Solutions to Inequality Problems from the Chinese Mathematical Olympiad

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In this article, we present solutions to the problems in *Inequality Problems from the Chinese Mathematical Olympiad* appearing in ***Crux*** 41(7), p. 299–302.

Problem 2005.

Let $a_1 = \frac{21}{16}$ and $2a_n - 3a_{n-1} = \dfrac{3}{2^{n+1}}$ for $n \geq 2$. Prove that for any integers $m \geq n \geq 2$,

$$\left(a_n + \frac{3}{2^{n+3}}\right)^{\frac{1}{m}} \left(m - \left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) < \frac{m^2-1}{m-n+1}.$$

Solution to Problem 2005.

The given recurrence relation may be rewritten as

$$2^n a_n + \frac{3}{8} = 3\left(2^{n-1}a_{n-1} + \frac{3}{8}\right).$$

Let $b_n = 2^n a_n + \frac{3}{8}$. Then $b_1 = 2a + \frac{3}{8} = 3$ and for $n \geq 2$, $b_n = 3b_{n-1}$. Iteration yields $b_n = 3^n$, so that $a_n = (\frac{3}{2})^n - \frac{3}{2^{n+3}}$. The desired inequality is equivalent to

$$\left(1 - \frac{n}{m+1}\right)\left(\frac{3}{2}\right)^{\frac{n}{m}} \left(m - \left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) < m-1.$$

Since $m \geq 2$, $(1+\frac{1}{m})^m \geq 1 + \binom{m}{1}\frac{1}{m} + \binom{m}{2}\frac{1}{m^2} \geq (\frac{3}{2})^2$. From the Arithmetic-Geometric Means Inequality,

$$\begin{aligned}
\left(1 - \frac{n}{m+1}\right)^m &= \left(1 - \frac{n}{m+1}\right)^m 1^{m(n-1)} \\
&< \left(\frac{m(1-\frac{n}{m+1}) + m(n-1)}{mn}\right)^{mn} \\
&= \left(\frac{m}{m+1}\right)^{mn} \\
&= \left(\frac{1}{(1+\frac{1}{m})^m}\right)^n \\
&< \left(\frac{2}{3}\right)^{2n}.
\end{aligned}$$

The desired inequality will follow from

$$\begin{aligned} m-1 &> \left(\frac{2}{3}\right)^{\frac{2n}{m}}\left(\frac{3}{2}\right)^{\frac{n}{m}}\left(m-\left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) \\ &= \left(\frac{2}{3}\right)^{\frac{n}{m}}\left(m-\left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) \\ &= t(m-t^{m-1}) \end{aligned}$$

where $t=(\frac{2}{3})^{\frac{n}{m}}<1$. This certainly holds since

$$mt-t^m-m+1=(t-1)(m-(t^{m-1}+t^{m-2}+\cdots+t+1))<0.$$

Problem 2006.

The real numbers $a_1,\ a_2,\ \ldots,\ a_n$ have sum 0. Prove that

$$\max_{1\le k\le n} a_k^2 \le \frac{n}{3}\sum_{i=1}^{n-1}(a_i-a_{i+1})^2.$$

Solution to Problem 2006.

For $1\le k\le n-1$, let $d_k=a_k-a_{k+1}$. For a specific k, we have

$$\begin{aligned} a_1 &= a_k+d_{k-1}+\cdots+d_2+d_1, \\ a_2 &= a_k+d_{k-1}+\cdots+d_2, \\ \cdots &= \cdots \\ a_{k-1} &= a_k+d_{k-1}, \\ a_k &= a_k, \\ a_{k+1} &= a_k-d_k, \\ \cdots &= \cdots \\ a_n &= a_k-d_k-d_{k-1}-\cdots-d_{n-1}. \end{aligned}$$

Summation yields

$$0=na_k-(n-1)d_k-(n-k-1)d_{k+1}-\cdots-d_{n-1}+(k-1)d_{k-1}+\cdots+2d_2+d_1.$$

By Cauchy's Inequality,

$$\begin{aligned} (na_k)^2=\left(\sum_{i=k}^{n-1}(n-i)d_i-\sum_{i=1}^{k-1}id_i\right)^2 &\le \left(\sum_{i=1}^{n-1}i^2\right)\left(\sum_{i=1}^{n-1}d_i^2\right) \\ &= \frac{n(n-1)(2n-1)}{6}\left(\sum_{i=1}^{n-1}d_i^2\right) \\ &\le \frac{n^3}{3}\left(\sum_{i=1}^{n-1}d_i^2\right). \end{aligned}$$

The desired result follows immediately.

Problem 2007.

Let $a,\ b$ and c be given complex numbers. Let $m = |a+b|$ and $n = |a-b|$. Prove that if $mn \neq 0$, then

$$\max\{|ac+b|, |bc+a|\} \geq \frac{mn}{\sqrt{m^2+n^2}}.$$

Solution to Problem 2007.

We have

$$\begin{aligned}\max\{|ac+b|, |bc+a|\} &\geq \frac{|b||ac+b| + |a||bc+a|}{|a|+|b|} \\ &\geq \frac{|b(ac+b) - a(bc+a)|}{|a|+|b|} \\ &\geq \frac{|b+a||b-a|}{\sqrt{2(|a|^2+|b|^2}} \\ &= \frac{mn}{\sqrt{m^2+n^2}}.\end{aligned}$$

Problem 2008.

For a positive integer n, let the real numbers $x_1 \leq \cdots \leq x_n$ and $y_1 \geq \cdots \geq y_n$ be such that $\sum_{i=1}^{n} ix_i \geq \sum_{i=1}^{n} iy_i$. Prove that for any real number λ,

$$\sum_{i=1}^{n} \lfloor i\lambda \rfloor x_i \geq \sum_{i=1}^{n} \lfloor i\lambda \rfloor y_i.$$

Solution to Problem 2008.

First note that we must have $x_n \geq y_n$. Otherwise, $y_1 \geq y_2 \geq \cdots \geq y_n > x_n \geq x_{n-1} \geq \cdots \geq x_1$, which contradicts $\sum_{i=1}^{n} ix_i \geq \sum_{i=1}^{n} iy_i$. We use mathematical induction on n. For $n = 1$, we have $x_1 \geq y_1$. Hence $\lfloor \lambda \rfloor x_1 \geq \lfloor \lambda \rfloor y_1$ for any positive real number λ. Suppose the result holds for some $n \geq 1$, so that there exists a real number λ with the desired properties. Consider the case $n+1$. We have $x_{n+1} \geq y_{n+1}$. For $1 \leq i \leq n$, define $a_i = x_i + x_{n+1}$ and $b_i = y_i + y_{n+1}$. Clearly, we have $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. Moreover,

$$\sum_{i-1}^{n} ia_i = \sum_{i=1}^{n} ix_i + \frac{n(n+1)}{2} x_{n+1} \geq \sum_{i=1}^{n} iy_i + \frac{n(n+1)}{2} y_{n+1} = \sum_{i=1}^{n} ib_i.$$

By the induction hypothesis, $\sum_{i=1}^{n}\lfloor i\lambda\rfloor a_i \geq \sum_{i=1}^{n}\lfloor i\lambda\rfloor b_i$. Now

$$\begin{aligned}\sum_{i=1}^{n+1}\lfloor i\lambda\rfloor x_i - \sum_{i=1}^{n}\lfloor i\lambda\rfloor a_i &= x_{n+1}\left(\lfloor (n+1)\lambda\rfloor - \sum_{i=1}^{n}\lfloor i\lambda\rfloor\right)\\ &= y_{n+1}\left(\lfloor (n+1)\lambda\rfloor - \sum_{i=1}^{n}\lfloor i\lambda\rfloor\right)\\ &= \sum_{i=1}^{n+1}\lfloor i\lambda\rfloor y_i - \sum_{i=1}^{n}\lfloor i\lambda\rfloor b_i.\end{aligned}$$

It follows that

$$\sum_{i=1}^{n+1}\lfloor i\lambda\rfloor x_i - \sum_{i=1}^{n+1}\lfloor i\lambda\rfloor y_i \geq \sum_{i=1}^{n}\lfloor i\lambda\rfloor a_i - \sum_{i=1}^{n}\lfloor i\lambda\rfloor b_i \geq 0.$$

Problem 2009.

The real numbers $a_1,\ a_2,\ \ldots,\ a_n$, $n \geq 3$, are such that

$$\min_{1\leq i<j\leq n} |a_i - a_j| = 1.$$

Determine in terms of n the minimum value of $\sum_{k=1}^{n}|a_k|^3$.

Solution to Problem 2009.

We may assume that $a_1 < a_2 < \cdots < a_n$. For $1 \leq k \leq n$,

$$\begin{aligned}|a_k| + |a_{n-k+1}| &\geq a_{n-k+1} - a_k\\ &= (a_{n-k+1} - a_{n-k}) + (a_{n-k} - a_{n-k-1}) + \cdots + (a_{k+1} - a_k)\\ &\geq |(n-k+1) - k|.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{k=1}^{n}|a_k|^3 &= \frac{1}{2}\sum_{k=1}^{n}(|a_k| + |a_{n-k+1}|)^3\\ &= \frac{1}{2}\sum_{k=1}^{n}(|a_k| + |a_{n-k+1}|)\left(\frac{1}{4}(|a_k| + |a_{n-k+1}|)^2 + \frac{3}{4}(|a_k| - |a_{n-k+1}|)^2\right)\\ &\geq \frac{1}{8}\sum_{k=1}^{n}(|a_k| + |a_{n-k+1}|)^3\\ &\geq \frac{1}{8}\sum_{k=1}^{n}|n-2k+1|^3\\ &= \frac{1}{8}((n-1)^3 + (n-3)^3 + \cdots + (-(n+1))^3).\end{aligned}$$

When n is odd,

$$\sum_{k=1}^{n}|a_k|^3 \geq \frac{1}{4}\left(2^3\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor} i^3\right) = \frac{(n^2-1)^2}{32}.$$

When n is even,

$$\sum_{k=1}^{n}|a_k|^3 = \frac{1}{4}\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}(2i-1)^3 = \frac{1}{4}\left(\sum_{j=1}^{n} j^3 - 2^3\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor} i^3\right) = \frac{n^2(n-2)^2}{32}.$$

The minimum value in each case can be attained by taking $a_i = i - \frac{n+1}{2}$ for $1 \leq i \leq k$.

Problem 2011.

Let $n \geq 4$ be an integer. Let a_k and b_k, $1 \leq k \leq n$ be non-negative real numbers such that $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k > 0$. Determine the maximum value of

$$\frac{\sum_{k=1}^{n} a_k(a_k+b_k)}{\sum_{k=1}^{n} b_k(a_k+b_k)}.$$

Solution to Problem 2011.

Since the given expression is homogeneous, we may take $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k = 1$. We may also assume that $b_1 \leq b_k$ for $2 \leq k \leq n$. Then

$$\begin{aligned}(n-1)\sum_{k=1}^{n} b_k^2 + (n-2)\sum_{k=1}^{n} a_k b_k &\geq (n-1)b_1^2 + (n-1)\sum_{k=2}^{n} b_k^2 + (n-2)b_1\sum_{k=1}^{n} a_k\\ &\geq (n-1)b_1^2 + \left(\sum_{k=2}^{n} b_k\right)^2 + (n-2)b_1\\ &= (n-1)b_1^2 + (1-b_1)^2 + (n-2)b_1\\ &= nb_1^2 + (n-4)b_1 + 1\\ &\geq 1\\ &= \sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} a_k^2.\end{aligned}$$

It follows that

$$\sum_{k=1}^{n} a_k(a_k+b_k) \leq (n-1)\sum_{k=1}^{n} b_k(a_k+b_k),$$

so that the maximum value of the given expression is $n-1$. This is attained when

$$a_1 = 1,\ a_2 = a_3 = \cdots = a_n = 0 = b_1 \ \text{ and } \ b_2 = b_3 = \cdots = b_n = \frac{1}{n-1}.$$