CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2014 : 40(8), p. 319–320.

CC136. A spiderweb is a square 100×100 grid with knots at each intersection. The spider sits at one corner of his spiderweb ; there are 100 flies caught in the web with at most one fly per knot. Can the spider get all the flies in no more than 2000 moves, if in one move it crawls to an adjacent knot ?

Originally from Tournament of Towns Fall Round 2014, A-level, Juniors.

We received no solutions to this problem.

CC137. An Emperor invited 2015 wizards to a festival. Each wizard, but not the Emperor, knows which wizards are good and which ones are evil. A good wizard always tells the truth, while an evil wizard can either tell the truth or lie. At the festival, the Emperor gives every wizard a card with one "yes-or-no" question (questions might be different for different wizards), learns all the answers and then expels one wizard through a magic door which shows if this wizard is good or evil. Then the Emperor makes new cards and repeats the procedure with the remaining wizards until he wants to stop (with or without expelling a wizard). Prove that the Emperor can devise his questions so that all the evil wizards are expelled while expelling at most one good wizard.

Originally from Tournament of Towns, 2015, Spring, A-level, Senior.

We received and present one solution by Josh Zukewich and others (unnamed).

This solution is in two parts. In the first part (Part A) we find a living good wizard, expelling at most one good wizard in the process. In the second (Part B), we use this fact to expel all remaining bad wizards.

Part A : Pick a wizard at random (called Jim). Ask every wizard : "Is Jim good ?" Jim's answer does not matter for our analysis. For the remaining wizards' answers, consider the two exhaustive possibilities :

(1) Every non-Jim wizard says 'no.'

(2) Some non-Jim wizard says 'yes.'

Our actions are as follows for each possibility :

(1) Expel Jim. Jim goes through the door and he is either :

  (1.1) Good. Everyone else lied and is therefore bad. Expel everyone else in turn. Then we are done.

  (1.2) Bad. You expelled a bad wizard. Return to A.

(2) Expel a wizard (called Jane) that said 'yes.' When Jane goes through the door, she is either :

(2.1) Good. Then you have identified that Jim is good. You are done step A.

(2.2) Bad. You have expelled a bad wizard. Return to A.

In every iteration of step A we either expel a bad wizard, and repeat, or expel a good wizard and identify a good wizard still in the group.

Part B : Arrange all the wizards in a line facing forward, with Jim at the back. Ask every wizard : "Is the wizard directly ahead of you good ?" If Jim says

(1) 'No,' expel the wizard in front of Jim. You have expelled a bad wizard. Return to B.

(2) 'Yes,' the wizard in front of Jim is also good. Keep moving your attention ahead until a wizard says 'no.' The first wizard in front of Jim that says 'no' identifies the wizard in front of them as bad. Expel that bad wizard and return to B.

Goodness is transitive : Each wizard that says 'yes' moving ahead from Jim confirms the goodness of another wizard. Once the second-to-front wizard in the line says 'yes,' you have confirmed that all the remaining wizards are good, and you are done. You choose not to expel a wizard once this condition is met.

CC138. Prove that the integer

$$\sum_{i=1}^{2^{n-1}} (2i-1)^{2i-1} = 1^1 + 3^3 + 5^5 + \cdots + (2^n-1)^{2^n-1}$$

is a multiple of 2^n but not a multiple of 2^{n+1}.

Originally from Tournament of Towns, Fall Round 2011, A-level, Senior.

No solutions were received. We present a hint to encourage readers.

Consider the problem modulo 2^{n+2}. Specifically, prove that $k^{2^n} \equiv 1 \pmod{2^{n+2}}$ for any odd positive integer k.

CC139. It is well-known that if in a quadrilateral the circumcircle and the incircle have the same centre, then the quadrilateral is a square. Is the similar statement true in 3 dimensions ? Namely, if a cuboid is inscribed into a sphere and circumscribed around a sphere and the centres of these spheres coincide, does it imply that the cuboid is a cube ? (A cuboid is a polyhedron with 6 quadrilateral faces such that each vertex belongs to 3 edges.)

Originally from Tournament of Towns, Spring Round 2015, A-level, Seniors.

We received one correct solution. We present the solution of the Missouri State University Problem Solving Group.

The answer is no.

Consider the polyhedron whose vertices are

$$A = (a^2; 1; a), \quad B = (a^2; -1; a), \quad C = (-a^2; -1; a), \quad D = (-a^2; 1; a),$$
$$E = (1; a^2; -a), \quad F = (1; -a^2; -a), \quad G = (-1; -a^2; -a), \quad H = (-1; a^2; -a).$$

These clearly lie on the sphere of radius $\sqrt{a^4 + a^2 + 1}$ centered at the origin.

The rectangular faces $ABCD$ and $EFGH$ are tangent to the sphere of radius a centered at the origin. The other four faces, $ABFE$, $BCGF$, $CDHG$, and $DAEH$ are congruent trapezoids lying in the planes

$$\begin{aligned} 2ax - (a^2 - 1)z &= a^3 + a, \\ -2ay + (a^2 - 1)z &= a^3 + a, \\ 2ax + (a^2 - 1)z &= -a^3 - a, \\ 2ay + (a^2 - 1)z &= a^3 + a, \end{aligned}$$

respectively.

We claim that the sphere of radius a centered at the origin is tangent to these planes at the points

$$\Big(\frac{2a2}{a2+1}, 0, \frac{a - a^3}{a^2+1}\Big), \quad \Big(0, -\frac{2a^2}{a^2+1}, \frac{a^3 - a}{a^2+1}\Big),$$
$$\Big(-\frac{2a^2}{a2+1}, 0, \frac{a - a^3}{a^2+1}\Big), \quad \Big(0, \frac{2a^2}{a^2+1}, \frac{a^3 - a}{a^2+1}\Big)$$

respectively. To see this, note that each point lies on its corresponding plane and its position vector has length a and is parallel to the normal vector of that plane. This cuboid satisfies the conditions of the problem, but is not a cube unless $a = 1$.

CC140. Let $P(x)$ be a polynomial with real coefficients so that the equation $P(m) + P(n) = 0$ has infinitely many pairs of integer solutions (m, n). Prove that the graph of $y = P(x)$ has a centre of symmetry.

Originally from Tournament of Towns, Fall Round 2008, A-level, Seniors.

We received no solutions to this problem.