THE CONTEST CORNER

No. 38

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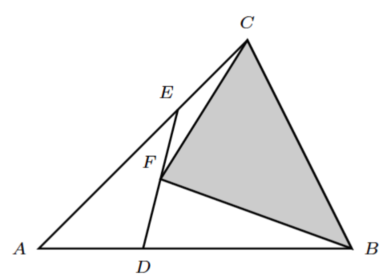
Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 octobre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

CC186. Soit n un entier positif. Faire le décompte des nombres $k \in \{0, 1, \dots, n\}$ pour lesquels $\binom{n}{k}$ est impair. Démontrer que ce nombre est une puissance de deux, c'est-à-dire est de la forme 2^p pour un certain entier non négatif p.

CC187. Dans le diagramme, la surface du triangle ABC est 1, $\overline{AD} = \frac{1}{3}\overline{AB}$, $\overline{EC} = \frac{1}{3}\overline{AC}$ et $\overline{DF} = \overline{FE}$. Déterminer la surface du triangle en gris BFC.



CC188. Un plan divise l'espace en deux régions. Deux plans intersectant en une ligne divisent l'espace en quatre régions. Supposer maintenant que douze plans sont donnés dans l'espace tel que

a) deux d'entre eux intersectent toujours en une ligne,

b) trois d'entre eux intersectent toujours en un point, et

c) quatre d'entre eux n'ont jamais un point en commun.

Dans combien de régions l'espace est-il divisé ? Justifier votre réponse.

CC189. Des pièces de monnaie sont placées sur certains des 100 carrés d'une grille 10×10. Tout carré est voisin à un autre carré couvert d'une pièce de monnaie. Déterminer le nombre minimum de pièces de monnaie. (Deux carrés distincts sont voisins s'ils partagent un côté.)

CC190. Les lettres du mot TRIANGLE forment un arrangement tel qu'indiqué.

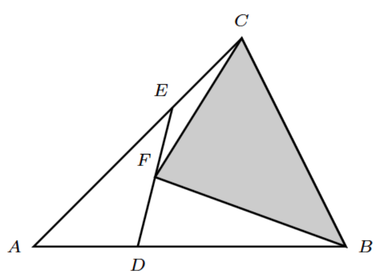
E
E L E
E L G L E
E L G N G L E
E L G N A N G L E
E L G N A I A N G L E
E L G N A I R I A N G L E
E L G N A I R T R I A N G L E

Déterminer le nombre de façons d'épeler le mot TRIANGLE, utilisant des lettres adjacentes dans cet arrangement et allant vers le haut, la gauche ou la droite.

. . . .

CC186. Let n be a positive integer. Count the number of $k \in \{0, 1, \ldots, n\}$ for which $\binom{n}{k}$ is odd. Prove that this number is a power of two, i.e. it is of the form 2^p for some non-negative integer p.

CC187. In the diagram the area of the triangle ABC is 1, $\overline{AD} = \frac{1}{3}\overline{AB}$, $\overline{EC} = \frac{1}{3}\overline{AC}$ and $\overline{DF} = \overline{FE}$. Find the area of the shaded triangle BFC.



CC188. A plane divides space into two regions. Two planes that intersect in a line divide space into four regions. Now suppose that twelve planes are given in space so that three conditions are met :

a) every two of them intersect in a line,

b) every three of them intersect in a point, and

c) no four of them have a common point.

Into how many regions is space divided ? Justify your answer.

CC189. Coins are placed on some of the 100 squares in a 10×10 grid. Every square is next to another square with a coin. Find the minimum possible number of coins. (We say that two squares are next to each other when they share a common edge but are not equal.)

CC190. An arrangement of the letters from the word TRIANGLE is shown.

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              E
            E L E
          E L G L E
        E L G N G L E
      E L G N A N G L E
    E L G N A I A N G L E
  E L G N A I R I A N G L E
E L G N A I R T R I A N G L E
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Find the number of ways that the word TRIANGLE can be spelled out, using adjacent letters, going up or left or right, in this arrangement.

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2014 : 40(8), p. 319–320.

CC136. A spiderweb is a square 100×100 grid with knots at each intersection. The spider sits at one corner of his spiderweb ; there are 100 flies caught in the web with at most one fly per knot. Can the spider get all the flies in no more than 2000 moves, if in one move it crawls to an adjacent knot ?

Originally from Tournament of Towns Fall Round 2014, A-level, Juniors.

We received no solutions to this problem.

CC137. An Emperor invited 2015 wizards to a festival. Each wizard, but not the Emperor, knows which wizards are good and which ones are evil. A good wizard always tells the truth, while an evil wizard can either tell the truth or lie. At the festival, the Emperor gives every wizard a card with one "yes-or-no" question (questions might be different for different wizards), learns all the answers and then expels one wizard through a magic door which shows if this wizard is good or evil. Then the Emperor makes new cards and repeats the procedure with the remaining wizards until he wants to stop (with or without expelling a wizard). Prove that the Emperor can devise his questions so that all the evil wizards are expelled while expelling at most one good wizard.

Originally from Tournament of Towns, 2015, Spring, A-level, Senior.

We received and present one solution by Josh Zukewich and others (unnamed).

This solution is in two parts. In the first part (Part A) we find a living good wizard, expelling at most one good wizard in the process. In the second (Part B), we use this fact to expel all remaining bad wizards.

Part A : Pick a wizard at random (called Jim). Ask every wizard : "Is Jim good ?" Jim's answer does not matter for our analysis. For the remaining wizards' answers, consider the two exhaustive possibilities :

(1) Every non-Jim wizard says 'no.'

(2) Some non-Jim wizard says 'yes.'

Our actions are as follows for each possibility :

(1) Expel Jim. Jim goes through the door and he is either :

- (1.1) Good. Everyone else lied and is therefore bad. Expel everyone else in turn. Then we are done.
- (1.2) Bad. You expelled a bad wizard. Return to A.

(2) Expel a wizard (called Jane) that said 'yes.' When Jane goes through the door, she is either :

(2.1) Good. Then you have identified that Jim is good. You are done step A.

(2.2) Bad. You have expelled a bad wizard. Return to A.

In every iteration of step A we either expel a bad wizard, and repeat, or expel a good wizard and identify a good wizard still in the group.

Part B : Arrange all the wizards in a line facing forward, with Jim at the back. Ask every wizard : "Is the wizard directly ahead of you good ?" If Jim says

(1) 'No,' expel the wizard in front of Jim. You have expelled a bad wizard. Return to B.

(2) 'Yes,' the wizard in front of Jim is also good. Keep moving your attention ahead until a wizard says 'no.' The first wizard in front of Jim that says 'no' identifies the wizard in front of them as bad. Expel that bad wizard and return to B.

Goodness is transitive : Each wizard that says 'yes' moving ahead from Jim confirms the goodness of another wizard. Once the second-to-front wizard in the line says 'yes,' you have confirmed that all the remaining wizards are good, and you are done. You choose not to expel a wizard once this condition is met.

CC138. Prove that the integer

$$\sum_{i=1}^{2^{n-1}} (2i-1)^{2i-1} = 1^1 + 3^3 + 5^5 + \cdots + (2^n-1)^{2^n-1}$$

is a multiple of 2^n but not a multiple of 2^{n+1}.

Originally from Tournament of Towns, Fall Round 2011, A-level, Senior.

No solutions were received. We present a hint to encourage readers.

Consider the problem modulo 2^{n+2}. Specifically, prove that $k^{2^n} \equiv 1 \pmod{2^{n+2}}$ for any odd positive integer k.

CC139. It is well-known that if in a quadrilateral the circumcircle and the incircle have the same centre, then the quadrilateral is a square. Is the similar statement true in 3 dimensions ? Namely, if a cuboid is inscribed into a sphere and circumscribed around a sphere and the centres of these spheres coincide, does it imply that the cuboid is a cube ? (A cuboid is a polyhedron with 6 quadrilateral faces such that each vertex belongs to 3 edges.)

Originally from Tournament of Towns, Spring Round 2015, A-level, Seniors.

We received one correct solution. We present the solution of the Missouri State University Problem Solving Group.

The answer is no.

Consider the polyhedron whose vertices are

$$A = (a^2; 1; a), \quad B = (a^2; -1; a), \quad C = (-a^2; -1; a), \quad D = (-a^2; 1; a),$$
$$E = (1; a^2; -a), \quad F = (1; -a^2; -a), \quad G = (-1; -a^2; -a), \quad H = (-1; a^2; -a).$$

These clearly lie on the sphere of radius $\sqrt{a^4 + a^2 + 1}$ centered at the origin.

The rectangular faces $ABCD$ and $EFGH$ are tangent to the sphere of radius a centered at the origin. The other four faces, $ABFE$, $BCGF$, $CDHG$, and $DAEH$ are congruent trapezoids lying in the planes

$$\begin{aligned} 2ax - (a^2 - 1)z &= a^3 + a, \\ -2ay + (a^2 - 1)z &= a^3 + a, \\ 2ax + (a^2 - 1)z &= -a^3 - a, \\ 2ay + (a^2 - 1)z &= a^3 + a, \end{aligned}$$

respectively.

We claim that the sphere of radius a centered at the origin is tangent to these planes at the points

$$\Big(\frac{2a2}{a2+1}, 0, \frac{a - a^3}{a^2+1}\Big), \quad \Big(0, -\frac{2a^2}{a^2+1}, \frac{a^3 - a}{a^2+1}\Big),$$
$$\Big(-\frac{2a^2}{a2+1}, 0, \frac{a - a^3}{a^2+1}\Big), \quad \Big(0, \frac{2a^2}{a^2+1}, \frac{a^3 - a}{a^2+1}\Big)$$

respectively. To see this, note that each point lies on its corresponding plane and its position vector has length a and is parallel to the normal vector of that plane. This cuboid satisfies the conditions of the problem, but is not a cube unless $a = 1$.

CC140. Let $P(x)$ be a polynomial with real coefficients so that the equation $P(m) + P(n) = 0$ has infinitely many pairs of integer solutions (m, n). Prove that the graph of $y = P(x)$ has a centre of symmetry.

Originally from Tournament of Towns, Fall Round 2008, A-level, Seniors.

We received no solutions to this problem.