Crux Mathematicorum

VOLUME 41, NO. 8 October / Octobre 2015

Editorial Board

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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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EDITORIAL

The future is here yet again for ***Crux*** : our problem proposals and solution submissions will now be processed using an electronic online form called Cruxbox! Our new automated system will allow for more efficient and reliable process. For all submissions, visit

`https://publications.cms.math.ca/cruxbox/`

The system accepts various file formats, will inform you of the successful upload and will send a confirmation email once the problem gets filed into our system. For problem proposals, your confirmation email will also contain the proposal's tracking number whose status you can now check at

`https://publications.cms.math.ca/cruxbox/proposals/status/`

As we would like to fully move to this system as quickly as possible, we now discourage submissions by email (in fact, we will simply redirect you to the new system if you try to submit by email). So I urge you to familiarize yourself with Cruxbox and email me at `crux-editors@cms.math.ca` if you have any comments about it.

If Cruxbox is our future, then inequalities are our past, present and future. When I first took over this job, the previous editor Shawn Godin warned me about the backlog of inequalities he had acquired. I did not quite understand the issue (namely, why not just publish the accepted ones as they get accepted?) and then I saw the overwhelming number of inequality submissions we receive. Yes, inequalities are popular and I understand why: they are like little puzzles that allow us to figure out which tools from our problem-solving toolbox to use when, how, in which order, to what extent. But we receive a truly staggering number of inequalities; in fact, nowadays a vast majority of our proposals are inequalities. So let me remind you that variety is the spice of life and encourage you to spice up your submissions by stepping outside the inequality comfort zone. After all, I do not believe that anyone wants our Problems section to become the Inequalities section.

Kseniya Garaschuk

THE CONTEST CORNER

No. 38

John McLoughlin

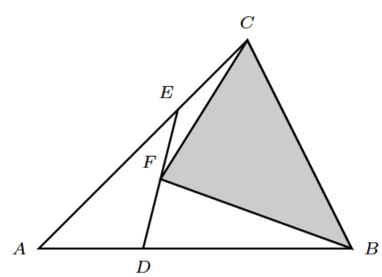
Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 octobre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

CC186. Soit n un entier positif. Faire le décompte des nombres $k \in \{0, 1, \ldots, n\}$ pour lesquels $\binom{n}{k}$ est impair. Démontrer que ce nombre est une puissance de deux, c'est-à-dire est de la forme 2^p pour un certain entier non négatif p.

CC187. Dans le diagramme, la surface du triangle ABC est 1, $\overline{AD} = \frac{1}{3}\overline{AB}$, $\overline{EC} = \frac{1}{3}\overline{AC}$ et $\overline{DF} = \overline{FE}$. Déterminer la surface du triangle en gris BFC.



CC188. Un plan divise l'espace en deux régions. Deux plans intersectant en une ligne divisent l'espace en quatre régions. Supposer maintenant que douze plans sont donnés dans l'espace tel que

a) deux d'entre eux intersectent toujours en une ligne,

b) trois d'entre eux intersectent toujours en un point, et

c) quatre d'entre eux n'ont jamais un point en commun.

Dans combien de régions l'espace est-il divisé ? Justifier votre réponse.

CC189. Des pièces de monnaie sont placées sur certains des 100 carrés d'une grille 10×10. Tout carré est voisin à un autre carré couvert d'une pièce de monnaie. Déterminer le nombre minimum de pièces de monnaie. (Deux carrés distincts sont voisins s'ils partagent un côté.)

CC190. Les lettres du mot TRIANGLE forment un arrangement tel qu'indiqué.

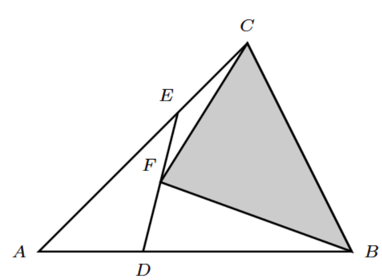
```
              E
            E L E
          E L G L E
        E L G N G L E
      E L G N A N G L E
    E L G N A I A N G L E
  E L G N A I R I A N G L E
E L G N A I R T R I A N G L E
```

Déterminer le nombre de façons d'épeler le mot TRIANGLE, utilisant des lettres adjacentes dans cet arrangement et allant vers le haut, la gauche ou la droite.

. . . .

CC186. Let n be a positive integer. Count the number of $k \in \{0, 1, \ldots, n\}$ for which $\binom{n}{k}$ is odd. Prove that this number is a power of two, i.e. it is of the form 2^p for some non-negative integer p.

CC187. In the diagram the area of the triangle ABC is 1, $\overline{AD} = \frac{1}{3}\overline{AB}$, $\overline{EC} = \frac{1}{3}\overline{AC}$ and $\overline{DF} = \overline{FE}$. Find the area of the shaded triangle BFC.



CC188. A plane divides space into two regions. Two planes that intersect in a line divide space into four regions. Now suppose that twelve planes are given in space so that three conditions are met :

a) every two of them intersect in a line,

b) every three of them intersect in a point, and

c) no four of them have a common point.

Into how many regions is space divided ? Justify your answer.

CC189. Coins are placed on some of the 100 squares in a 10×10 grid. Every square is next to another square with a coin. Find the minimum possible number of coins. (We say that two squares are next to each other when they share a common edge but are not equal.)

CC190. An arrangement of the letters from the word TRIANGLE is shown.

```
              E
            E L E
          E L G L E
        E L G N G L E
      E L G N A N G L E
    E L G N A I A N G L E
  E L G N A I R I A N G L E
E L G N A I R T R I A N G L E
```

Find the number of ways that the word TRIANGLE can be spelled out, using adjacent letters, going up or left or right, in this arrangement.

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2014 : 40(8), p. 319–320.

CC136. A spiderweb is a square 100×100 grid with knots at each intersection. The spider sits at one corner of his spiderweb ; there are 100 flies caught in the web with at most one fly per knot. Can the spider get all the flies in no more than 2000 moves, if in one move it crawls to an adjacent knot ?

Originally from Tournament of Towns Fall Round 2014, A-level, Juniors.

We received no solutions to this problem.

CC137. An Emperor invited 2015 wizards to a festival. Each wizard, but not the Emperor, knows which wizards are good and which ones are evil. A good wizard always tells the truth, while an evil wizard can either tell the truth or lie. At the festival, the Emperor gives every wizard a card with one "yes-or-no" question (questions might be different for different wizards), learns all the answers and then expels one wizard through a magic door which shows if this wizard is good or evil. Then the Emperor makes new cards and repeats the procedure with the remaining wizards until he wants to stop (with or without expelling a wizard). Prove that the Emperor can devise his questions so that all the evil wizards are expelled while expelling at most one good wizard.

Originally from Tournament of Towns, 2015, Spring, A-level, Senior.

We received and present one solution by Josh Zukewich and others (unnamed).

This solution is in two parts. In the first part (Part A) we find a living good wizard, expelling at most one good wizard in the process. In the second (Part B), we use this fact to expel all remaining bad wizards.

Part A : Pick a wizard at random (called Jim). Ask every wizard : "Is Jim good ?" Jim's answer does not matter for our analysis. For the remaining wizards' answers, consider the two exhaustive possibilities :

(1) Every non-Jim wizard says 'no.'

(2) Some non-Jim wizard says 'yes.'

Our actions are as follows for each possibility :

(1) Expel Jim. Jim goes through the door and he is either :

  (1.1) Good. Everyone else lied and is therefore bad. Expel everyone else in turn. Then we are done.

  (1.2) Bad. You expelled a bad wizard. Return to A.

(2) Expel a wizard (called Jane) that said 'yes.' When Jane goes through the door, she is either :

(2.1) Good. Then you have identified that Jim is good. You are done step A.

(2.2) Bad. You have expelled a bad wizard. Return to A.

In every iteration of step A we either expel a bad wizard, and repeat, or expel a good wizard and identify a good wizard still in the group.

Part B : Arrange all the wizards in a line facing forward, with Jim at the back. Ask every wizard : "Is the wizard directly ahead of you good ?" If Jim says

(1) 'No,' expel the wizard in front of Jim. You have expelled a bad wizard. Return to B.

(2) 'Yes,' the wizard in front of Jim is also good. Keep moving your attention ahead until a wizard says 'no.' The first wizard in front of Jim that says 'no' identifies the wizard in front of them as bad. Expel that bad wizard and return to B.

Goodness is transitive : Each wizard that says 'yes' moving ahead from Jim confirms the goodness of another wizard. Once the second-to-front wizard in the line says 'yes,' you have confirmed that all the remaining wizards are good, and you are done. You choose not to expel a wizard once this condition is met.

CC138. Prove that the integer

$$\sum_{i=1}^{2^{n-1}} (2i-1)^{2i-1} = 1^1 + 3^3 + 5^5 + \cdots + (2^n - 1)^{2^n - 1}$$

is a multiple of 2^n but not a multiple of 2^{n+1}.

Originally from Tournament of Towns, Fall Round 2011, A-level, Senior.

No solutions were received. We present a hint to encourage readers.

Consider the problem modulo 2^{n+2}. Specifically, prove that $k^{2^n} \equiv 1 \pmod{2^{n+2}}$ for any odd positive integer k.

CC139. It is well-known that if in a quadrilateral the circumcircle and the incircle have the same centre, then the quadrilateral is a square. Is the similar statement true in 3 dimensions ? Namely, if a cuboid is inscribed into a sphere and circumscribed around a sphere and the centres of these spheres coincide, does it imply that the cuboid is a cube ? (A cuboid is a polyhedron with 6 quadrilateral faces such that each vertex belongs to 3 edges.)

Originally from Tournament of Towns, Spring Round 2015, A-level, Seniors.

We received one correct solution. We present the solution of the Missouri State University Problem Solving Group.

The answer is no.

Consider the polyhedron whose vertices are

$$A = (a^2; 1; a), \quad B = (a^2; -1; a), \quad C = (-a^2; -1; a), \quad D = (-a^2; 1; a),$$
$$E = (1; a^2; -a), \quad F = (1; -a^2; -a), \quad G = (-1; -a^2; -a), \quad H = (-1; a^2; -a).$$

These clearly lie on the sphere of radius $\sqrt{a^4 + a^2 + 1}$ centered at the origin.

The rectangular faces $ABCD$ and $EFGH$ are tangent to the sphere of radius a centered at the origin. The other four faces, $ABFE$, $BCGF$, $CDHG$, and $DAEH$ are congruent trapezoids lying in the planes

$$\begin{aligned} 2ax - (a^2 - 1)z &= a^3 + a, \\ -2ay + (a^2 - 1)z &= a^3 + a, \\ 2ax + (a^2 - 1)z &= -a^3 - a, \\ 2ay + (a^2 - 1)z &= a^3 + a, \end{aligned}$$

respectively.

We claim that the sphere of radius a centered at the origin is tangent to these planes at the points

$$\left(\frac{2a2}{a2+1}, 0, \frac{a - a^3}{a^2+1}\right), \quad \left(0, -\frac{2a^2}{a^2+1}, \frac{a^3 - a}{a^2+1}\right),$$
$$\left(-\frac{2a^2}{a2+1}, 0, \frac{a - a^3}{a^2+1}\right), \quad \left(0, \frac{2a^2}{a^2+1}, \frac{a^3 - a}{a^2+1}\right)$$

respectively. To see this, note that each point lies on its corresponding plane and its position vector has length a and is parallel to the normal vector of that plane. This cuboid satisfies the conditions of the problem, but is not a cube unless $a = 1$.

CC140. Let $P(x)$ be a polynomial with real coefficients so that the equation $P(m) + P(n) = 0$ has infinitely many pairs of integer solutions (m, n). Prove that the graph of $y = P(x)$ has a centre of symmetry.

Originally from Tournament of Towns, Fall Round 2008, A-level, Seniors.

We received no solutions to this problem.

THE OLYMPIAD CORNER

No. 336

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 octobre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC246. Déterminer tous les entiers positifs x, y and z tels que $x^3 = 3^y 7^z + 8$.

OC247. Soit $a_1 \leq a_2 \leq \cdots$ une suite non décroissante d'entiers positifs. Un entier positif n est dit *bon* s'il existe i tel que $n = \dfrac{i}{a_i}$. Démontrer que si 2013 est bon, 20 l'est aussi.

OC248. Soit B et C deux points spécifiques sur un cercle centré à O, mais pas sur un même diamètre. Soit A qui bouge sur le cercle tout en étant distinct de B et C et n'appartenant pas à la bissectrice perpendiculaire de BC. Soit H l'orthocentre de $\triangle ABC$ et soit M et N les mi points des segments BC et AH respectivement. La ligne AM intersecte le cercle de nouveau à D ; enfin, NM et OC intersectent au point P. Déterminer le locus des points P au fur et à mesure que A bouge sur le cercle.

OC249. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ vérifiant

$$f(xf(x) + f(x)f(y) + y - 1) = f(xf(x) + xy) + y - 1.$$

OC250. Alberte et Bertrand s'amusent à un jeu de nombres. Commençant avec un entier positif n, ils prennent leur tour à modifier ce nombre, Alberte allant première. Chaque joueur change le nombre courant k à $k - 1$ ou $\lceil k/2 \rceil$. La personne qui change le nombre à 0 ou 1 gagne. Déterminer n où Alberte a une stratégie gagnante.

. . . .

OC246. Find all positive integers x, y and z such that $x^3 = 3^y 7^z + 8$.

OC247. Let $a_1 \leq a_2 \leq \cdots$ be a non-decreasing sequence of positive integers. A positive integer n is called *good* if there is an index i such that $n = \dfrac{i}{a_i}$. Prove that if 2013 is good, then so is 20.

OC248. Let B and C be two fixed points on a circle centered at O that are not diametrically opposite. Let A be a variable point on the circle distinct from B and C and not belonging to the perpendicular bisector of BC. Let H be the orthocenter of $\triangle ABC$, and M and N be the midpoints of the segments BC and AH, respectively. The line AM intersects the circle again at D, and finally, NM and OD intersect at P. Determine the locus of points P as A moves around the circle.

OC249. Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfies the following

$$f(xf(x) + f(x)f(y) + y - 1) = f(xf(x) + xy) + y - 1.$$

OC250. Alice and Bob play a number game. Starting with a positive integer n they take turns changing the number with Alice going first. Each player may change the current number k to either $k - 1$ or $\lceil k/2 \rceil$. The person who changes 1 to 0 wins. Determine all n such that Alice has a winning strategy.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2014 : 40(6), p. 240–241.

OC186. Show that if for non-negative integers m, n, N, k, the equation

$$(n^2+1)^{2^k} \cdot (44n^3+11n^2+10n+2) = N^m$$

holds, then necessarily $m = 1$.

Originally problem 1 of the 2013 Austria Federal Competition for Advanced Students.

We received two correct submissions. We present the solution by Oliver Geupel.

We consider the cases for even and odd numbers n in succession.

If $n = 2q$ is an even number, then $(n^2+1)^{2^k}$ is odd and

$$44n^3+11n^2+10n+2 = 2(22n^3+22q^2+10q+1) \equiv 2 \pmod 4.$$

Hence $N^m \equiv 2 \pmod 4$, that is, the multiplicity of the prime factor 2 in N^m is 1, so that $m = 1$.

It remains to consider n to be an odd number. Then n^2+1 is even. Since $n^2 \not\equiv -1 \pmod 4$, we have $n^2+1 \equiv 2 \pmod 4$. Thus, the multiplicity of the prime factor 2 in $(n^2+1)^{2^k}$ is 2^k. Moreover, the number

$$44n^3+11n^2+10n+2 = 2(22n^3+5n+1)+11n^2$$

is odd. Therefore the multiplicity of the prime factor 2 in N^m is 2^k. As a consequence, $m = 2^\ell$ with $\ell \leq k$, so that the number $44n^3+11n^2+10n+2$ has the form r^{2^ℓ} with a rational number r. It is enough to prove that $\ell = 0$.

The proof is by contradiction. Suppose $\ell \geq 1$. We deduce that the odd number $44n^3+11n^2+10n+2$ is a perfect square, which implies that it is congruent to 1 (mod 4). On the other hand, we have

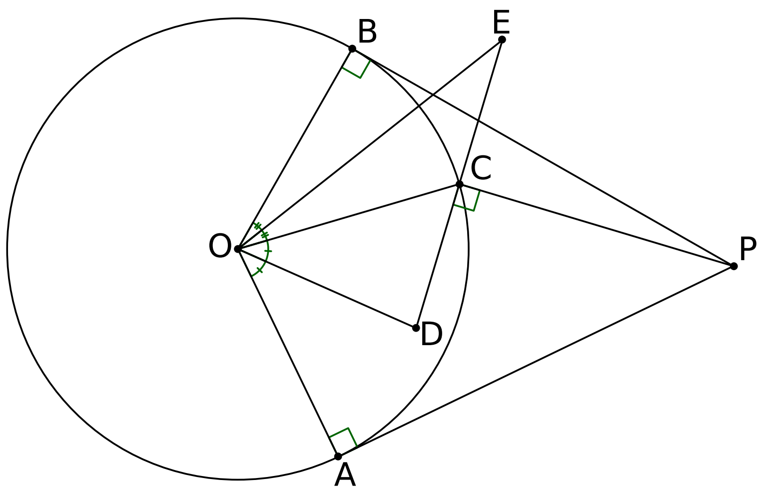
$$44n^3+11n^2+10n+2 \equiv -1 \pmod 4,$$

a contradiction. The proof is complete.

OC187. Let PA and PB be tangents to a circle centered at O and C a point on the minor arc AB. The perpendicular from C to PC intersects internal angle bisectors of $\angle AOC$ and $\angle BOC$ at D and E. Show that $CD = CE$.

Originally problem 6 of the 2013 China Western Mathematical Olympiad.

We received four correct submissions. We present the solution by Oliver Geupel.



Consider the problem in the plane of complex numbers where the circle is the unit circle and $a, b, c, \ldots$ are the coordinates of points $A, B, C, \ldots$. We have $\bar{a} = 1/a$, $\bar{b} = 1/b$, $\bar{c} = 1/c$. The intersection P of the tangents from A and B has coordinate $p = 2ab/(a+b)$. Hence $\bar{p} = 2/(a+b)$. Since d and e are on the angle bisectors, there are real numbers λ, μ such that $d = \lambda(a+c)$ and $e = \mu(b+c)$.

Since CD is perpendicular to CP, we obtain

$$\begin{aligned}
0 &= (c-d)(\bar{c}-\bar{p}) + \left(\bar{c}-\bar{d}\right)(c-p) = 2 - c\bar{p} - \bar{c}p - c\bar{d} - \bar{c}d + d\bar{p} + \bar{d}p \\
&= 2 - \frac{2c}{a+b} - \frac{2ab}{(a+b)c} - \lambda\left(\frac{a+c}{a} + \frac{a+c}{c} - \frac{2(a+c)}{a+b} - \frac{2(a+c)b}{(a+b)c}\right) \\
&= \frac{2(c-b)(a-c)}{(a+b)c} - \lambda\frac{(a+c)(a-b)(a-c)}{(a+b)ac},
\end{aligned}$$

whence

$$\lambda = \frac{2a(c-b)}{(a-b)(a+c)}.$$

Similarly,

$$\mu = \frac{2b(c-a)}{(b-a)(b+c)}.$$

We compute

$$d + e = \lambda(a+c) + \mu(b+c) = \frac{2a(c-b)}{a-b} + \frac{2b(c-a)}{b-a} = 2c,$$

that is $c = (d+e)/2$. Consequently, C is the midpoint of the line segment DE. The proof is complete.

OC188. Find the maximum value of M for which for all positive real numbers a, b, c, we have

$$a^3 + b^3 + c^3 - 3abc \geq M(ab^2 + bc^2 + ca^2 - 3abc).$$

Originally problem 2 from day 2 of the 2013 Turkey Mathematical Olympiad Second Round.

We received one correct submission and four incorrect submissions. We present the solution by Michel Bataille.

Editor's note. Some solutions correctly determined the maximum value M provided the inequality holds but failed to prove that when $M = 3 \cdot 2^{-2/3}$, then the inequality does indeed hold for all positive real numbers a, b and c. The solution below justifies this.

We show that the required maximal value is $\alpha = \dfrac{3\sqrt[3]{2}}{2}$.

First, suppose that M satisfies

$$a^3 + b^3 + c^3 - 3abc \geq M(ab^2 + bc^2 + ca^2 - 3abc)$$

for all $a, b, c > 0$. Taking the limit as $c \to 0$ on both sides, we must have $a^3 + b^3 \geq Mab^2$ for all $a, b > 0$. Taking $x = \frac{b}{a}$, it follows that $M \leq \phi(x)$ for all $x > 0$ where $\phi(x) = x + x^{-2}$. However, for any positive x,

$$\phi(x) = \frac{x}{2} + \frac{x}{2} + \frac{1}{x^2} \geq 3\sqrt[3]{\frac{x}{2} \cdot \frac{x}{2} \cdot \frac{1}{x^2}} = \frac{3}{\sqrt[3]{4}} = \alpha,$$

with equality if $\dfrac{x}{2} = \dfrac{1}{x^2}$, that is, $x = \sqrt[3]{2}$. In particular, $M \leq \phi(\sqrt[3]{2}) = \alpha$.

To complete the proof, we consider $a, b, c > 0$ and show that

$$a^3 + b^3 + c^3 - 3abc \geq \alpha(ab^2 + bc^2 + ca^2 - 3abc). \qquad (1)$$

Without loss of generality, we suppose that $c = \min(a, b, c)$. It is easily checked that

$$2(a^3 + b^3 + c^3 - 3abc) = (a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2),$$
$$2(ab^2 + bc^2 + ca^2 - 3abc) = a(b - c)^2 + b(c - a)^2 + c(a - b)^2 + (a - b)(b - c)(c - a).$$

As a result, the inequality (1) is equivalent to

$$(a + b - (\alpha - 1)c)(a - b)^2 + (b + c - (\alpha - 1)a)(b - c)^2 + (c + a - (\alpha - 1)b)(c - a)^2$$
$$\geq \alpha(a - b)(b - c)(c - a). \qquad (2)$$

Now, let $\mathcal{L}(a, b, c)$ be the left-hand side of (2) and $\mathcal{R}(a, b, c)$ be its right-hand side. If $a_1 = a - c$, $b_1 = b - c$, and $c_1 = 0$, then

$$a_1 - b_1 = a - b,\ b_1 - c_1 = b - c,\ c_1 - a_1 = c - a,$$

hence $\mathcal{R}(a_1, b_1, c_1) = \mathcal{R}(a, b, c)$. On the other hand, we observe that

$$\begin{aligned} a+b-(\alpha-1)c-(a_1+b_1-(\alpha-1)c_1) &= b+c-(\alpha-1)a-(b_1+c_1-(\alpha-1)a_1) \\ &= c+a-(\alpha-1)b-(c_1+a_1-(\alpha-1)b_1) \\ &= (3-\alpha)c > 0 \end{aligned}$$

and so $\mathcal{L}(a, b, c) \geq \mathcal{L}(a_1, b_1, c_1)$. To conclude, it is sufficient that the inequality

$$\mathcal{L}(a_1, b_1, c_1) \geq \mathcal{R}(a_1, b_1, c_1)$$

holds. But, since $c_1 = 0$, the latter is equivalent to

$$a_1^3 + b_1^3 \geq \alpha a_1 b_1^2,$$

which holds if $a_1 = 0$ or $b_1 = 0$ and also if $a_1, b_1 > 0$ since $\alpha \leq \phi(\frac{b_1}{a_1})$, so we are done.

OC189. Let n, k be positive integers with $n \geq k$. There are n people, each person belongs to exactly one of group 1, group $2, \cdots$, group k and more than or equal to one person belong to any groups. Show that n^2 candies can be delivered to n persons in such way that all of the following conditions are satisfied.

1. At least one candy is delivered to each person.
2. The integers a_i are candies that are delivered to each person belonging to group i $(1 \leq i \leq k)$ such that if $1 \leq i < j \leq k$, then $a_i > a_j$.

Originally problem 1 of the 2013 Japan Mathematical Olympiad Finals.

We present the solution by Oliver Geupel. There were no other submissions.

Suppose that there are b_j persons in group j $(1 \leq j \leq k)$, so that

$$\sum_{j=1}^{k} b_j = n.$$

We shall show that the numbers

$$a_1 = n + \sum_{j=2}^{k} b_j, \qquad a_{i+1} = a_i - b_i - b_{i+1} \tag{1}$$

for $i \geq 2$ satisfy the conditions of the problem. We have for any $j \in \mathbb{N}$,

$$a_j = n + \sum_{i=j+1}^{k} b_i - \sum_{i=1}^{j-1} b_i$$

and

$$a_1 > a_2 > \cdots > a_k = b_k \geq 1.$$

Moreover,

$$\begin{aligned}\sum_{j=1}^{k} a_j b_j &= \sum_{j=1}^{k}\left(n + \sum_{i=j+1}^{k} b_i - \sum_{i=1}^{j-1} b_i\right) b_j \\ &= n\sum_{j=1}^{k} b_j + \sum_{j=1}^{k}\sum_{i=j+1}^{k} b_j b_i - \sum_{j=1}^{k}\sum_{i=1}^{j-1} b_i b_j \\ &= n^2 + \sum_{1\le j<i\le k} b_j b_i - \sum_{1\le i<j\le k} b_i b_j \\ &= n^2.\end{aligned}$$

Consequently, (1) is a solution of the problem.

OC190. Let a, m and n be positive integers with $m \le n$. Prove that if one of the numbers a^m and a^n is divisible by m, then the other number must also be divisible by n.

Originally problem 4 of the 2013 Philippines Mathematical Olympiad.

We present the solution by Oliver Geupel. There were no other submissions.

Editor's note. The above statement contains a typo. It should read

> Let a, m and n be positive integers with $m \le n$. Prove that if one of the numbers a^m and a^n is divisible by m, then the other number must also be divisible by m.

Thanks to Oliver Geupel for the catch!

By assumption we have either $m \mid a^m$ or $m \mid a^n$. If $m \mid a^m$, then $m \le n$ immediately implies $m \mid a^m \mid a^n$ and we are done. Now assume $m \mid a^n$. The number a^m has the same prime divisors as a^n, each with multiplicity at least m. Since the multiplicity of every prime divisor of m is not greater than m, we conclude $m \mid a^m$. This completes the proof.

BOOK REVIEWS

Robert Bilinski

The Ellipse : A Historical and Mathematical Journey by Arthur Mazer
ISBN 978-0-470-58718-8, hardcover, 303 pages
Published by Wiley, 2010

Reviewed by ***Robert Bilinski****, Collège Montmorency.*

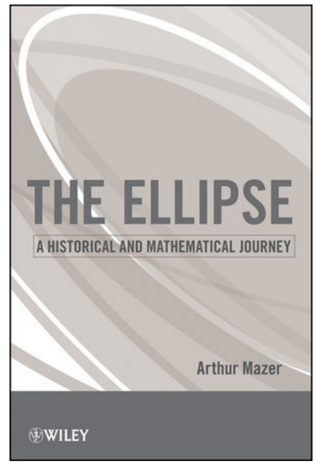
Arthur Mazer started his career at Utah State University as an assistant professor and then transferred to the energy industry. He currently works in industry as a manager of the Quantitative Analytics Department at Southern California Edison. He has written three books : a manuscript on power planning, and two mathematical history books that came out roughly at the same time. The first is the subject of this book review and the second is titled *Quest to understand the motion of the universe*. In the preface, we see Mazer's motivation :

> There is of course the opposite argument [...] that is, mathematics and science are the drivers of historical evolution and in Darwinian fashion, philosophies and political entities that promote scientific excellence flourish, while those that do not fade away. This latter argument provides the perspective for this book.

This book seemed to be a bit of a paradox to me. In fact, that is what attracted me to it. The title implies that the book is specialized ; yet, it is published by Wiley as a general interest book. After reading the book, it becomes clear that the book is not about the ellipse. Rather, it uses the ellipse as its main thread to explore the world of mathematics. The chapters are organized roughly in chronological order as the author explores the evolution of thought in society and the interactions between the society and mathematical ideas. Mazer gives witty titles to his chapters, such as "Chapter 6 : The Slayer Calculus" or "Chapter 7 : Eight minutes that changed history".

The first two chapters focus mainly on history with a bit of a primer on numbers and modular arithmetic that is explored through cycles on graphs. The following five chapters start off with a short introduction, which is then followed by about 50 pages of math. Each of these chapters is an exploration of small math morsels that range in complexity and subject. The link to the ellipse is not always clear at the beginning, but all the threads are neatly tied together into a tapestry about the ellipse by the end of the book.

It may make for some intense reading. I will not go into detail for each chapter, but to give an idea, I will provide a synopsis of "Chapter 4 : The language Algebra". We start off with Cartesian coordinates, then continue to translations, polynomials, conics and the quadratic formula, the circle and the four dimensional sphere, Cavalieri's theorem, Pyramids, higher dimensions, areas and volumes of spheres, finite series, induction, linear algebra, transformations, determinants, and the ellipse. No spoilers !



The book is well written. It is abundantly illustrated and, to the author's credit, not superfluously. With the hustle and bustle of teaching and the mass of information presented, I am pretty sure I missed a thing or two and a second re-read is a must. A minus for me was that the knowledge about the ellipse exposed in the book ends around the 18th century without any mention of more recent results. It makes it a side project for me to look into in the coming months, unless a ***Crux*** reader sends me useful references to more recent results and more technical manuscripts about the ellipse. As an avid ***Crux*** reader, I also hoped the book would make me a better problem solver, but there are no posed problems to be found.

The range of mathematics presented in the book is far-reaching, but at an introductory level. It passes through some number theory, classical geometry, stereometry, calculus, combinatorics, analysis, differential geometry, projective geometry, vector calculus, etc. It could serve as a great run through or "transversal" review of a student's mathematical studies. If teacher colleges across the country have taken the same path as Quebec's, this book should be a mandatory read for all new secondary level teachers as it would widen their mathematical horizons greatly, yet at a manageable pace. It could be used in a high school honours math class or for a math camp. If you have read a lot of general interest math books, it depends more on you and what you are looking for in a book. Personally, I am happy to have it on my bookshelf as a reference book and I will definitely try to find the time to re-read it when I am less pressed for time. Happy reading!

Solutions to Inequality Problems from the Chinese Mathematical Olympiad

Huawei Zhu
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In this article, we present solutions to the problems in *Inequality Problems from the Chinese Mathematical Olympiad* appearing in ***Crux*** 41(7), p. 299–302.

Problem 2005.

Let $a_1 = \frac{21}{16}$ and $2a_n - 3a_{n-1} = \dfrac{3}{2^{n+1}}$ for $n \geq 2$. Prove that for any integers $m \geq n \geq 2$,

$$\left(a_n + \frac{3}{2^{n+3}}\right)^{\frac{1}{m}} \left(m - \left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) < \frac{m^2-1}{m-n+1}.$$

Solution to Problem 2005.

The given recurrence relation may be rewritten as

$$2^n a_n + \frac{3}{8} = 3\left(2^{n-1}a_{n-1} + \frac{3}{8}\right).$$

Let $b_n = 2^n a_n + \frac{3}{8}$. Then $b_1 = 2a + \frac{3}{8} = 3$ and for $n \geq 2$, $b_n = 3b_{n-1}$. Iteration yields $b_n = 3^n$, so that $a_n = (\frac{3}{2})^n - \frac{3}{2^{n+3}}$. The desired inequality is equivalent to

$$\left(1 - \frac{n}{m+1}\right)\left(\frac{3}{2}\right)^{\frac{n}{m}} \left(m - \left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) < m-1.$$

Since $m \geq 2$, $(1+\frac{1}{m})^m \geq 1 + \binom{m}{1}\frac{1}{m} + \binom{m}{2}\frac{1}{m^2} \geq (\frac{3}{2})^2$. From the Arithmetic-Geometric Means Inequality,

$$\begin{aligned}
\left(1 - \frac{n}{m+1}\right)^m &= \left(1 - \frac{n}{m+1}\right)^m 1^{m(n-1)} \\
&< \left(\frac{m(1-\frac{n}{m+1}) + m(n-1)}{mn}\right)^{mn} \\
&= \left(\frac{m}{m+1}\right)^{mn} \\
&= \left(\frac{1}{(1+\frac{1}{m})^m}\right)^n \\
&< \left(\frac{2}{3}\right)^{2n}.
\end{aligned}$$

The desired inequality will follow from

$$\begin{aligned} m-1 &> \left(\frac{2}{3}\right)^{\frac{2n}{m}}\left(\frac{3}{2}\right)^{\frac{n}{m}}\left(m-\left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) \\ &= \left(\frac{2}{3}\right)^{\frac{n}{m}}\left(m-\left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) \\ &= t(m-t^{m-1}) \end{aligned}$$

where $t=(\frac{2}{3})^{\frac{n}{m}}<1$. This certainly holds since

$$mt-t^m-m+1=(t-1)(m-(t^{m-1}+t^{m-2}+\cdots+t+1))<0.$$

Problem 2006.

The real numbers $a_1,\ a_2,\ \ldots,\ a_n$ have sum 0. Prove that

$$\max_{1\le k\le n} a_k^2 \le \frac{n}{3}\sum_{i=1}^{n-1}(a_i-a_{i+1})^2.$$

Solution to Problem 2006.

For $1\le k\le n-1$, let $d_k=a_k-a_{k+1}$. For a specific k, we have

$$\begin{aligned} a_1 &= a_k+d_{k-1}+\cdots+d_2+d_1, \\ a_2 &= a_k+d_{k-1}+\cdots+d_2, \\ \cdots &= \cdots \\ a_{k-1} &= a_k+d_{k-1}, \\ a_k &= a_k, \\ a_{k+1} &= a_k-d_k, \\ \cdots &= \cdots \\ a_n &= a_k-d_k-d_{k-1}-\cdots-d_{n-1}. \end{aligned}$$

Summation yields

$$0=na_k-(n-1)d_k-(n-k-1)d_{k+1}-\cdots-d_{n-1}+(k-1)d_{k-1}+\cdots+2d_2+d_1.$$

By Cauchy's Inequality,

$$\begin{aligned} (na_k)^2=\left(\sum_{i=k}^{n-1}(n-i)d_i-\sum_{i=1}^{k-1}id_i\right)^2 &\le \left(\sum_{i=1}^{n-1}i^2\right)\left(\sum_{i=1}^{n-1}d_i^2\right) \\ &= \frac{n(n-1)(2n-1)}{6}\left(\sum_{i=1}^{n-1}d_i^2\right) \\ &\le \frac{n^3}{3}\left(\sum_{i=1}^{n-1}d_i^2\right). \end{aligned}$$

The desired result follows immediately.

Problem 2007.

Let $a,\ b$ and c be given complex numbers. Let $m = |a+b|$ and $n = |a-b|$. Prove that if $mn \neq 0$, then

$$\max\{|ac+b|, |bc+a|\} \geq \frac{mn}{\sqrt{m^2+n^2}}.$$

Solution to Problem 2007.

We have

$$\begin{aligned}\max\{|ac+b|, |bc+a|\} &\geq \frac{|b||ac+b|+|a||bc+a|}{|a|+|b|} \\ &\geq \frac{|b(ac+b)-a(bc+a)|}{|a|+|b|} \\ &\geq \frac{|b+a||b-a|}{\sqrt{2(|a|^2+|b|^2}} \\ &= \frac{mn}{\sqrt{m^2+n^2}}.\end{aligned}$$

Problem 2008.

For a positive integer n, let the real numbers $x_1 \leq \cdots \leq x_n$ and $y_1 \geq \cdots \geq y_n$ be such that $\sum_{i=1}^{n} ix_i \geq \sum_{i=1}^{n} iy_i$. Prove that for any real number λ,

$$\sum_{i=1}^{n} \lfloor i\lambda \rfloor x_i \geq \sum_{i=1}^{n} \lfloor i\lambda \rfloor y_i.$$

Solution to Problem 2008.

First note that we must have $x_n \geq y_n$. Otherwise, $y_1 \geq y_2 \geq \cdots \geq y_n > x_n \geq x_{n-1} \geq \cdots \geq x_1$, which contradicts $\sum_{i=1}^{n} ix_i \geq \sum_{i=1}^{n} iy_i$. We use mathematical induction on n. For $n = 1$, we have $x_1 \geq y_1$. Hence $\lfloor \lambda \rfloor x_1 \geq \lfloor \lambda \rfloor y_1$ for any positive real number λ. Suppose the result holds for some $n \geq 1$, so that there exists a real number λ with the desired properties. Consider the case $n+1$. We have $x_{n+1} \geq y_{n+1}$. For $1 \leq i \leq n$, define $a_i = x_i + x_{n+1}$ and $b_i = y_i + y_{n+1}$. Clearly, we have $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. Moreover,

$$\sum_{i-1}^{n} ia_i = \sum_{i=1}^{n} ix_i + \frac{n(n+1)}{2} x_{n+1} \geq \sum_{i=1}^{n} iy_i + \frac{n(n+1)}{2} y_{n+1} = \sum_{i=1}^{n} ib_i.$$

By the induction hypothesis, $\sum_{i=1}^{n} \lfloor i\lambda \rfloor a_i \geq \sum_{i=1}^{n} \lfloor i\lambda \rfloor b_i$. Now

$$\begin{aligned}\sum_{i=1}^{n+1} \lfloor i\lambda \rfloor x_i - \sum_{i=1}^{n} \lfloor i\lambda \rfloor a_i &= x_{n+1}\left(\lfloor (n+1)\lambda \rfloor - \sum_{i=1}^{n} \lfloor i\lambda \rfloor\right) \\ &= y_{n+1}\left(\lfloor (n+1)\lambda \rfloor - \sum_{i=1}^{n} \lfloor i\lambda \rfloor\right) \\ &= \sum_{i=1}^{n+1} \lfloor i\lambda \rfloor y_i - \sum_{i=1}^{n} \lfloor i\lambda \rfloor b_i.\end{aligned}$$

It follows that

$$\sum_{i=1}^{n+1} \lfloor i\lambda \rfloor x_i - \sum_{i=1}^{n+1} \lfloor i\lambda \rfloor y_i \geq \sum_{i=1}^{n} \lfloor i\lambda \rfloor a_i - \sum_{i=1}^{n} \lfloor i\lambda \rfloor b_i \geq 0.$$

Problem 2009.

The real numbers $a_1,\ a_2,\ \ldots,\ a_n$, $n \geq 3$, are such that

$$\min_{1 \leq i < j \leq n} |a_i - a_j| = 1.$$

Determine in terms of n the minimum value of $\sum_{k=1}^{n} |a_k|^3$.

Solution to Problem 2009.

We may assume that $a_1 < a_2 < \cdots < a_n$. For $1 \leq k \leq n$,

$$\begin{aligned}|a_k| + |a_{n-k+1}| &\geq a_{n-k+1} - a_k \\ &= (a_{n-k+1} - a_{n-k}) + (a_{n-k} - a_{n-k-1}) + \cdots + (a_{k+1} - a_k) \\ &\geq |(n-k+1) - k|.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{k=1}^{n} |a_k|^3 &= \frac{1}{2}\sum_{k=1}^{n} (|a_k| + |a_{n-k+1}|)^3 \\ &= \frac{1}{2}\sum_{k=1}^{n} (|a_k| + |a_{n-k+1}|)\left(\frac{1}{4}(|a_k| + |a_{n-k+1}|)^2 + \frac{3}{4}(|a_k| - |a_{n-k+1}|)^2\right) \\ &\geq \frac{1}{8}\sum_{k=1}^{n} (|a_k| + |a_{n-k+1}|)^3 \\ &\geq \frac{1}{8}\sum_{k=1}^{n} |n-2k+1|^3 \\ &= \frac{1}{8}((n-1)^3 + (n-3)^3 + \cdots + (-(n+1))^3).\end{aligned}$$

When n is odd,

$$\sum_{k=1}^{n} |a_k|^3 \geq \frac{1}{4}\left(2^3 \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} i^3\right) = \frac{(n^2-1)^2}{32}.$$

When n is even,

$$\sum_{k=1}^{n} |a_k|^3 = \frac{1}{4}\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}(2i-1)^3 = \frac{1}{4}\left(\sum_{j=1}^{n} j^3 - 2^3 \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} i^3\right) = \frac{n^2(n-2)^2}{32}.$$

The minimum value in each case can be attained by taking $a_i = i - \frac{n+1}{2}$ for $1 \leq i \leq k$.

Problem 2011.

Let $n \geq 4$ be an integer. Let a_k and b_k, $1 \leq k \leq n$ be non-negative real numbers such that $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k > 0$. Determine the maximum value of

$$\frac{\sum_{k=1}^{n} a_k(a_k+b_k)}{\sum_{k=1}^{n} b_k(a_k+b_k)}.$$

Solution to Problem 2011.

Since the given expression is homogeneous, we may take $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k = 1$. We may also assume that $b_1 \leq b_k$ for $2 \leq k \leq n$. Then

$$\begin{aligned}(n-1)\sum_{k=1}^{n} b_k^2 + (n-2)\sum_{k=1}^{n} a_k b_k &\geq (n-1)b_1^2 + (n-1)\sum_{k=2}^{n} b_k^2 + (n-2)b_1 \sum_{k=1}^{n} a_k \\ &\geq (n-1)b_1^2 + \left(\sum_{k=2}^{n} b_k\right)^2 + (n-2)b_1 \\ &= (n-1)b_1^2 + (1-b_1)^2 + (n-2)b_1 \\ &= nb_1^2 + (n-4)b_1 + 1 \\ &\geq 1 \\ &= \sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} a_k^2.\end{aligned}$$

It follows that

$$\sum_{k=1}^{n} a_k(a_k+b_k) \leq (n-1)\sum_{k=1}^{n} b_k(a_k+b_k),$$

so that the maximum value of the given expression is $n-1$. This is attained when

$$a_1 = 1,\ a_2 = a_3 = \cdots = a_n = 0 = b_1 \ \text{ and } \ b_2 = b_3 = \cdots = b_n = \frac{1}{n-1}.$$

Angle Bisectors in a Triangle

I. F. Sharygin

In this article, we have collected some geometric facts which are directly or tangentially related to the angle bisectors in a triangle. These results vary from easy lemmas to serious theorems, but we will not classify them; rather, we will just number them. Every statement that occurs without a proof is considered as an exercise for the reader. In fact, even the presented proofs are rather concise, which allows the reader to fill in the details.

Everyone should know this

Let us first recall some standard notation : let ABC be the given triangle, S_{ABC} is its area, $|BC| = a$, $|CA| = b$, $|AB| = c$, $2p = a + b + c$, O and R are the circumcentre and the circumradius, I and r are the incentre and the inradius. Furthermore, the triangle has three excircles, each of which touches one side of a triangle and the extensions of the other two sides. Their centres and their radii will be denoted by I_a, I_b, I_c, r_a, r_b, r_c (I_a denotes the centre of the excircle touching the side BC and extensions of the sides AB and AC with r_a being its radius). Further notation will be presented as needed.

1. Suppose the internal bisector of $\angle A$ intersects the side BC at a point A_1. Then

$$\frac{|BA_1|}{|A_1C|} = \frac{|BA|}{|AC|} = \frac{c}{b}.$$

2. Suppose the external angle bisector of $\angle A$ intersects the line BC at the point A_2. Then

$$\frac{|BA_2|}{|A_2C|} = \frac{|BA|}{|AC|} = \frac{c}{b}.$$

3. $S_{ABC} = pr$.

4. $S_{ABC} = (p - a)r_a$.

5. Let M be the point of tangency of the incircle with the side AB. Then $|AM| = p - a$.

6. Let M be the point of tangency of the excircle with the centre I_a and the line AB. Then $|AM| = p$.

7. Points B and C lie on the circle with diameter II_a and the centre of that circle lies on a circumcircle (see Figure 1.)

Therefore, the centre I of the incircle has the following property : the lines AI, BI and CI (that is, the angle bisectors of the triangle) go through the centres of the circumcircles of triangles BIC, CIA and AIB, respectively. The converse is true as well, namely :

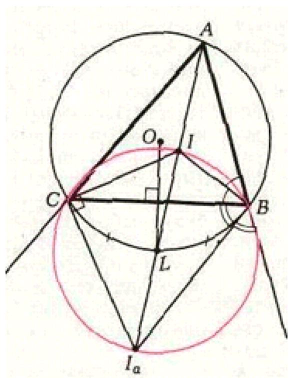


FIGURE 1: See Problem 7.

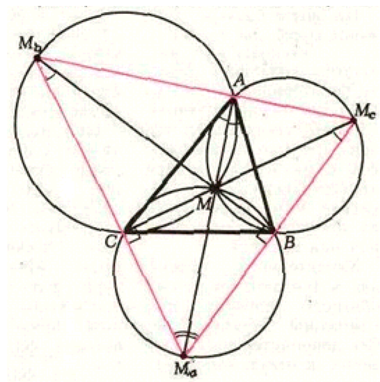


FIGURE 2: See Problem 8.

8. If the lines AM, BM and CM go through the centres of the circumcircles of triangles BMC, CMA and AMB, then M is the centre of the incircle of ABC.

Indeed, let M_a, M_b and M_c be the points of intersection (different from M) of the lines AM, BM and CM with the corresponding circles (see Figure 2). Then MM_a, MM_b and MM_c are diameters of those circles ; therefore, M_aA, M_bB and M_cC are altitudes of the triangle $M_aM_bM_c$. This implies that $\angle BAM = \angle BM_cM = 90^\circ - \angle BM_aC = \angle CM_bM = \angle CAM$, which means that M lies on the angle bisector of angle A and, analogously, on the angle bisectors of angles B and C.

Distances between centres of special circles

9. $|OI|^2 = R^2 - 2Rr$ (Euler's formula).

10. $|OI_a|^2 = R^2 + 2Rr_a$.

11. $|II_a|^2 = 4R(r_a - r)$.

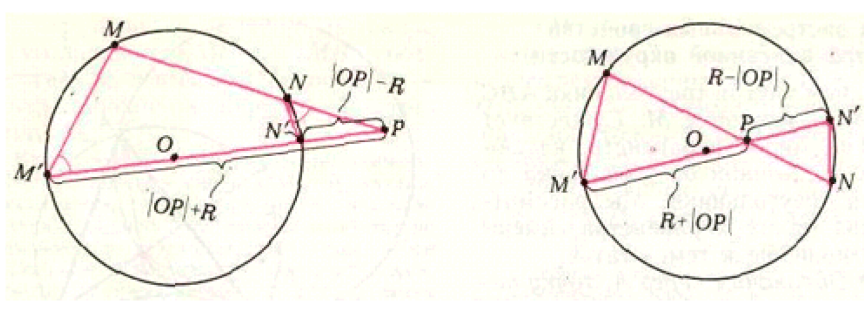


FIGURE 3: See Problems 9 and 10.

For the proof of **9** and **10**, recall that if M and N are points of intersection of a line passing through an arbitrary point P with the circle of radius R and centre O, then $|PM| \cdot |PN| = |R^2 - |OP|^2|$; this follows from the similarity of triangles

PMM' and PNN', where M' and N' are points of intersection of the line OP with the circle (see Figure 3). It implies that $R^2 - |OI|^2 = |IA| \cdot |IL|$, where L is the point of intersection of the angle bisector of angle A and the circumcircle (see Figure 4). But $|IA| = r/\sin(\angle A/2)$ and, by **7**, $|IL| = |LB| = 2R \times \sin(\angle A/2)$, so $R^2 - |OI|^2 = 2Rr$. Analogously,

$$|OI_a|^2 - R^2 = |I_aL| \cdot |I_aA| = 2R \times \sin(\angle A/2) \times \frac{r_a}{\sin(\angle A/2)} = 2Rr_a.$$

Finally,

$$|II_a|^2 = 2|IL| \cdot (|I_aA| - |IA|) = 4R\sin(\angle A/2) \cdot \frac{r_a - r}{\sin(\angle A/2)} = 4R(r_a - r).$$

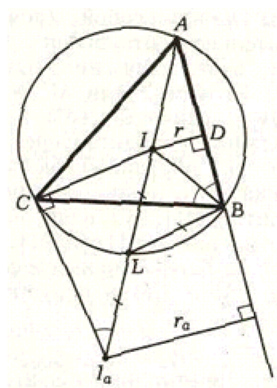


FIGURE 4: See Problems 9 and 10.

12. Consider the points symmetric to the centres of the excircles with respect to the centre of the circumcircle. These points lie on the circle of radius $2R$ with the centre I.

Two extremal properties of the centre of the incircle

Consider an arbitrary point M inside the triangle ABC. There are many inequalities concerning the distances between M and the vertices of the triangle. We will consider two such inequalities.

13. Let A_1 be the point of intersection of the line AM and the circumcircle. Then

$$\frac{|BM| \cdot |CM|}{|A_1M|} \geq 2r$$

and equality holds if M coincides with I.

Suppose that the smallest value of $f(M) = \frac{|BM|\cdot|CM|}{|A_1M|}$ is achieved when M is some point inside ABC. We will show that $M = I$. Then, since $f(I) = 2r$ (this follows, for example, from similar triangles BID and I_aIC in Figure 4), it would imply that *if* $f(M)$ *achieves its minimum inside* ABC, then $f(M) \geq 2r$. The italicized statement is far from trivial and should be carefully proven.

Construct the circumcircle of triangle AMC (see Figure 5). Consider triangles CMA_1 formed by moving the point M along the arc AC — they are all similar (why ?) and hence the ratio $|CM|/|A_1M|$ is constant for all of them. Therefore, if the minimum of $f(M)$ is achieved at M, then the line BM must go through the circumcentre of the triangle AMC (otherwise, we could reduce $|BM|$ while keeping $|CM|/|A_1M|$ constant). Now, let B_1 and C_1 be the points of intersection of lines BM and CM with the circumcircle of ABC. Then, as we saw in the proof of **9**, we have $|MA| \cdot |MA_1| = |MB| \cdot |MB_1| = |MC| \cdot |MC_1|$ and hence

$$\frac{|BM| \cdot |CM|}{|A_1M|} = \frac{|CM| \cdot |AM|}{|B_1M|} = \frac{|AM| \cdot |BM|}{|C_1M|}.$$

Therefore, lines AM and CM must also pass through the circumcentres of triangles BMC and AMB respectively. Then M is the circumcentre of ABC (by **8**).

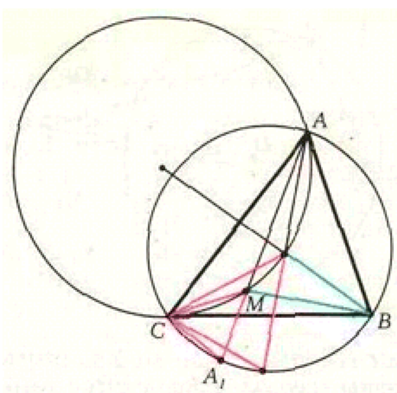


FIGURE 5: See Problem 13.

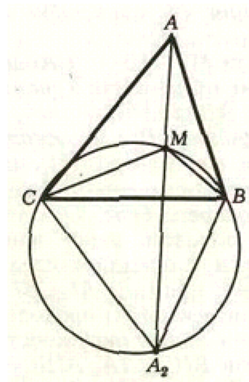


FIGURE 6: See Problem 14.

One must always be careful when using indirect proofs like the one above (where we did not directly prove that $f(M) \geq f(I)$ for all points M inside ABC) since a function does not always achieve its minimum and maximum.

14. Show that

$$|AM| \sin \angle BMC + |BM| \sin \angle CMA + |CM| \sin \angle AMB \leq p,$$

and equality holds if M coincides with I.

The proof of this statement will also be indirect : we will show that the point M where the left-hand side achieves its maximum (if it exists !) coincides with I.

Construct the circumcircle of triangle BMC and extend the line AM until the second point of intersection A_2 (see Figure 6). Apply Ptolemy's theorem to the quadrangle $BMCA_2$ to get :

$$|BM| \cdot |A_2C| + |CM| \cdot |A_2B| = |BC| \cdot |A_2M|.$$

Since the lengths of chords of a circle are proportional to the sines of angles subtended by those chords, we have

$$|BM|\sin\angle A_2MC + |CM|\sin\angle A_2MB = |A_2M|\sin\angle BMC$$

or

$$|BM|\sin\angle AMC + |CM|\sin\angle AMB = |A_2M|\sin\angle BMC.$$

Comparing the last equation to **14**, we see that the left side of the inequality equals $|AA_2|\sin\angle BMC$. Therefore, the line AM must go through the circumcentre of BMC since otherwise we can increase the value of the left-hand side of **14** by moving M along the arc BC. The rest of the proof is similar to that of **13**.

We leave it to the reader to prove that if $M = I$, then $|AA_2|\sin\angle BMC = p$. To see that, you can use **6** and **7** and the fact that $\angle BIC = 90^\circ + \angle A/2$.

When intuition fails

When two similar elements of a triangle are equal (such as two angles or two medians), it seems natural to expect the triangle to be isosceles. Among the problems of this type, one of the hardest to prove is the Steiner-Lehmus theorem.

15. If a triangle has two angle bisectors of equal lengths, then it is isosceles.

This problem is well-known, whereas the following amusing variation is not usually familiar even to geometry buffs.

16. Suppose a triangle ABC has angle bisectors AA_1, BB_1 and CC_1. If the triangle $A_1B_1C_1$ is isosceles, is ABC isosceles as well ?

Experiments with the graphic software suggest that the answer is no ; $\triangle A_1B_1C_1$ can be isosceles while $\triangle ABC$ is not. We do not know of any brief, elegant construction of a counterexample.

And for now, some more problems.

17. Prove that the angle bisector in a triangle bisects the angle between the circumradius and the altitude from the same vertex.

18. Let AA_1 be the angle bisector of $\angle A$ in a triangle ABC. Show that

$$|AA_1| = \sqrt{bc - |BA_1|\cdot|CA_1|} = \frac{2bc\cos(\angle A/2)}{b+c}.$$

19. Suppose a triangle ABC has angle bisectors AA_1, BB_1 and CC_1. Show that the altitudes of ABC are angle bisectors of $A_1B_1C_1$.

20. Let M and N be the projections of the point of intersection of the altitudes of ABC onto the internal and external angle bisector of angle A. Show that the line MN divides the side BC in half.

21. Let S be the sum of the areas of the three triangles whose vertices are the points where an excircle touches the sides (or their extensions) of the given triangle

ABC. Let T be the area of the triangle formed by the points where the incircle touches the sides of ABC. Prove that $S = S_{ABC} + T$.

22. Suppose a triangle ABC has angle bisectors AA_1, BB_1 and CC_1 ; let L and K be the points of intersection of the lines AA_1 with B_1C_1 and CC_1 with A_1B_1, respectively. Show that BB_1 bisects angle LBK.

23. Let M and N be the midpoints of the diagonals AC and BC of a cyclic quadrilateral $ABCD$. Prove that if BC bisects angle ANC, then AC bisects angle BMD.

24. In a triangle ABC, let M be the point of intersection of the angle bisector of angle B with the line passing through the midpoint of AC and the midpoint of the altitude from the vertex B. Let N be the midpoint of the angle bisector of angle B. Show that the angle bisector of angle C also bisects angle MCN.

25. Suppose a triangle ABC has angle bisectors AA_1, BB_1 and CC_1, and construct the circle O through the points A_1, B_1 and C_1. Consider the three chords of O formed by the segments of the sides of ABC lying inside O. Prove that the length of one of these chords is equal to the sum of the other two.

26. In a triangle ABC, let K and L be points on the sides AB and BC, respectively, such that $|AK| = |KL| = |LC|$. Draw the line parallel to the angle bisector of angle B through the point of intersection of the lines AL and CK. Let M be the point of intersection of this line with the line AB. Show that $|AM| = |BC|$.

27. Let $ABCD$ be a cyclic quadrilateral. Let K be the point of intersection of the extensions of the sides AB and CD ; let L be the point of intersection of the extensions of the sides BC and AD. Show that the bisectors of the angles BKC and BLA are perpendicular and intersect on the line connecting the midpoints of AC and BD.

. . . .

This article appeared in Russian in Kvant, 1983(8), p. 32–36. It has been translated and adapted with permission.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 octobre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

4071. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that if $a, b, c \in (0, 1)$, then $a^{a+1}b^{b+1}c^{c+1} < e^{2(a+b+c)-6}$.

4072. *Proposed by Michel Bataille.*

Let a, b be distinct positive real numbers and $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $L = \frac{a-b}{\ln a - \ln b}$. Prove that

$$\frac{L}{G} > \frac{4A + 5G}{A + 8G}.$$

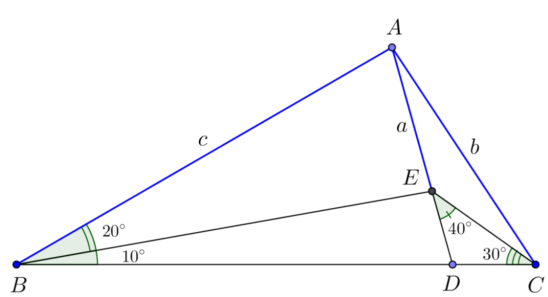
4073. *Proposed by Daniel Sitaru.*

Solve the following system :

$$\begin{cases} \sin 2x + \cos 3y = -1, \\ \sqrt{\sin^2 x + \sin^2 y} + \sqrt{\cos^2 x + \cos^2 y} = 1 + \sin(x + y). \end{cases}$$

4074. *Proposed by Abdilkadir Altinaş.*

Consider the triangle ABC with the following measures :



Show that $a + b = c$; that is, $|AE| + |AC| = |AB|$.

4075. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that in any triangle ABC with $BC = a$, $CA = b$, $AB = c$ the following inequality holds :

$$\sqrt[3]{abc}\cdot\sqrt{a^2+b^2+c^2}\geq 4[ABC],$$

where $[ABC]$ is the area of triangle ABC.

4076. *Proposed by Mehtaab Sawhney.*

Prove that

$$(x^2+y^2+z^2)^3\geq (x^3+y^3+z^3+3(\sqrt{3}-1)xyz)^2$$

for all nonnegative reals x, y, and z.

4077. *Proposed by George Apostolopoulos.*

Let ABC be a triangle. Prove that

$$\sin\frac{A}{2}\cdot\sin B\cdot\sin C+\sin A\cdot\sin\frac{B}{2}\cdot\sin C+\sin A\cdot\sin B\cdot\sin\frac{C}{2}\leq\frac{9}{8}.$$

4078. *Proposed by Michel Bataille.*

Given θ such that $\frac{\pi}{3}\leq\theta\leq\frac{5\pi}{3}$, let M_0 be a point of a circle with centre O and radius R and M_k its image under the counterclockwise rotation with centre O and angle $k\theta$. If M is the point diametrically opposite to M_0 and n is a positive integer, show that

$$\sum_{k=0}^{n}MM_k\geq(2n+1)\cdot\frac{R}{2}.$$

4079. *Proposed by Mihaela Berindeanu.*

Let $x,y,z>0$ and $x+y+z=2016$. Prove that :

$$x\sqrt{\frac{yz}{y+2015z}}+y\sqrt{\frac{xz}{z+2015x}}+z\sqrt{\frac{xy}{x+2015z}}\leq\frac{2016}{\sqrt{3}}.$$

4080. *Proposed by Alina Sîntămărian and Ovidiu Furdui.*

Let $a,b\in\mathbb{R}$, with $ab>0$. Calculate

$$\int_0^\infty x^2\mathrm{e}^{-\left(ax-\frac{b}{x}\right)^2}\mathrm{d}x.$$

. . . .

4071. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soit $a, b, c \in (0,1)$. Démontrer que

$$a^{a+1}b^{b+1}c^{c+1} < e^{2(a+b+c)-6}.$$

4072. *Proposé par Michel Bataille.*

Soit a et b des réels strictement positifs distincts et soit $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ et $L = \frac{a-b}{\ln a - \ln b}$. Démontrer que

$$\frac{L}{G} > \frac{4A+5G}{A+8G}.$$

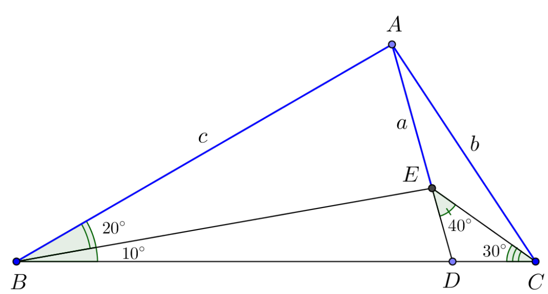
4073. *Proposé par Daniel Sitaru.*

Résoudre ce système d'équations :

$$\begin{cases} \sin 2x + \cos 3y = -1, \\ \sqrt{\sin^2 x + \sin^2 y} + \sqrt{\cos^2 x + \cos^2 y} = 1 + \sin(x+y). \end{cases}$$

4074. *Proposé par Abdilkadir Altinaş.*

On considère le triangle ABC dans la figure suivante :



Démontrer que $a + b = c$.

4075. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soit un triangle ABC avec $BC = a$, $CA = b$ et $AB = c$. Démontrer que

$$\sqrt[3]{abc} \cdot \sqrt{a^2+b^2+c^2} \geq 4[ABC],$$

$[ABC]$ étant l'aire du triangle ABC.

4076. *Proposé par Mehtaab Sawhney.*

Démontrer que pour tous réels non négatifs x, y et z,

$$(x^2+y^2+z^2)^3 \geq (x^3+y^3+z^3+3(\sqrt{3}-1)xyz)^2.$$

4077. *Proposé par George Apostolopoulos.*

Étant donné un triangle ABC, démontrer que

$$\sin\frac{A}{2}\cdot\sin B\cdot\sin C+\sin A\cdot\sin\frac{B}{2}\cdot\sin C+\sin A\cdot\sin B\cdot\sin\frac{C}{2}\le\frac{9}{8}.$$

4078. *Proposé par Michel Bataille.*

Soit θ tel que $\frac{\pi}{3}\le\theta\le\frac{5\pi}{3}$, soit M_0 un point sur un cercle de centre O et de rayon R et soit M_k son image par une rotation de centre O et d'angle $k\theta$ dans le sens contraire des aiguilles d'une montre . Sachant que M est le point diamétralement opposé à M_0 et que n est un entier strictement positif, démontrer que

$$\sum_{k=0}^{n} MM_k\ge(2n+1)\cdot\frac{R}{2}.$$

4079. *Proposé par Mihaela Berindeanu.*

Soit x, y et z des réels strictement positifs tels que $x+y+z=2016$. Démontrer que

$$x\sqrt{\frac{yz}{y+2015z}}+y\sqrt{\frac{xz}{z+2015x}}+z\sqrt{\frac{xy}{x+2015z}}\le\frac{2016}{\sqrt{3}}.$$

4080. *Proposé par Alina Sîntămărian et Ovidiu Furdui.*

Soit a et b des réels tels que $ab>0$. Déterminer la valeur de

$$\int_0^{\infty}x^2\mathrm{e}^{-\left(ax-\frac{b}{x}\right)^2}\mathrm{d}x.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014 : 40(8), p. 345–348.

3971. *Proposed by Marcel Chiriţă.*

Let x, y, z be positive real numbers such that $4(x+y+z)^2 = 9(x^2+y^2+z^2)$. Show that

$$4-\sqrt{15} \le \frac{x}{z} \le 4+\sqrt{15}.$$

We received 21 submissions of which 20 were correct and complete. We present the solution by Roy Barbara.

By the hypothesis we have

$$8(xy+yz+zx) = 5(x^2+y^2+z^2). \qquad (1)$$

Set $k = \frac{x}{z}$. Replacing x by kz in (1) and rearranging yields :

$$5y^2 - 8z(k+1)y + z^2(5k^2-8k+5) = 0. \qquad (2)$$

Looking at (2) as a quadratic equation in y (with root y a real number), we see that its discriminant must be non-negative, that is,

$$z^2[16(k+1)^2 - 5(5k^2-8k+5)] \ge 0.$$

Hence, since $z^2 > 0$, $16(k+1)^2 - 5(5k^2-8k+5) \ge 0$, which simplifies to

$$k^2 - 8k + 1 \le 0. \qquad (3)$$

Inequality (3) shows that k must lie between the two zeros of $k^2-8k+1=0$, that is,

$$4-\sqrt{15} \le \frac{x}{z} \le 4+\sqrt{15}.$$

3972. *Proposed by Michel Bataille.*

Find all functions $f : (0,\infty) \to (0,\infty)$ such that

$$f\left(xf\left(\frac{1}{y}\right)\right) = xf\left(\frac{1}{x+y}\right)$$

for all positive real numbers x, y.

We received five solutions, of which three were correct and complete. We present two solutions.

Solution 1, by Roy Barbara.

The function $f(x) = \frac{x}{x+1}$ is a unique solution. Indeed, let f be any solution and let y be any positive real number. Set

$$x = \sqrt{\frac{y^2}{4} + f\left(\frac{1}{y}\right)^{-1}} - \frac{y}{2}. \qquad (1)$$

Then, $x > 0$ and $x + yx - f\left(\frac{1}{y}\right)^{-1} = 0$, whence

$$xf\left(\frac{1}{y}\right) = \frac{1}{x+y}.$$

Hence,

$$f\left(xf\left(\frac{1}{y}\right)\right) = f\left(\frac{1}{x+y}\right).$$

Since $f\left(xf\left(\frac{1}{y}\right)\right) = xf\left(\frac{1}{x+y}\right)$ by hypothesis, we conclude that $x = 1$ and therefore by (1)

$$\sqrt{\frac{y^2}{4} + f\left(\frac{1}{y}\right)^{-1}} = \frac{y}{2} + 1.$$

Squaring yields $f\left(\frac{1}{y}\right)^{-1} = y + 1$, that is, $f\left(\frac{1}{y}\right) = \frac{1}{y+1}$. Since $y > 0$ is arbitrary, replacing y by $\frac{1}{y}$ yields $f(y) = \frac{y}{y+1}$.

Solution 2, by Mohammed Aassila.

Let

$$(x, y) = \left(\frac{-1 + \sqrt{1 + \frac{4t^2}{f(t)}}}{2t}, \frac{1}{t}\right),$$

where $t > 0$. Clearly, $x > 0$ and $y > 0$. Then we obtain

$$1 = \frac{-1 + \sqrt{1 + \frac{4t^2}{f(t)}}}{2t}.$$

Thus, $f(t) = \frac{t}{t+1}$ and it is easy to check that it satisfies the given functional equation.

Editor's Comments. We don't know anything about the injectivity of f, so it cannot be used to solve the problem! If f were injective, the problem would be trivial because if we set $x = 1$, then we get

$$f\left(f\left(\frac{1}{y}\right)\right) = f\left(\frac{1}{1+y}\right) \implies f\left(\frac{1}{y}\right) = \frac{1}{1+y},$$

which yields $f(y) = \frac{y}{y+1}$.

3973. *Proposed by Dragoljub Milošević.*

Let m_a, m_b, m_c be the medians, r_a, r_b, r_c the exradii, R the circumradius and r the inradius of a triangle. Prove that

$$\frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c} \leq \frac{2R}{r} - 1.$$

We received four correct solutions. We present the one by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, expanded slightly by the editor.

Let p_a, p_b, and p_c be the distances from the circumcentre O of the triangle to the sides BC, CA, and AB, respectively. Then it is known (see page 73 of O. Bottema et al., *Geometric inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969) that

$$m_a \leq R + p_a, \quad m_b \leq R + p_b, \quad m_c \leq R + p_c. \tag{1}$$

Now let K be the foot of the perpendicular from O to BC. Then

$$p_a = R\cos(\angle BOK) = R\cos\left(\frac{1}{2}\angle BOC\right) = R\cos A,$$

$$\text{and similarly, } \; p_b = R\cos B \text{ and } \; p_c = R\cos C. \tag{2}$$

From (1) and (2) we have

$$m_a \leq R(a + \cos A), \quad m_b \leq R(1 + \cos B), \quad m_c \leq R(1 + \cos C). \tag{3}$$

It is well known that

$$rs = (s-a)r_a = (s-b)r_b = (s-c)r_c, \tag{4}$$

where $s = \frac{1}{2}(a+b+c)$. From (3) and (4) we then obtain

$$\begin{aligned}
\sum_{cyclic} \frac{m_a}{r_a} &\leq \frac{R}{rs}\sum_{cyc}(1+\cos A)(s-a) \\
&= \frac{R}{r}\sum_{cyc}(1+\cos A) - \frac{R}{rs}\left(2s + \sum_{cyc} a\cos A\right) \\
&= \frac{R}{r}\left(3 + \sum_{cyc}\cos A\right) - \frac{2R}{r} - \frac{R}{rs}\sum_{cyc} a\cos A \\
&= \frac{R}{r} + \frac{R}{r}\sum_{cyc}\cos A - \frac{R}{rs}\sum_{cyc} a\cos A.
\end{aligned} \tag{5}$$

Now,

$$\begin{aligned}
\sum_{cyc}\cos A &= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1 + \frac{4(s-a)(s-b)(s-c)}{abc} \\
&= 1 + 4\left(\frac{F^2/s}{4RF}\right) = 1 + \frac{F}{sR} = 1 + \frac{rs}{sR} = 1 + \frac{r}{R},
\end{aligned} \tag{6}$$

where F denotes the area of the triangle. Using (2) we have

$$\sum_{cyc} a\cos A = \frac{1}{R}\sum_{cyc} ap_a = \frac{2F}{R}. \tag{7}$$

Substituting (6) and (7) into (5) we finally have

$$\sum_{cyc}\frac{m_a}{r_a} \le \frac{R}{r} + \frac{R}{r}\left(1+\frac{r}{R}\right) - \frac{R}{rs}\left(\frac{2F}{R}\right) = \frac{2R}{r} + 1 - \frac{2F}{rs} = \frac{2R}{r} - 1$$

since $F = rs$, and our proof is complete.

3974. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a+b+c=3$. Prove that

$$\sqrt{\frac{a}{b}+\frac{1}{a}} + \sqrt{\frac{b}{c}+\frac{1}{b}} + \sqrt{\frac{c}{a}+\frac{1}{c}} \ge 3\sqrt{2}.$$

We received 22 solutions and 5 incorrect submissions (mostly due to arithmetic errors). We present the solution by Mohammed Aassila.

By the power mean inequality we have

$$\frac{a+b+c}{3} \ge \left(\frac{a^{-1/4}+b^{-1/4}+c^{-1/4}}{3}\right)^{-4}.$$

Using the hypothesis $a+b+c=3$ and rearranging, we get

$$a^{-1/4}+b^{-1/4}+c^{-1/4} \ge 3.$$

Finally, applying the AM-GM inequality to each term, we have

$$\begin{aligned}\sqrt{\tfrac{a}{b}+\tfrac{1}{a}} + \sqrt{\tfrac{b}{c}+\tfrac{1}{b}} + \sqrt{\tfrac{c}{a}+\tfrac{1}{c}} &\ge \sqrt{2\sqrt{\tfrac{1}{b}}} + \sqrt{2\sqrt{\tfrac{1}{c}}} + \sqrt{2\sqrt{\tfrac{1}{a}}} \\ &= \sqrt{2}\left(a^{-1/4}+b^{-1/4}+c^{-1/4}\right) \\ &\ge 3\sqrt{2}.\end{aligned}$$

Note that equality holds if and only if $a=b=c=1$.

3975. *Proposed by Ovidiu Furdui.*

Let $k \ge 1$ be an integer. Calculate

$$\int_0^\infty \frac{e^x-1}{e^x+1}\ln^k\left(\frac{e^x+1}{e^x-1}\right)dx.$$

We received six correct solutions. We present the solution by Anastasios Kotronis.

With the change of the order of integration and summation being justified by the constant sign of the summand, the change of variables $y = \ln\left(\frac{e^x+1}{e^x-1}\right)$ gives

$$\begin{aligned}\int_0^\infty \frac{e^x-1}{e^x+1}\ln^k\left(\frac{e^x+1}{e^x-1}\right)dx &= 2\int_0^\infty \frac{e^{-2y}y^k}{1-e^{-2y}}\,dy\\ &= 2\int_0^\infty \sum_{n=0}^\infty e^{-2(n+1)y}y^k\,dy\\ &= 2\sum_{n=0}^\infty \int_0^\infty e^{-2(n+1)y}y^k\,dy.\end{aligned}$$

With the substitution $z = 2(n+1)y$ the above is equal to

$$2^{-k}\sum_{n=1}^\infty \frac{1}{n^{k+1}}\int_0^\infty e^{-z}z^k\,dz = 2^{-k}\Gamma(k+1)\zeta(k+1).$$

3976. *Proposed by Cristinel Mortici and Leonard Giugiuc.*

Find positive integer solutions for the following equation :

$$\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = \frac{x}{y-z}.$$

We received two correct and one incorrect solution. Presented is the solution by Madhav R. Modak, slightly modified by the editor.

First, let positive integers x, y, z with $y \neq z$, satisfy the given equation. Then

$$(yz - xz + xy)(y - z) = x^2yz. \tag{1}$$

Let $d = (y, z)$ so that $y = ad,\ z = bd,\ (a, b) = 1$, where a, b are positive integers. Substituting into (1) and dividing by d^2 yields

$$(dab - xb + xa)(a - b) = x^2ab. \tag{2}$$

Since $(a, b) = 1$, a is co-prime to b and $a - b$. As a divides the left hand side, it must also divide x. Similarly, $b|x$. Thus $ab|x$. Hence let $x = kab$ where k is a positive integer. Substituting into (2) and dividing by ab yields

$$(d - kb + ka)(a - b) = k^2a^2b^2. \tag{3}$$

Since $a - b$ is co-prime to both a and b, (3) shows that $(a - b)|k^2$. Hence (3) determines the integer d as follows :

$$d = -k(a - b) + a^2b^2 \cdot \frac{k^2}{a-b}. \tag{4}$$

Note that $|a-b| < \max\{|a|,|b|\}$. and therefore $|a-b|^2 < a^2b^2$. It follows that the magnitude of the left summand in (4) is less than the magnitude of the right summand. As d is positive, it must be $a > b$.

Thus a solution (x,y,z) of (1) necessarily satisfies the following conditions. There exist positive integers a,b,k with

$$\begin{gathered} x = kab,\ y = ad,\ z = bd,\ \text{where} \\ a > b,\ (a,b) = 1,\ (a-b)|k^2,\ \text{and} \\ d = -k(a-b) + a^2b^2\frac{k^2}{a-b}. \end{gathered} \tag{5}$$

Conversely it is easy to check that substituting x,y,z from (5) into the original equation yields a feasible solution. Hence (5) gives all solutions to the given equation.

As an example, for $a = 2$, $b = 1$, $k = 1$, we obtain the solution $(x,y,z) = (2,6,3)$.

Editor's comments. The solution can be easily extended to all integers x,y,z by allowing a,b,k to be negative. In that case the condition $a > b$ can also be omitted in (5) without changing the solution set.

3977. *Proposed by Dragoljub Milošević.*

Given a triangle ABC, prove that

$$\frac{1}{3-2\cos A} + \frac{1}{3-2\cos B} + \frac{1}{3-2\cos C} \geq \frac{3}{2}.$$

We received 15 correct solutions and three incorrect submissions. We present the joint solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, which is one of only a few using elementary methods.

Let S denote the semiperimeter of ΔABC. Then using the fact that

$$\sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc},$$

we have

$$\begin{aligned} 3 - 2\cos A &= 1 + 2(1-\cos A) = 1 + 4\sin^2\frac{A}{2} \\ &= 1 + \frac{4(s-b)(s-c)}{bc} = \frac{bc + 4(s-b)(s-c)}{bc}. \end{aligned}$$

Hence,

$$\sum_{cyc}\frac{1}{3-\cos A} = \sum_{cyc}\frac{bc}{bc+4(s-b)(s-c)} = 3 - \sum_{cyc}\frac{4(s-b)(s-c)}{bc+4(s-b)(s-c)}$$

so it suffices to show that

$$\sum_{cyc} \frac{4(s-b)(s-c)}{bc+4(s-b)(s-c)} \le \frac{3}{2}.$$

By the AM-GM inequality, we have $bc + 4(s-b)(s-c) \ge 2\sqrt{4bc(s-b)(s-c)}$, and thus,

$$\sum_{cyc} \frac{4(s-b)(s-c)}{bc+4(s-b)(s-c)} \le \sum_{cyc} \sqrt{\frac{(s-b)(s-c)}{bc}} = \sum_{cyc} \sin\frac{A}{2}.$$

Since it is well known (see item 2.9 on p.20 of [1] O. Bottema et al., *Geometric inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969) that $\sum_{cyc} \sin\frac{A}{2} \le \frac{3}{2}$, our proof is complete. The equality holds if and only if ΔABC is equilateral.

Editor's comments. The proofs of most other solvers made use of various 'heavy' machineries. These include : convexity of a function ; Lagrange's Multipliers ; Muirhead's Inequality ; Sturm method ; Gerretsen's Inequality ; and Half-convex Function Theorem.

3978. *Proposed by Billy Jin and Edward T.H. Wang.*

Let $n > 2$ be a positive integer. A permutation $\sigma = (a_1, a_2, \ldots, a_n)$ of $S(n) = \{1, 2, \ldots, n\}$ is called a *zigzag permutation* if, when reading from left to right, the a_i's alternately change their parity and increase/decrease in magnitudes. For example, $(5, 6, 3, 4, 1, 2)$ is a zigzag permutation of $S(6)$. Determine the number of zigzag permutations of $S(n)$.

We received six correct submissions. We present the solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal.

Let us prove that there exist 2 zigzag permutations of $S(n)$ if n is even and none if n is odd.

For $1 < j < n$, we say a_j is a local minimum for the permutation $(a_1, a_2, \ldots, a_n)$ if $a_j < \min\{a_{j-1}, a_{j+1}\}$. A local maximum is defined analogously, and these definitions can be adapted for a_1 and a_n in an obvious manner.

Because in a zigzag permutation the parity alternates, we deduce that numbers of the same parity must be all local minima or all local maxima. But 1 is a minimum, whence the odd numbers must all be local minima while the even numbers must all be local maxima. Note, as an immediate consequence, that n (which is the largest element of $S(n)$) must automatically be a local maximum, which means that n *is necessarily even.* Because there is only one element of $S(n)$ less than 2, 2 must take an extreme position and 1 must be next to it ; that is, we either have $a_1 = 2$ or $a_n = 2$. Let us first examine the case in which 2 comes first, so that $a_2 = 1$. The same observation now holds for the remaining elements of $S(n)$: 3 is the minimum of those remaining numbers and, therefore, the only remaining number less than

4. Because 4 must be surrounded by the two smallest odd numbers 1 and 3, we deduce that $a_3 = 4$ and $a_4 = 3$. When n is even, this construction will continue up to the largest even number n (and therefore smallest remaining even number) as a_{n-1} and with $n-1$ appearing last as a_n. This explicit construction proves the existence and uniqueness of a zigzag permutation starting with $a_1 = 2$. It remains to observe that the reverse of a zigzag permutation is also a zigzag permutation (having $a_n = 2$), from which we conclude that when n is even there exist exactly two zigzag permutations of $S(n)$.

Editor's Comments. Trey Smith of Angelo State University observed that our problem bears a striking resemblance to "Problem 4755 : A Permutation Problem", which appeared in *The American Mathematical Monthly*, proposed by Chandler Davis (**64** (1957), p. 596) with a solution by W.J. Blundon (**65** (1958), p. 533-534). That problem calls for the number of permutations of $S(n)$ in which relative minima alternate with relative maxima (without requiring alternating parity). It is quite remarkable how dropping the parity requirement turns our elementary problem into the rather difficult *Monthly* problem, whose solution involves the Bernoulli and Euler numbers! For that problem, the number of permutations for $n = 1, 2, 3, 4, 5, 6, 7\ldots$ turns out to be $1, 2, 4, 10, 32, 122, 544, \ldots$, which is sequence A001250 in the *On-Line Encyclopedia of Integer Sequences.*

3979. *Proposed by George Apostolopoulos.*

Let AD, BE and CF be the internal bisectors of the triangle ABC. Prove that

$$\frac{[DEF]}{[ABC]} \leq \frac{R_1}{4r},$$

where R_1 denotes the circumradius of DEF, r denotes the inradius of ABC and $[\cdot]$ represents the area of the corresponding triangle.

We received eight correct solutions. We present the solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal.

The bisector theorem gives, for example, $BF = \frac{ac}{a+b}$ and $BD = \frac{ac}{b+c}$. Then, denoting $[BDF] = S_b$ and $[ABC] = S$, we have

$$S_b = \frac{1}{2} BF \cdot BD \cdot \sin B = \frac{1}{2} \frac{a^2c^2}{(a+b)(b+c)} \cdot \sin B.$$

Since $S = \frac{1}{2} ac \sin B$,

$$S_b = \frac{ac}{(a+b)(b+c)} S.$$

Analogously,

$$S_a := [AFE] = \frac{bc}{(a+b)(a+c)} S \quad \text{and} \quad S_c := [CDE] = \frac{ab}{(a+c)(b+c)} S.$$

From this,

$$\frac{[DEF]}{[ABC]} = \frac{S - S_a - S_b - S_c}{S}$$
$$= 1 - \sum_{cyc} \frac{ac}{(a+b)(b+c)} = \frac{2abc}{(a+b)(a+c)(b+c)}\,. \qquad (1)$$

On the other hand, denoting $FD = e$, the law of cosines on $\triangle BDF$ gives

$$e^2 = \left(\frac{ac}{b+c}\right)^2 + \left(\frac{ac}{a+b}\right)^2 - 2\frac{a^2c^2}{(a+b)(b+c)} \cdot \cos B$$
$$= \frac{a^2c^2}{(a+b)^2(b+c)^2}\left((b+c)^2 + (a+b)^2 - 2(a+b)(b+c)\cos B\right)$$
$$= \frac{a^2c^2}{(a+b)^2(b+c)^2}\left(a^2 + c^2 - 2ac\cos B + 2b(a+b+c)(1-\cos B)\right).$$

Now the law of cosines on $\triangle ABC$ gives $b^2 = a^2 + c^2 - 2ac\cos B$, so

$$e^2 = \frac{a^2c^2}{(a+b)^2(b+c)^2}\left(b^2 + 2b(a+b+c)(1-\cos B)\right)$$
$$= \frac{a^2c^2}{(a+b)^2(b+c)^2} \cdot b\left(b + 8p\sin^2\frac{B}{2}\right)$$
$$= \frac{abc}{(a+b)^2(b+c)^2}\left(abc + 8p(p-a)(p-c)\right),$$

where p is the semiperimeter of the triangle ABC and we have used the well known identity $\sin^2\frac{B}{2} = \frac{(p-a)(p-c)}{ac}$. Analogously, denoting $DE = f$ and $EF = d$, we have

$$f^2 = \frac{abc}{(a+c)^2(b+c)^2}\left(abc + 8p(p-a)(p-b)\right)$$

and

$$d^2 = \frac{abc}{(a+b)^2(a+c)^2}\left(abc + 8p(p-b)(p-c)\right).$$

From this,

$$d^2e^2f^2 = \frac{a^3b^3c^3}{(a+b)^4(a+c)^4(b+c)^4}\prod_{\text{cyc}}\left(abc + 8p(p-a)(p-b)\right). \qquad (2)$$

By using the identities $[DEF] = \frac{def}{4R_1}$ and $[ABC] = pr$, we obtain

$$\frac{R_1}{4r} = \frac{defp}{16[DEF]\,[ABC]}$$

and the proposed inequality becomes

$$16[DEF]^2 \le defp.$$

Now, applying (1) and Heron's formula $[ABC]^2 = p(p-a)(p-b)(p-c)$ in the previous inequality, we have

$$\frac{64a^2b^2c^2(p-a)(p-b)(p-c)}{(a+b)^2(a+c)^2(b+c)^2} \le def,$$

which becomes

$$2^{12}abc(p-a)^2(p-b)^2(p-c)^2 \le \prod_{\text{cyc}}\big(abc+8p(p-a)(p-b)\big), \tag{3}$$

after using (2).

In order to prove this inequality (3), we will use the following estimation obtained from the AM-GM inequality :

$$\begin{aligned} abc+8p(p-a)(p-b) &= abc+4(a+b+c)(p-a)(p-b) \\ &= abc+4a(p-a)(p-b)+4b(p-a)(p-b)+4c(p-a)(p-b) \\ &\ge 4\sqrt[4]{4^3a^2b^2c^2(p-a)^3(p-b)^3}. \end{aligned}$$

In a similar way, we can prove that

$$abc+8p(p-a)(p-c) \ge 4\sqrt[4]{4^3a^2b^2c^2(p-a)^3(p-c)^3}$$

and

$$abc+8p(p-b)(p-c) \ge 4\sqrt[4]{4^3a^2b^2c^2(p-b)^3(p-c)^3}.$$

Multiplying the three inequalities, we obtain the estimation

$$\prod_{\text{cyc}}\big(abc+8p(p-a)(p-b)\big) \ge 2^{21/2}\big(abc(p-a)(p-b)(p-c)\big)^{3/2},$$

and to deduce (3) it is sufficient prove the inequality

$$2^{21/2}\big(abc(p-a)(p-b)(p-c)\big)^{3/2} \ge 2^{12}abc(p-a)^2(p-b)^2(p-c)^2,$$

or, equivalently,

$$abc \ge 8(p-a)(p-b)(p-c). \tag{4}$$

But $abc = 4R[ABC]$, where R denotes the circumradius of ABC, and

$$(p-a)(p-b)(p-c) = r[ABC],$$

and (4) becomes

$$R \ge 2r,$$

which is well known and true.

3980. *Proposed by S. Viswanathan.*

Let a, b, c be distinct real numbers. Prove that

$$\left(\sum_{cyc}\frac{a+b}{a-b}\right)\left(\prod_{cyc}\frac{a+b}{a-b}\right)<\frac{1}{3}.$$

We received seven correct solutions. We present the solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal.

Let us prove that

$$\left(\sum_{\text{cyc}}\frac{a+b}{a-b}\right)\left(\prod_{\text{cyc}}\frac{a+b}{a-b}\right)<\frac{1}{4}.$$

If $a=0$, the previous inequality becomes

$$-\left(\frac{b+c}{b-c}\right)^2<\frac{1}{4},$$

which is obviously true, and we can assume that $abc\neq 0$. With the change of variables

$$\alpha=\frac{a+b}{a-b},\quad \beta=\frac{b+c}{b-c},\quad \gamma=\frac{c+a}{c-a},$$

the inequality to be proved is equivalent to

$$\alpha\beta\gamma(\alpha+\beta+\gamma)<\frac{1}{4},$$

with the restriction $\alpha\beta+\beta\gamma+\gamma\alpha+1=0$. The above inequality is of course true if $\alpha\beta\gamma=0$, so we can assume without loss of generality that $\alpha\beta\gamma\neq 0$, and $\alpha\beta>0$. From the restriction, we obtain

$$\alpha\beta\left(-1-\alpha\beta+\left(\frac{1+\alpha\beta}{\alpha+\beta}\right)^2\right)<\frac{1}{4}.$$

Since $(\alpha+\beta)^2\geq 4\alpha\beta$, setting $t=\alpha\beta$, it is enough to prove

$$\alpha\beta(1+\alpha\beta)\left(-1+\frac{1+\alpha\beta}{4\alpha\beta}\right)<\frac{1}{4}\Longleftrightarrow(1+t)(1-3t)<1, t>0,$$

which is obviously true.

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations