Ovidiu Furdui.*

Let $k \ge 1$ be an integer. Calculate

$$\int_0^\infty \frac{e^x-1}{e^x+1}\ln^k\left(\frac{e^x+1}{e^x-1}\right)dx.$$

We received six correct solutions. We present the solution by Anastasios Kotronis.

With the change of the order of integration and summation being justified by the constant sign of the summand, the change of variables $y = \ln\left(\frac{e^x+1}{e^x-1}\right)$ gives

$$\begin{aligned}\int_0^\infty \frac{e^x-1}{e^x+1}\ln^k\left(\frac{e^x+1}{e^x-1}\right)dx &= 2\int_0^\infty \frac{e^{-2y}y^k}{1-e^{-2y}}\,dy\\ &= 2\int_0^\infty \sum_{n=0}^\infty e^{-2(n+1)y}y^k\,dy\\ &= 2\sum_{n=0}^\infty \int_0^\infty e^{-2(n+1)y}y^k\,dy.\end{aligned}$$

With the substitution $z = 2(n+1)y$ the above is equal to

$$2^{-k}\sum_{n=1}^\infty \frac{1}{n^{k+1}}\int_0^\infty e^{-z}z^k\,dz = 2^{-k}\Gamma(k+1)\zeta(k+1).$$

3976. *Proposed by Cristinel Mortici and Leonard Giugiuc.*

Find positive integer solutions for the following equation :

$$\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = \frac{x}{y-z}.$$

We received two correct and one incorrect solution. Presented is the solution by Madhav R. Modak, slightly modified by the editor.

First, let positive integers x, y, z with $y \neq z$, satisfy the given equation. Then

$$(yz - xz + xy)(y - z) = x^2yz. \tag{1}$$

Let $d = (y, z)$ so that $y = ad,\ z = bd,\ (a, b) = 1$, where a, b are positive integers. Substituting into (1) and dividing by d^2 yields

$$(dab - xb + xa)(a - b) = x^2ab. \tag{2}$$

Since $(a, b) = 1$, a is co-prime to b and $a - b$. As a divides the left hand side, it must also divide x. Similarly, $b|x$. Thus $ab|x$. Hence let $x = kab$ where k is a positive integer. Substituting into (2) and dividing by ab yields

$$(d - kb + ka)(a - b) = k^2a^2b^2. \tag{3}$$

Since $a - b$ is co-prime to both a and b, (3) shows that $(a - b)|k^2$. Hence (3) determines the integer d as follows :

$$d = -k(a - b) + a^2b^2 \cdot \frac{k^2}{a - b}. \tag{4}$$

Note that $|a-b| < \max\{|a|,|b|\}$. and therefore $|a-b|^2 < a^2b^2$. It follows that the magnitude of the left summand in (4) is less than the magnitude of the right summand. As d is positive, it must be $a > b$.

Thus a solution (x, y, z) of (1) necessarily satisfies the following conditions. There exist positive integers a, b, k with

$$\begin{gathered} x = kab,\ y = ad,\ z = bd,\ \text{where} \\ a > b,\ (a, b) = 1,\ (a-b)|k^2,\ \text{and} \\ d = -k(a-b) + a^2b^2\frac{k^2}{a-b}. \end{gathered} \tag{5}$$

Conversely it is easy to check that substituting x, y, z from (5) into the original equation yields a feasible solution. Hence (5) gives all solutions to the given equation.

As an example, for $a = 2$, $b = 1$, $k = 1$, we obtain the solution $(x, y, z) = (2, 6, 3)$.

Editor's comments. The solution can be easily extended to all integers x, y, z by allowing a, b, k to be negative. In that case the condition $a > b$ can also be omitted in (5) without changing the solution set.

3977. *Proposed by Dragoljub Milošević.*

Given a triangle ABC, prove that

$$\frac{1}{3-2\cos A} + \frac{1}{3-2\cos B} + \frac{1}{3-2\cos C} \ge \frac{3}{2}.$$

We received 15 correct solutions and three incorrect submissions. We present the joint solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, which is one of only a few using elementary methods.

Let S denote the semiperimeter of ΔABC. Then using the fact that

$$\sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc},$$

we have

$$\begin{aligned} 3 - 2\cos A &= 1 + 2(1-\cos A) = 1 + 4\sin^2\frac{A}{2} \\ &= 1 + \frac{4(s-b)(s-c)}{bc} = \frac{bc + 4(s-b)(s-c)}{bc}. \end{aligned}$$

Hence,

$$\sum_{cyc}\frac{1}{3-\cos A} = \sum_{cyc}\frac{bc}{bc+4(s-b)(s-c)} = 3 - \sum_{cyc}\frac{4(s-b)(s-c)}{bc+4(s-b)(s-c)}$$

so it suffices to show that

$$\sum_{cyc} \frac{4(s-b)(s-c)}{bc+4(s-b)(s-c)} \leq \frac{3}{2}.$$

By the AM-GM inequality, we have $bc + 4(s-b)(s-c) \geq 2\sqrt{4bc(s-b)(s-c)}$, and thus,

$$\sum_{cyc} \frac{4(s-b)(s-c)}{bc+4(s-b)(s-c)} \leq \sum_{cyc} \sqrt{\frac{(s-b)(s-c)}{bc}} = \sum_{cyc} \sin\frac{A}{2}.$$

Since it is well known (see item 2.9 on p.20 of [1] O. Bottema et al., *Geometric inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969) that $\sum_{cyc} \sin\frac{A}{2} \leq \frac{3}{2}$, our proof is complete. The equality holds if and only if ΔABC is equilateral.

Editor's comments. The proofs of most other solvers made use of various 'heavy' machineries. These include : convexity of a function ; Lagrange's Multipliers ; Muirhead's Inequality ; Sturm method ; Gerretsen's Inequality ; and Half-convex Function Theorem.

3978. *Proposed by Billy Jin and Edward T.H. Wang.*

Let $n > 2$ be a positive integer. A permutation $\sigma = (a_1, a_2, \ldots, a_n)$ of $S(n) = \{1, 2, \ldots, n\}$ is called a *zigzag permutation* if, when reading from left to right, the a_i's alternately change their parity and increase/decrease in magnitudes. For example, $(5, 6, 3, 4, 1, 2)$ is a zigzag permutation of $S(6)$. Determine the number of zigzag permutations of $S(n)$.

We received six correct submissions. We present the solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal.

Let us prove that there exist 2 zigzag permutations of $S(n)$ if n is even and none if n is odd.

For $1 < j < n$, we say a_j is a local minimum for the permutation $(a_1, a_2, \ldots, a_n)$ if $a_j < \min\{a_{j-1}, a_{j+1}\}$. A local maximum is defined analogously, and these definitions can be adapted for a_1 and a_n in an obvious manner.

Because in a zigzag permutation the parity alternates, we deduce that numbers of the same parity must be all local minima or all local maxima. But 1 is a minimum, whence the odd numbers must all be local minima while the even numbers must all be local maxima. Note, as an immediate consequence, that n (which is the largest element of $S(n)$) must automatically be a local maximum, which means that *n is necessarily even.* Because there is only one element of $S(n)$ less than 2, 2 must take an extreme position and 1 must be next to it ; that is, we either have $a_1 = 2$ or $a_n = 2$. Let us first examine the case in which 2 comes first, so that $a_2 = 1$. The same observation now holds for the remaining elements of $S(n)$: 3 is the minimum of those remaining numbers and, therefore, the only remaining number less than

4. Because 4 must be surrounded by the two smallest odd numbers 1 and 3, we deduce that $a_3 = 4$ and $a_4 = 3$. When n is even, this construction will continue up to the largest even number n (and therefore smallest remaining even number) as a_{n-1} and with $n-1$ appearing last as a_n. This explicit construction proves the existence and uniqueness of a zigzag permutation starting with $a_1 = 2$. It remains to observe that the reverse of a zigzag permutation is also a zigzag permutation (having $a_n = 2$), from which we conclude that when n is even there exist exactly two zigzag permutations of $S(n)$.

Editor's Comments. Trey Smith of Angelo State University observed that our problem bears a striking resemblance to "Problem 4755 : A Permutation Problem", which appeared in *The American Mathematical Monthly*, proposed by Chandler Davis (**64** (1957), p. 596) with a solution by W.J. Blundon (**65** (1958), p. 533-534). That problem calls for the number of permutations of $S(n)$ in which relative minima alternate with relative maxima (without requiring alternating parity). It is quite remarkable how dropping the parity requirement turns our elementary problem into the rather difficult *Monthly* problem, whose solution involves the Bernoulli and Euler numbers! For that problem, the number of permutations for $n = 1, 2, 3, 4, 5, 6, 7\ldots$ turns out to be $1, 2, 4, 10, 32, 122, 544, \ldots$, which is sequence A001250 in the *On-Line Encyclopedia of Integer Sequences*.

3979. *Proposed by George Apostolopoulos.*

Let AD, BE and CF be the internal bisectors of the triangle ABC. Prove that

$$\frac{[DEF]}{[ABC]} \le \frac{R_1}{4r},$$

where R_1 denotes the circumradius of DEF, r denotes the inradius of ABC and $[\cdot]$ represents the area of the corresponding triangle.

We received eight correct solutions. We present the solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal.

The bisector theorem gives, for example, $BF = \frac{ac}{a+b}$ and $BD = \frac{ac}{b+c}$. Then, denoting $[BDF] = S_b$ and $[ABC] = S$, we have

$$S_b = \frac{1}{2} BF \cdot BD \cdot \sin B = \frac{1}{2}\frac{a^2c^2}{(a+b)(b+c)} \cdot \sin B.$$

Since $S = \frac{1}{2}ac\sin B$,

$$S_b = \frac{ac}{(a+b)(b+c)} S.$$

Analogously,

$$S_a := [AFE] = \frac{bc}{(a+b)(a+c)} S \quad \text{and} \quad S_c := [CDE] = \frac{ab}{(a+c)(b+c)} S.$$

From this,

$$\frac{[DEF]}{[ABC]} = \frac{S - S_a - S_b - S_c}{S}$$
$$= 1 - \sum_{cyc} \frac{ac}{(a+b)(b+c)} = \frac{2abc}{(a+b)(a+c)(b+c)}\,. \tag{1}$$

On the other hand, denoting $FD = e$, the law of cosines on $\triangle BDF$ gives

$$e^2 = \left(\frac{ac}{b+c}\right)^2 + \left(\frac{ac}{a+b}\right)^2 - 2\frac{a^2c^2}{(a+b)(b+c)} \cdot \cos B$$
$$= \frac{a^2c^2}{(a+b)^2(b+c)^2}\big((b+c)^2 + (a+b)^2 - 2(a+b)(b+c)\cos B\big)$$
$$= \frac{a^2c^2}{(a+b)^2(b+c)^2}\big(a^2 + c^2 - 2ac\cos B + 2b(a+b+c)(1-\cos B)\big)\,.$$

Now the law of cosines on $\triangle ABC$ gives $b^2 = a^2 + c^2 - 2ac\cos B$, so

$$e^2 = \frac{a^2c^2}{(a+b)^2(b+c)^2}\big(b^2 + 2b(a+b+c)(1-\cos B)\big)$$
$$= \frac{a^2c^2}{(a+b)^2(b+c)^2} \cdot b\left(b + 8p\sin^2\frac{B}{2}\right)$$
$$= \frac{abc}{(a+b)^2(b+c)^2}\big(abc + 8p(p-a)(p-c)\big),$$

where p is the semiperimeter of the triangle ABC and we have used the well known identity $\sin^2\frac{B}{2} = \frac{(p-a)(p-c)}{ac}$. Analogously, denoting $DE = f$ and $EF = d$, we have

$$f^2 = \frac{abc}{(a+c)^2(b+c)^2}\big(abc + 8p(p-a)(p-b)\big)$$

and

$$d^2 = \frac{abc}{(a+b)^2(a+c)^2}\big(abc + 8p(p-b)(p-c)\big).$$

From this,

$$d^2e^2f^2 = \frac{a^3b^3c^3}{(a+b)^4(a+c)^4(b+c)^4}\prod_{\text{cyc}}\big(abc + 8p(p-a)(p-b)\big). \tag{2}$$

By using the identities $[DEF] = \frac{def}{4R_1}$ and $[ABC] = pr$, we obtain

$$\frac{R_1}{4r} = \frac{defp}{16[DEF]\,[ABC]}$$

and the proposed inequality becomes

$$16[DEF]^2 \le defp.$$

Now, applying (1) and Heron's formula $[ABC]^2 = p(p-a)(p-b)(p-c)$ in the previous inequality, we have

$$\frac{64a^2b^2c^2(p-a)(p-b)(p-c)}{(a+b)^2(a+c)^2(b+c)^2} \le def,$$

which becomes

$$2^{12}abc(p-a)^2(p-b)^2(p-c)^2 \le \prod_{\text{cyc}}\big(abc+8p(p-a)(p-b)\big), \tag{3}$$

after using (2).

In order to prove this inequality (3), we will use the following estimation obtained from the AM-GM inequality :

$$\begin{aligned} abc+8p(p-a)(p-b) &= abc+4(a+b+c)(p-a)(p-b) \\ &= abc+4a(p-a)(p-b)+4b(p-a)(p-b)+4c(p-a)(p-b) \\ &\ge 4\sqrt[4]{4^3a^2b^2c^2(p-a)^3(p-b)^3}. \end{aligned}$$

In a similar way, we can prove that

$$abc+8p(p-a)(p-c) \ge 4\sqrt[4]{4^3a^2b^2c^2(p-a)^3(p-c)^3}$$

and

$$abc+8p(p-b)(p-c) \ge 4\sqrt[4]{4^3a^2b^2c^2(p-b)^3(p-c)^3}.$$

Multiplying the three inequalities, we obtain the estimation

$$\prod_{\text{cyc}}\big(abc+8p(p-a)(p-b)\big) \ge 2^{21/2}\big(abc(p-a)(p-b)(p-c)\big)^{3/2},$$

and to deduce (3) it is sufficient prove the inequality

$$2^{21/2}\big(abc(p-a)(p-b)(p-c)\big)^{3/2} \ge 2^{12}abc(p-a)^2(p-b)^2(p-c)^2,$$

or, equivalently,

$$abc \ge 8(p-a)(p-b)(p-c). \tag{4}$$

But $abc = 4R[ABC]$, where R denotes the circumradius of ABC, and

$$(p-a)(p-b)(p-c) = r[ABC],$$

and (4) becomes

$$R \ge 2r,$$

which is well known and true.

3980. *Proposed by S. Viswanathan.*

Let a, b, c be distinct real numbers. Prove that

$$\left(\sum_{cyc} \frac{a+b}{a-b}\right)\left(\prod_{cyc} \frac{a+b}{a-b}\right) < \frac{1}{3}.$$

We received seven correct solutions. We present the solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal.

Let us prove that

$$\left(\sum_{\text{cyc}} \frac{a+b}{a-b}\right)\left(\prod_{\text{cyc}} \frac{a+b}{a-b}\right) < \frac{1}{4}.$$

If $a = 0$, the previous inequality becomes

$$-\left(\frac{b+c}{b-c}\right)^2 < \frac{1}{4},$$

which is obviously true, and we can assume that $abc \neq 0$. With the change of variables

$$\alpha = \frac{a+b}{a-b}, \quad \beta = \frac{b+c}{b-c}, \quad \gamma = \frac{c+a}{c-a},$$

the inequality to be proved is equivalent to

$$\alpha\beta\gamma(\alpha+\beta+\gamma) < \frac{1}{4},$$

with the restriction $\alpha\beta + \beta\gamma + \gamma\alpha + 1 = 0$. The above inequality is of course true if $\alpha\beta\gamma = 0$, so we can assume without loss of generality that $\alpha\beta\gamma \neq 0$, and $\alpha\beta > 0$. From the restriction, we obtain

$$\alpha\beta\left(-1 - \alpha\beta + \left(\frac{1+\alpha\beta}{\alpha+\beta}\right)^2\right) < \frac{1}{4}.$$

Since $(\alpha+\beta)^2 \geq 4\alpha\beta$, setting $t = \alpha\beta$, it is enough to prove

$$\alpha\beta(1+\alpha\beta)\left(-1 + \frac{1+\alpha\beta}{4\alpha\beta}\right) < \frac{1}{4} \iff (1+t)(1-3t) < 1, t > 0,$$

which is obviously true.