OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2014 : 40(6), p. 240–241.

OC186. Show that if for non-negative integers m, n, N, k, the equation

$$(n^2+1)^{2^k} \cdot (44n^3+11n^2+10n+2) = N^m$$

holds, then necessarily $m = 1$.

Originally problem 1 of the 2013 Austria Federal Competition for Advanced Students.

We received two correct submissions. We present the solution by Oliver Geupel.

We consider the cases for even and odd numbers n in succession.

If $n = 2q$ is an even number, then $(n^2+1)^{2^k}$ is odd and

$$44n^3+11n^2+10n+2 = 2(22n^3+22q^2+10q+1) \equiv 2 \pmod 4.$$

Hence $N^m \equiv 2 \pmod 4$, that is, the multiplicity of the prime factor 2 in N^m is 1, so that $m = 1$.

It remains to consider n to be an odd number. Then n^2+1 is even. Since $n^2 \not\equiv -1 \pmod 4$, we have $n^2+1 \equiv 2 \pmod 4$. Thus, the multiplicity of the prime factor 2 in $(n^2+1)^{2^k}$ is 2^k. Moreover, the number

$$44n^3+11n^2+10n+2 = 2(22n^3+5n+1)+11n^2$$

is odd. Therefore the multiplicity of the prime factor 2 in N^m is 2^k. As a consequence, $m = 2^\ell$ with $\ell \le k$, so that the number $44n^3+11n^2+10n+2$ has the form r^{2^ℓ} with a rational number r. It is enough to prove that $\ell = 0$.

The proof is by contradiction. Suppose $\ell \ge 1$. We deduce that the odd number $44n^3+11n^2+10n+2$ is a perfect square, which implies that it is congruent to 1 (mod 4). On the other hand, we have

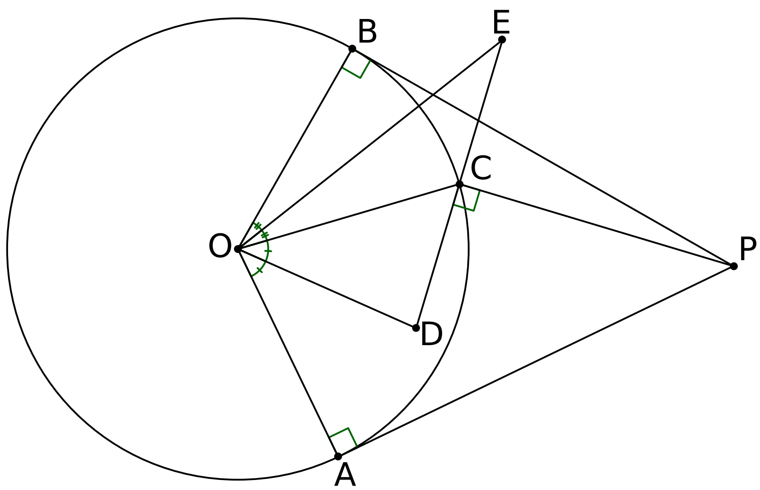
$$44n^3+11n^2+10n+2 \equiv -1 \pmod 4,$$

a contradiction. The proof is complete.

OC187. Let PA and PB be tangents to a circle centered at O and C a point on the minor arc AB. The perpendicular from C to PC intersects internal angle bisectors of $\angle AOC$ and $\angle BOC$ at D and E. Show that $CD = CE$.

Originally problem 6 of the 2013 China Western Mathematical Olympiad.

We received four correct submissions. We present the solution by Oliver Geupel.



Consider the problem in the plane of complex numbers where the circle is the unit circle and $a, b, c, \ldots$ are the coordinates of points $A, B, C, \ldots$. We have $\bar{a} = 1/a$, $\bar{b} = 1/b$, $\bar{c} = 1/c$. The intersection P of the tangents from A and B has coordinate $p = 2ab/(a+b)$. Hence $\bar{p} = 2/(a+b)$. Since d and e are on the angle bisectors, there are real numbers λ, μ such that $d = \lambda(a+c)$ and $e = \mu(b+c)$.

Since CD is perpendicular to CP, we obtain

$$\begin{aligned} 0 &= (c-d)\left(\bar{c}-\bar{p}\right) + \left(\bar{c}-\bar{d}\right)(c-p) = 2 - c\bar{p} - \bar{c}p - c\bar{d} - \bar{c}d + d\bar{p} + \bar{d}p \\ &= 2 - \frac{2c}{a+b} - \frac{2ab}{(a+b)c} - \lambda\left(\frac{a+c}{a} + \frac{a+c}{c} - \frac{2(a+c)}{a+b} - \frac{2(a+c)b}{(a+b)c}\right) \\ &= \frac{2(c-b)(a-c)}{(a+b)c} - \lambda\frac{(a+c)(a-b)(a-c)}{(a+b)ac}, \end{aligned}$$

whence

$$\lambda = \frac{2a(c-b)}{(a-b)(a+c)}.$$

Similarly,

$$\mu = \frac{2b(c-a)}{(b-a)(b+c)}.$$

We compute

$$d + e = \lambda(a+c) + \mu(b+c) = \frac{2a(c-b)}{a-b} + \frac{2b(c-a)}{b-a} = 2c,$$

that is $c = (d+e)/2$. Consequently, C is the midpoint of the line segment DE. The proof is complete.

OC188. Find the maximum value of M for which for all positive real numbers a, b, c, we have

$$a^3 + b^3 + c^3 - 3abc \geq M(ab^2 + bc^2 + ca^2 - 3abc).$$

Originally problem 2 from day 2 of the 2013 Turkey Mathematical Olympiad Second Round.

We received one correct submission and four incorrect submissions. We present the solution by Michel Bataille.

Editor's note. Some solutions correctly determined the maximum value M provided the inequality holds but failed to prove that when $M = 3 \cdot 2^{-2/3}$, then the inequality does indeed hold for all positive real numbers a, b and c. The solution below justifies this.

We show that the required maximal value is $\alpha = \dfrac{3\sqrt[3]{2}}{2}$.

First, suppose that M satisfies

$$a^3 + b^3 + c^3 - 3abc \geq M(ab^2 + bc^2 + ca^2 - 3abc)$$

for all $a, b, c > 0$. Taking the limit as $c \to 0$ on both sides, we must have $a^3 + b^3 \geq Mab^2$ for all $a, b > 0$. Taking $x = \frac{b}{a}$, it follows that $M \leq \phi(x)$ for all $x > 0$ where $\phi(x) = x + x^{-2}$. However, for any positive x,

$$\phi(x) = \frac{x}{2} + \frac{x}{2} + \frac{1}{x^2} \geq 3\sqrt[3]{\frac{x}{2} \cdot \frac{x}{2} \cdot \frac{1}{x^2}} = \frac{3}{\sqrt[3]{4}} = \alpha,$$

with equality if $\dfrac{x}{2} = \dfrac{1}{x^2}$, that is, $x = \sqrt[3]{2}$. In particular, $M \leq \phi(\sqrt[3]{2}) = \alpha$.

To complete the proof, we consider $a, b, c > 0$ and show that

$$a^3 + b^3 + c^3 - 3abc \geq \alpha(ab^2 + bc^2 + ca^2 - 3abc). \qquad (1)$$

Without loss of generality, we suppose that $c = \min(a, b, c)$. It is easily checked that

$$2(a^3 + b^3 + c^3 - 3abc) = (a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2),$$
$$2(ab^2 + bc^2 + ca^2 - 3abc) = a(b - c)^2 + b(c - a)^2 + c(a - b)^2 + (a - b)(b - c)(c - a).$$

As a result, the inequality (1) is equivalent to

$$(a + b - (\alpha - 1)c)(a - b)^2 + (b + c - (\alpha - 1)a)(b - c)^2 + (c + a - (\alpha - 1)b)(c - a)^2$$
$$\geq \alpha(a - b)(b - c)(c - a). \qquad (2)$$

Now, let $\mathcal{L}(a, b, c)$ be the left-hand side of (2) and $\mathcal{R}(a, b, c)$ be its right-hand side. If $a_1 = a - c$, $b_1 = b - c$, and $c_1 = 0$, then

$$a_1 - b_1 = a - b,\ b_1 - c_1 = b - c,\ c_1 - a_1 = c - a,$$

hence $\mathcal{R}(a_1, b_1, c_1) = \mathcal{R}(a, b, c)$. On the other hand, we observe that

$$\begin{aligned} a + b - (\alpha - 1)c - (a_1 + b_1 - (\alpha - 1)c_1) &= b + c - (\alpha - 1)a - (b_1 + c_1 - (\alpha - 1)a_1) \\ &= c + a - (\alpha - 1)b - (c_1 + a_1 - (\alpha - 1)b_1) \\ &= (3 - \alpha)c > 0 \end{aligned}$$

and so $\mathcal{L}(a, b, c) \geq \mathcal{L}(a_1, b_1, c_1)$. To conclude, it is sufficient that the inequality

$$\mathcal{L}(a_1, b_1, c_1) \geq \mathcal{R}(a_1, b_1, c_1)$$

holds. But, since $c_1 = 0$, the latter is equivalent to

$$a_1^3 + b_1^3 \geq \alpha a_1 b_1^2,$$

which holds if $a_1 = 0$ or $b_1 = 0$ and also if $a_1, b_1 > 0$ since $\alpha \leq \phi(\frac{b_1}{a_1})$, so we are done.

OC189. Let n, k be positive integers with $n \geq k$. There are n people, each person belongs to exactly one of group 1, group $2, \cdots$, group k and more than or equal to one person belong to any groups. Show that n^2 candies can be delivered to n persons in such way that all of the following conditions are satisfied.

1. At least one candy is delivered to each person.
2. The integers a_i are candies that are delivered to each person belonging to group i $(1 \leq i \leq k)$ such that if $1 \leq i < j \leq k$, then $a_i > a_j$.

Originally problem 1 of the 2013 Japan Mathematical Olympiad Finals.

We present the solution by Oliver Geupel. There were no other submissions.

Suppose that there are b_j persons in group j $(1 \leq j \leq k)$, so that

$$\sum_{j=1}^{k} b_j = n.$$

We shall show that the numbers

$$a_1 = n + \sum_{j=2}^{k} b_j, \qquad a_{i+1} = a_i - b_i - b_{i+1} \tag{1}$$

for $i \geq 2$ satisfy the conditions of the problem. We have for any $j \in \mathbb{N}$,

$$a_j = n + \sum_{i=j+1}^{k} b_i - \sum_{i=1}^{j-1} b_i$$

and

$$a_1 > a_2 > \cdots > a_k = b_k \geq 1.$$

Moreover,

$$\begin{aligned}\sum_{j=1}^{k} a_j b_j &= \sum_{j=1}^{k}\left(n+\sum_{i=j+1}^{k} b_i - \sum_{i=1}^{j-1} b_i\right) b_j \\ &= n\sum_{j=1}^{k} b_j + \sum_{j=1}^{k}\sum_{i=j+1}^{k} b_j b_i - \sum_{j=1}^{k}\sum_{i=1}^{j-1} b_i b_j \\ &= n^2 + \sum_{1\le j<i\le k} b_j b_i - \sum_{1\le i<j\le k} b_i b_j \\ &= n^2.\end{aligned}$$

Consequently, (1) is a solution of the problem.

OC190. Let a, m and n be positive integers with $m \le n$. Prove that if one of the numbers a^m and a^n is divisible by m, then the other number must also be divisible by n.

Originally problem 4 of the 2013 Philippines Mathematical Olympiad.

We present the solution by Oliver Geupel. There were no other submissions.

Editor's note. The above statement contains a typo. It should read

> Let a, m and n be positive integers with $m \le n$. Prove that if one of the numbers a^m and a^n is divisible by m, then the other number must also be divisible by m.

Thanks to Oliver Geupel for the catch!

By assumption we have either $m \mid a^m$ or $m \mid a^n$. If $m \mid a^m$, then $m \le n$ immediately implies $m \mid a^m \mid a^n$ and we are done. Now assume $m \mid a^n$. The number a^m has the same prime divisors as a^n, each with multiplicity at least m. Since the multiplicity of every prime divisor of m is not greater than m, we conclude $m \mid a^m$. This completes the proof.