SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014 : 40(10), p. 435–438.

3991. *Proposed by Michel Bataille.*

Let ABC be a triangle with $BC = a, CA = b, AB = c, \angle BAC = \alpha, \angle CBA = \beta, \angle ACB = \gamma$ and let $m_a = AA', m_b = BB', m_c = CC'$ where A', B', C' are the midpoints of BC, CA, AB. Prove that

$$\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \geq \frac{3(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)}{\sin\alpha + \sin\beta + \sin\gamma}.$$

We received eight correct solutions, from which two will be featured.

Solution 1, by Šefket Arslanagić and Dragoljub Milošević (done independently).

Without loss of generality we can suppose that $a \leq b \leq c$. This implies both

$$\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c} \quad \text{and} \quad m_a \geq m_b \geq m_c,$$

so that we can apply the Chebyshev Sum Inequality to get

$$m_a \cdot \frac{1}{a} + m_b \cdot \frac{1}{b} + m_c \cdot \frac{1}{c} \ \geq\ \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(m_a + m_b + m_c)\,. \tag{1}$$

We use the AM-HM inequality,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c},$$

together with the inequality

$$m_a + m_b + m_c \geq \frac{1}{2R}\left(a^2 + b^2 + c^2\right)$$

(where R is the circumradius of ΔABC), which can be found on p.13 of [1] or p. 213 of [2], to reduce inequality (1) to

$$m_a \cdot \frac{1}{a} + m_b \cdot \frac{1}{b} + m_c \cdot \frac{1}{c} \ \geq\ \frac{3(a^2 + b^2 + c^2)}{2R(a+b+c)}. \tag{2}$$

Finally, plug $a = 2R\sin\alpha,\ b = 2R\sin\beta,$ and $c = 2R\sin\gamma$ into the right-hand-side of (2) to finish the proof.

The equality holds if and only if $a = b = c$ (and ΔABC is equilateral).

Solution 2, a composite of the solutions of Arkady Alt and of Andrea Fanchini.

The inequality continues to hold when the medians m_x are replaced by the altitudes h_x ; more precisely, we shall prove that

$$\frac{m_a}{a}+\frac{m_b}{b}+\frac{m_c}{c}\geq\frac{h_a}{a}+\frac{h_b}{b}+\frac{h_c}{c}\geq\frac{3(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)}{\sin\alpha+\sin\beta+\sin\gamma},$$

with equality if and only if $a=b=c$.

The left inequality is clear because $m_x\geq h_x$ for each side x. For the right inequality, in terms of the area K of ΔABC we know that

$$\sin\alpha=\frac{2K}{bc},\quad \sin\beta=\frac{2K}{ac},\quad \sin\gamma=\frac{2K}{ab},$$

so that

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\frac{4K^2(a^2+b^2+c^2)}{a^2b^2c^2},\quad \sin\alpha+\sin\beta+\sin\gamma=\frac{2K(a+b+c)}{abc}.$$

Since the altitudes satisfy

$$h_a=\frac{2K}{a},\quad h_b=\frac{2K}{b},\quad h_c=\frac{2K}{c},$$

the inequality on the right becomes

$$\frac{2K}{a^2}+\frac{2K}{b^2}+\frac{2K}{c^2}\geq\frac{6K(a^2+b^2+c^2)}{abc(a+b+c)},$$

which reduces to

$$(a+b+c)(a^2b^2+b^2c^2+c^2a^2)\geq 3abc(a^2+b^2+c^2). \qquad (3)$$

Warning! Inequality (3) is guaranteed to hold only when a,b,c are the sides of a triangle. It might not hold for an arbitrary triple of positive real numbers; for example, when $b=c=1$ the inequality fails for a sufficiently large.

Because a,b,c are the sides of a triangle, we can set

$$a=\frac{y+z}{2},\quad b=\frac{z+x}{2},\quad c=\frac{x+y}{2},$$

and (3) expands to

$$x^5+y^5+z^5+x^2y^2z+x^2yz^2+xy^2z^2\geq x^3y^2+x^3z^2+x^2y^3+y^3z^2+y^2z^3+x^2z^3.$$

Let us write $[k,\ell,m]=\sum p^kq^\ell r^m$, the sum being taken over the six permutations (p,q,r) of (x,y,z). In this notation our inequality becomes

$$[5,0,0]+[2,2,1]\geq 2[3,2,0],$$

which holds by Muirhead's Theorem (or, if you prefer, by an inequality of Schur). By the equality condition of Muirhead's Theorem, the equality holds if and only if ΔABC is equilateral.

Editor's Comments. The proposer observed that his inequality resembles a known one :

$$\frac{m_a}{a}+\frac{m_b}{b}+\frac{m_c}{c}\geq\frac{3\sqrt{3}}{2}$$

(see, for example, p. 211 of [2]). However, neither is a refinement of the other ; for example,

$$\text{with } \alpha=\tfrac{\pi}{2},\ \beta=\gamma=\tfrac{\pi}{4},\text{ then } \frac{3(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)}{\sin\alpha+\sin\beta+\sin\gamma}<\frac{3\sqrt{3}}{2},$$

$$\text{while if } \alpha=\tfrac{\pi}{2},\ \beta=\tfrac{\pi}{12},\ \gamma=\tfrac{5\pi}{12},\text{ then } \frac{3(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)}{\sin\alpha+\sin\beta+\sin\gamma}>\frac{3\sqrt{3}}{2}.$$

References :

[**1**] Marin Chirciu, *Inegalităti Geometriće.* Editura Paralela 45, Pitesti Romania, 2015.

[**2**] D.S. Mitrinović et al. *Recent Advances in Geometric Inequalities*, Kluwer, 1989.

3992. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let α, a, b, c be positive real numbers such that $a+b+c+3=6abc$. Find the maximum value of the expression

$$\frac{1}{a^\alpha+b^\alpha+1}+\frac{1}{b^\alpha+c^\alpha+1}+\frac{1}{c^\alpha+a^\alpha+1}.$$

Eight correct solutions were received. There was one incomplete solution. The solution presented draws on ideas from several solvers.

By the AM-GM inequality, we have that

$$6abc=a+b+c+1+1+1\geq 6(abc)^{1/6},$$

whence $abc\geq 1$. Let $a^\alpha=x^3$, $b^\alpha=y^3$ and $c^\alpha=z^3$, so that $xyz\geq 1$.

Observe that, by the AM-GM inequality,

$$x^3+y^3+1=\frac{1}{3}(2x^3+y^3)+\frac{1}{3}(x^3+2y^3)+1\geq xy(x+y)+1\geq\frac{x+y+z}{z}.$$

Apply this and analogous inequalities for y^3+z^3+1 and z^3+x^3+1 to obtain

$$\frac{1}{x^3+y^3+1}+\frac{1}{y^3+z^3+1}+\frac{1}{z^3+x^3+1}\leq\frac{z+y+x}{x+y+z}=1,$$

with equality if and only if $x=y=z=1$.

3993. *Proposed by Dragoljub Milošević.*

Let h_a, h_b and h_c be the altitudes and r the inradius of a triangle. Prove that

$$\frac{h_a - 2r}{h_a + 2r} + \frac{h_b - 2r}{h_b + 2r} + \frac{h_c - 2r}{h_c + 2r} \geq \frac{3}{5}.$$

We received 21 correct solutions. We present the solution by Kee-Wai Lau.

Let s be the semiperimeter of the triangle. The area equals

$$\frac{ah_a}{2} = \frac{bh_b}{2} = \frac{ch_c}{2} = rs.$$

Hence

$$\frac{h_a - 2r}{h_a + 2r} = \frac{2s}{s+a} - 1, \quad \frac{h_b - 2r}{h_b + 2r} = \frac{2s}{s+b} - 1 \text{ and } \frac{h_c - 2r}{h_c + 2r} = \frac{2s}{s+c} - 1,$$

so that the inequality of the problem is equivalent to

$$\frac{1}{s+a} + \frac{1}{s+b} + \frac{1}{s+c} \geq \frac{9}{5s}.$$

For $x > 0$, the function $\frac{1}{x}$ is convex. Hence

$$\frac{1}{s+a} + \frac{1}{s+b} + \frac{1}{s+c} \geq 3\left(\frac{1}{\frac{(s+a)+(s+b)+(s+c)}{3}}\right) = \frac{9}{5s},$$

and this completes the solution.

3994. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers with $a + b + c = 1$. Prove that

$$a^4 + b^4 + c^4 \geq abc.$$

We received 32 *correct solutions. We present three of the shorter solutions.*

Solution 1, by AN-anduud Problem Solving Group.

Using the AM-GM inequality we have

$$\begin{aligned} a^4 + b^4 + c^4 &= \frac{a^4 + a^4 + b^4 + c^4}{4} + \frac{a^4 + b^4 + b^4 + c^4}{4} + \frac{a^4 + b^4 + c^4 + c^4}{4} \\ &\geq \sqrt[4]{a^4a^4b^4c^4} + \sqrt[4]{a^4b^4b^4c^4} + \sqrt[4]{a^4b^4c^4c^4} \\ &= a^2bc + ab^2c + abc^2 \\ &= abc(a+b+c) = abc. \end{aligned}$$

Equality holds if and only if $a = b = c = \frac{1}{3}$.

Solution 2, by Cao Minh Quang.

For all positive x, y, z, by the rearrangement inequality we have

$$x^2 + y^2 + z^2 \geq xy + yz + zx.$$

Hence,

$$\begin{aligned} a^4 + b^4 + c^4 &\geq (ab)^2 + (bc)^2 + (ca)^2 \\ &\geq abbc + bcca + caab = abc(a+b+c) = abc. \end{aligned}$$

Solution 3, by Joel Schlosberg.

By Muirhead's inequality,

$$\sum_{\text{symmetric}} a^4 b^0 c^0 \geq \sum_{\text{symmetric}} a^2 b^1 c^1,$$

as $(4, 0, 0)$ majorizes $(2, 1, 1)$. That is,

$$2(a^4 + b^4 + c^4) \geq 2(a^2bc + b^2ac + c^2ab) = 2abc(a+b+c) = 2abc,$$

which via cancellation of the factor 2 yields the inequality.

3995. *Proposed by Michel Bataille.*

For positive x and y, let $\mathcal{M}_0(x, y) = \sqrt{xy}$ and $\mathcal{M}_\alpha(x, y) = \left(\frac{x^\alpha + y^\alpha}{2}\right)^{\frac{1}{\alpha}}$ if α is a nonzero real number. Given an equilateral triangle ABC, determine for which values of α the following property holds : $\mathcal{M}_\alpha(PB, PC) \leq PA$ for every point P distinct from B and C on the line BC.

Of the 4 submissions we received, only the one from Joel Schlosberg was complete and correct. We present his solution.

The desired property holds if and only if $\alpha \leq 4$. Set Cartesian coordinates so that the vertices of the equilateral triangle are

$$A = (0, \sqrt{3}), \quad B = (-1, 0), \quad C = (1, 0).$$

Any point P on line BC, the x-axis, has coordinates $(x, 0)$ for some $x \in \mathbb{R}$.

For $P = (x, 0)$, $PB = |x+1|$ and $PC = |x-1|$, so

$$\mathcal{M}_\alpha(PB, PC) = \begin{cases} \left(\frac{|x+1|^\alpha + |x-1|^\alpha}{2}\right)^{1/\alpha} & \text{if } \alpha \neq 0, \\ \sqrt{|x+1| \cdot |x-1|} & \text{if } \alpha = 0. \end{cases}$$

moreover, by the Pythagorean Theorem,

$$PA = \sqrt{x^2 + 3}.$$

Then because $2(x^2+3)^2 - (x+1)^4 - (x-1)^4 = 16 > 0$, we have

$$\mathcal{M}_4(PB, PC) = \left(\frac{(x+1)^4 + (x-1)^4}{2}\right)^{1/4} < \sqrt{x^2+3} = PA.$$

If $\alpha \le 4$, by the power mean inequality,

$$\mathcal{M}_\alpha(PB, PC) \le \mathcal{M}_4(PB, PC) < PA.$$

This includes $\alpha = 0$, since the power mean inequality includes the case of one of the power means being the zeroth power, defined by convention as the geometric mean. But the inequality is also easily proved directly : Because

$$-(x^2+3) \ \le \ x^2 - 1 \ \le \ x^2 + 3,$$

we deduce that

$$\mathcal{M}_0(PB, PC) = \sqrt{|x^2-1|} \le \sqrt{x^2+3} = PA.$$

For the converse, consider the function

$$f : z \mapsto \left(\frac{(1+z)^\alpha + (1-z)^\alpha}{2}\right)^{2/\alpha}.$$

By straightforward, if sometimes tedious calculations, $f(0) = 1$, $f'(0) = 0$ and $f''(0) = 2(\alpha - 1)$. By Taylor's theorem,

$$f(z) = f(0) + f'(0)z + h(z)z^2 = 1 + h(z)z^2,$$

with $\lim_{z \to 0} h(z) = \frac{1}{2} f''(0) = \alpha - 1$.

Suppose that the desired inequality holds for $\alpha \ne 0$. Then for all $x > 1$, we have

$$\left(\frac{(x+1)^\alpha + (x-1)^\alpha}{2}\right)^{1/\alpha} = \mathcal{M}_\alpha(PB, PC) \le PA = \sqrt{x^2+3},$$

$$\left(\frac{x^\alpha[(1+x^{-1})^\alpha + (1-x^{-1})^\alpha]}{2}\right)^{2/\alpha} \le x^2 + 3,$$

$$\left(\frac{(1+x^{-1})^\alpha + (1-x^{-1})^\alpha}{2}\right)^{2/\alpha} \le 1 + 3x^{-2}.$$

So for $z = x^{-1} \in (0, 1)$, we have $f(z) = 1 + h(z)z^2 \le 1 + 3z^2$, whence $h(z) \le 3$. Taking $z \to 0^+$,

$$\alpha - 1 = \lim_{z \to 0^+} h(z) \le 3.$$

Therefore $\alpha \le 4$, as claimed.

Editor's Comments. Two of the incomplete solutions restricted P to the interior of the edge BC, in which case the segment PA is longer than PB and PC

(because the longer side is opposite the larger angle) and, therefore, greater than $\mathcal{M}_\alpha(PB, PC)$ for all α. It is only when P is outside ΔABC on the line BC that the problem becomes interesting.

3996. *Proposed by Marcel Chiriţă.*

Let $a \in (1, \infty)$ and $b, c \in \mathbb{R}$. For each $\lambda \in (0, \infty)$, find all differentiable functions $f : [1, \infty) \to \mathbb{R}$ such that

$$f(a^{\lambda^2 x}) + 2f(a^{\lambda x}) + f(a^x) = bx + c$$

for all $x \in [1, \infty)$.

We received three submissions, all of which were correct and complete. We present the solution by Michel Bataille.

Since $a^x \geq 1$ is equivalent to $x \geq 0$, I suppose that "$x \in [1, \infty)$" at the end of the statement is to be replaced by "$x \in [0, \infty)$". This said, here are two solutions. The first one answers the problem as set. The second one answers the problem modified so as to make it more interesting (and probably closer to the one intended).

1. $a > 1$ and b, c given, suppose the differentiable function $f : [1, \infty) \to \mathbb{R}$ satisfies

$$f(a^{\lambda^2 x}) + 2f(a^{\lambda x}) + f(a^x) = bx + c \tag{1}$$

for all $\lambda \in (0, \infty)$ and $x \in [0, \infty)$.

Taking $\lambda = 1$, we see that $4f(a^x) = bx + c$ for all $x \geq 0$ and it follows that f defined by $f(y) = \frac{1}{4}(c + b \log_a(y))$ is the only possible solution.

Conversely, let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(y) = \frac{1}{4}(c + b \log_a(y))$ for $y \geq 1$. Then f is differentiable and a simple calculation gives

$$f(a^{\lambda^2 x}) + 2f(a^{\lambda x}) + f(a^x) = \frac{bx(\lambda + 1)^2}{4} + c.$$

It readily follows that for (1) to be verified for any $\lambda > 0$ and $x \geq 0$, we must have $b = 0$.

In conclusion, there is no solution if $b \neq 0$ and one solution if $b = 0$, namely the constant function f defined by $f(y) = \frac{c}{4}$ for all $y \geq 1$.

2. We modify the problem as follows :

> Let $a \in (1, \infty)$, $\lambda \in (0, \infty)$ and $b, c \in \mathbb{R}$. Find all differentiable functions $f : [1, \infty) \to \mathbb{R}$ such that
>
> $$f(a^{\lambda^2 x}) + 2f(a^{\lambda x}) + f(a^x) = bx + c$$
>
> for all $x \in [0, \infty)$.

We prove that the unique solution is the function f defined by

$$f(y) = \frac{b\log_a(y)}{(\lambda+1)^2} + \frac{c}{4} \tag{2}$$

for all $y \in [1, \infty)$. It is easy to check that this function satisfies all the requirements.

Conversely, let f be differentiable on $[1, \infty)$ such that (1) holds for all $x \in [0, \infty)$. If $\lambda = 1$, the calculations of part 1 show that f must be defined by $f(y) = \frac{1}{4}(c + b\log_a(y))$, in accordance with (2).

Let $g(x) = f(a^x)$ and $h(x) = g(x) + g(\lambda x) = f(a^x) + f(a^{\lambda x})$ $(x \geq 0)$. Note that h satisfies

$$h(x) + h(\lambda x) = bx + c \tag{3}$$

for all $x \in [0, \infty)$.

Suppose now that $\lambda \in (0, 1)$. Let x be an arbitrary nonnegative real number. For all nonnegative integers k, we have $h(\lambda^k x) + h(\lambda^{k+1}x) = b\lambda^k x + c$ so that

$$h(x) - h(\lambda^{2n}x) = \sum_{k=0}^{2n-1} (-1)^k \left(h(\lambda^k x) + h(\lambda^{k+1}x)\right) = bx\sum_{k=0}^{2n-1}(-\lambda)^k + c\sum_{k=0}^{2n-1}(-1)^k$$

for any $n \in \mathbb{N}$. Thus,

$$h(\lambda^{2n}x) + \frac{bx(1-(-\lambda)^{2n})}{1+\lambda} = h(x).$$

Letting n go to ∞ and because h is continuous at 0 with $h(0) = \frac{c}{2}$ ((3) with $x = 0$), we obtain $\frac{c}{2} + \frac{bx}{1+\lambda} = h(x)$.

If $\lambda > 1$, then from (3) we obtain $h(x) + h(\mu x) = b\mu x + c$ where $\mu = \frac{1}{\lambda} < 1$. A similar method then gives $h(x) = \frac{c}{2} + \frac{b\mu x}{1+\mu}$, that is, $h(x) = \frac{c}{2} + \frac{bx}{1+\lambda}$ again. We deduce that in the case $\lambda \neq 1$, we have

$$g(x) + g(\lambda x) = \frac{bx}{\lambda+1} + \frac{c}{2}.$$

Applying the above method once more easily leads to

$$g(x) = \frac{bx}{(\lambda+1)^2} + \frac{c}{4}.$$

Since x is arbitrary and $g(x) = f(a^x)$ we may conclude that f is defined by

$$f(y) = \frac{b\log_a(y)}{(\lambda+1)^2} + \frac{c}{4},$$

which completes the proof.

Note : the hypothesis "f differentiable" can be replaced by "f continuous at 1".

3997. *Proposed by Mihaela Berindeanu.*

Let a, b, c be positive numbers with product 8. Prove that

$$\frac{a^4+b^4}{c^3}+\frac{a^4+c^4}{b^3}+\frac{b^4+c^4}{a^3} \geq 64\left(\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}\right)+6.$$

We received 20 *correct and two incorrect solutions. We present the solution by the AN-anduud Problem Solving Group.*

Using the AM-GM inequality, we have

$$\begin{aligned}\sum_{cyc}\frac{a^4+b^4}{c^3} \geq \sum_{cyc}\frac{2\sqrt{a^4b^4}}{c^3} &= \sum_{cyc}\frac{a^2b^2}{c^3}+\sum_{cyc}\frac{a^2b^2}{c^3} \\ &\geq \sum_{cyc}\frac{(\frac{8}{c})^2}{c^3}+3\sqrt[3]{\frac{a^2b^2}{c^3}\cdot\frac{b^2c^2}{a^3}\cdot\frac{c^2a^2}{b^3}} \\ &= 64\left(\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}\right)+6.\end{aligned}$$

Equality holds if and only if $a=b=c=2$.

3998. *Proposed by George Apostolopoulos.*

Let a_i, $i=1,2,\ldots,n$ be positive real numbers such that $\sum_{i=1}^n a_i = n$. Prove that

$$\sum_{i=1}^n \left(\frac{a_i^3+1}{a_i^2+1}\right)^4 \geq n.$$

We received 17 submissions, all of which were correct and complete. We present two solutions.

Solution 1, by Joel Schlosberg.

For $x \in [0,\infty)$, define

$$f : x \mapsto \left(\frac{x^3+1}{x^2+1}\right)^4$$

and

$$g : x \mapsto 2x^2-2x+1.$$

For all $x \geq 0$, clearly $(x^2+1)^4 > 0$ and

$$\begin{aligned}&(x^3+1)^4-(x^2+1)^4(2x^2-2x+1) = \\ &x(x-1)^2(x^9+2x^8+x^7+6x^6+2x^5+6x^4+5x^2+1+(x-1)^2) \geq 0\end{aligned}$$

so $f(x) \geq g(x)$. Since $g''(x) = 4 > 0$, g is convex.

By Jensen's inequality,

$$\sum_{i=1}^{n} \left(\frac{a_i^3+1}{a_i^2+1}\right)^4 = \sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} g(a_i) \geq n \cdot g\left(\frac{\sum_{i=1}^{n} a_i}{n}\right) = n \cdot g(1) = n.$$

Note that the proof works if real $a_1, \ldots, a_n$ summing to n are nonnegative, rather than the stricter assumption that they are positive.

Solution 2, by Titu Zvonaru.

First, we will prove that for all positive real x we have the inequality

$$\frac{x^3+1}{x^2+1} \geq \sqrt{x}. \qquad (1)$$

Denoting $t = \sqrt{x}$, we obtain

$$\begin{aligned} \frac{t^6+1}{t^4+1} \geq t &\iff t^6 - t^5 - t + 1 \geq 0 \\ &\iff (t-1)(t^5-1) \geq 0 \\ &\iff (t-1)^2(t^4+t^3+t^2+t+1) \geq 0, \end{aligned}$$

which is true. Using (1) yields

$$\sum_{i=1}^{n} \left(\frac{a_i^3+1}{a_i^2+1}\right)^4 \geq \sum_{i=1}^{n} a_i^2. \qquad (2)$$

Applying Cauchy-Schwarz inequality we deduce that

$$\sum_{i=1}^{n} 1 \sum_{i=1}^{n} a_i^2 \geq \left(\sum_{i=1}^{n} a_i\right)^2 \implies n \sum_{i=1}^{n} a_i^2 \geq n^2,$$

so that

$$\sum_{i=1}^{n} a_i^2 \geq n. \qquad (3)$$

By (2) and (3) the desired inequality follows.

3999. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Consider real numbers a, b, c such that $a \geq 1 \geq b \geq c > -3$ and

$$ab + bc + ca = 3.$$

Prove that $a + b + c \geq 3$.

We received 11 submissions, of which 10 were correct and complete. We present two solutions.

Solution 1, by Michel Bataille, slightly modified by the editor.

By expanding and rearranging $(x-y)^2+(y-z)^2+(z-x)^2 \geq 0$, we get that $x^2+y^2+z^2 \geq xy+yz+zx$ for all real numbers x, y, z. Hence we have

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca) = 9. \qquad (1)$$

Thus, it is sufficient to prove that $b+c \geq -1$, since then $a+b+c \geq 0$ (as $a \geq 1$), which combined with (1) gives $a+b+c \geq 3$, as desired.

Towards contradiction, assume that $b+c < -1$. We will recursively construct a sequence $\{\alpha_n\}$ such that $c < \alpha_n$ for all n. To start with, let $\alpha_0 = -\frac{1}{2}$. Since $b+c < -1$ and $b \geq c$, we must have $c < -\frac{1}{2} = \alpha_0$; also note that $b+c < 2\alpha_0$.

Now, suppose that $b+c < 2\alpha_n$ for some α_n a negative real number ($n \geq 0$). Then

$$a(b+c) < 2a\alpha_n \leq 2\alpha_n,$$

hence

$$3 = bc + a(b+c) < bc + 2\alpha_n.$$

Since $bc > 0$ and $\frac{1}{c} < -\frac{1}{3}$, we deduce that

$$b = (bc)\left(\frac{1}{c}\right) < -\frac{1}{3}(bc) < -\frac{1}{3}(3-2\alpha_n) = \alpha_{n+1}$$

where α_{n+1} is the negative real number defined by $\alpha_{n+1} = \frac{2\alpha_n}{3} - 1$. We conclude that $b, c \leq \alpha_n$ for all positive integers n, where the sequence $\{\alpha_n\}$ is defined by $\alpha_0 = -\frac{1}{2}$ and the recursion $\alpha_{n+1} = \frac{2\alpha_n}{3} - 1$ for $n \geq 0$.

Using the fact that this recursion rewrites as

$$\alpha_{n+1} + 3 = \frac{2}{3}(\alpha_n + 3),$$

we readily see that

$$\alpha_n = \frac{5}{2} \cdot \left(\frac{2}{3}\right)^n - 3$$

for $n \geq 0$. Since $\lim_{n\to\infty} \alpha_n = -3$ and $c < \alpha_n$ for any n, we would have $c \leq -3$, in contradiction with the hypothesis.

Solution 2, by Digby Smith.

Using $(x+y+z)^2 \geq 3(xy+yz+zx)$, which holds for all real numbers x, y, z, we get that

$$(a+b+c)^2 \geq 3(xy+yz+zx) = 9,$$

which means that either $a+b+c \geq 3$ or $a+b+c \leq -3$. Use proof by contradiction to show that $a+b+c \geq 3$ is the only condition consistent with the problem.

Assume $a + b + c \le -3$. The given condition $a \ge 1$ implies $b + c \le -4$, whence $-3 < c \le b < -1$. Using these inequalities, as well as $b + c < 0$, we have

$$ab + bc + ca = a(b + c) + bc \le b + c + bc = (b + 1)(c + 1) - 1 < (-2)(-2) - 1 = 3.$$

Thus $a + b + c \le -3$ implies $ab + bc + ca < 3$, contradicting $ab + bc + ca = 3$. It follows that $a + b + c \ge 3$, with equality if and only if $a = b = c = 1$.

4000. *Proposed by Marcel Chiriţă.*

Let $x_1, x_2, \ldots x_n$ with $x_1 > x_2 > \ldots > x_n > 0$, $x_1 x_2 \ldots x_n = 1$ and $n \ge 3$. Show that

$$\frac{x_1^2 + x_2^2}{x_1 - x_2} \cdot \frac{x_2^2 + x_3^2}{x_2 - x_3} \cdots \frac{x_{n-1}^2 + x_n^2}{x_{n-1} - x_n} \cdot \frac{x_1^2 + x_n^2}{x_1 - x_n} > 2^{3/n}.$$

While the inequality holds as written, the right hand side was supposed to be the much stronger $2^{3n/2}$; we apologize for the error. As a consequence however, the solutions we received (11 in total) varied greatly in their approach. The strength of the proven results also varied, with the right hand side being variously replaced by $2^{n/3}$, 2^{n-1}, 2^n or $(2 + 2\sqrt{2})^{n-1}$, as well as the intended $2^{3n/2}$.

We present two of the solutions.

Solution 1, by Arkady Alt, modified by the editor. This solution had the best bound.

For $1 \le k \le n - 1$, let $t_k := \frac{x_k}{x_{k+1}}$. Note that $t_k > 1$, and

$$\frac{x_k^2 + x_{k+1}^2}{x_k - x_{k+1}} = \frac{x_{k+1}^2 \left(\frac{x_k^2}{x_{k+1}^2} + 1\right)}{x_{k+1} \left(\frac{x_k}{x_{k+1}} - 1\right)} = x_{k+1} \cdot \frac{\left(\frac{x_k}{x_{k+1}}\right)^2 + 1}{\frac{x_k}{x_{k+1}} - 1} = x_{k+1} \cdot \frac{t_k^2 + 1}{t_k - 1}.$$

Let $a := \frac{x_n}{x_1}$ and note $0 < a < 1$; then

$$\frac{x_1^2 + x_n^2}{x_1 - x_n} = \frac{x_1^2(1 + a^2)}{x_1(1 - a)} = x_1 \cdot \frac{1 + a^2}{1 - a}.$$

Since $x_1 x_2 \ldots x_n = 1$, it follows that

$$\prod_{k=1}^{n-1} \frac{x_k^2 + x_{k+1}^2}{x_k - x_{k+1}} \cdot \frac{x_1^2 + x_n^2}{x_1 - x_n} = \prod_{k=1}^{n-1} \frac{t_k^2 + 1}{t_k - 1} \cdot \frac{a^2 + 1}{1 - a}. \qquad (1)$$

For any $t > 1$ we claim that $\frac{t^2+1}{t-1} \ge 2\sqrt{2} + 2$.
To see this, note that

$$\begin{aligned} t^2 + 1 - (2\sqrt{2} + 2)(t - 1) &= t^2 + 1 - 2\sqrt{2}t - 2t + 2\sqrt{2} + 2 \\ &= (t - \sqrt{2} - 1)^2 \ge 0, \end{aligned}$$

whence, dividing by $t - 1$, we can obtain the claimed inequality.

On the other hand, for $0 < a < 1$, it is easy to check that $\frac{a^2+1}{1-a} > 1$.

In (1), we had $t_k > 1$ and $a < 1$, so using the last two observations, we get

$$\prod_{k=1}^{n-1} \frac{x_k^2 + x_{k+1}^2}{x_k - x_{k+1}} \cdot \frac{x_1^2 + x_n^2}{x_1 - x_n} > (2\sqrt{2} + 2)^{n-1}.$$

Solution 2, by Joel Schlosberg, slightly modified by the editor. This solution addressed the question of when the minimum is achieved.

Suppose $a, b \in \mathbb{R}$ with $0 < a < b$. Consider the function $g(x) = \frac{(b^2+x^2)(x^2+a^2)}{x(b-x)(x-a)}$. We claim that on the interval (a, b), $g(x)$ has a unique minimum at $x = \sqrt{ab}$.

Multiplying,

$$g(x) = \frac{b^2x^2 + b^2a^2 + x^4 + x^2a^2}{x(bx + xa - (ba + x^2))}.$$

For $x > 0$ we can use the arithmetic mean-geometric mean inequality. In the numerator, we have

$$b^2a^2 + x^4 \geq 2\sqrt{b^2a^2 \cdot x^4} = 2bax^2$$

(with equality if and only if $x = \sqrt{ab}$), and in the denominator

$$ba + x^2 \geq 2\sqrt{bax^2} = 2x\sqrt{ba}$$

(once again, with equality if and only if $x = \sqrt{ab}$). Moreover, for $x \in (a, b)$ we have that the denominator

$$x(b - x)(x - a) = x(bx + xa - (ba + x^2)) > 0$$

and hence

$$g(x) \geq \frac{b^2x^2 + x^2a^2 + 2bax^2}{x(bx + xa - 2x\sqrt{ba})} = \frac{b^2 + a^2 + 2ba}{b + a - 2\sqrt{ba}}.$$

Note that the right-hand side does not depend on x, and equality holds if and only if $x = \sqrt{ab}$. This concludes the proof that, for $x \in (a, b)$, $g(x)$ has a unique minimum at $x = \sqrt{ab}$. Note that g is differentiable on (a, b), and so $x = \sqrt{ab}$ must also be the unique critical point of g on (a, b).

Now suppose $z_1, z_n \in \mathbb{R}$ are fixed, with $z_1 > z_n > 0$, and define $f : \mathbb{R}^{n-2} \to \mathbb{R}$ by

$$f : (z_2, \ldots, z_{n-1}) \to \frac{1}{z_1 \cdot \ldots \cdot z_n} \cdot \frac{z_1^2 + z_2^2}{z_1 - z_2} \cdot \ldots \cdot \frac{z_{n-1}^2 + z_n^2}{z_{n-1} - z_n}.$$

Consider the domain

$$D := \{(t_2, \ldots, t_{n-1}) \in \mathbb{R}^{n-2} : z_1 > t_2 > \ldots > t_{n-1} > z_n\}.$$

As $(z_2, \ldots, z_{n-1})$ approaches the boundary of D, some $z_{i-1} - z_i \to 0$, and so $f \to +\infty$. Since f is real-valued and continuous on the open domain D, an absolute minimum value of f must be attained at a critical point. The partial derivatives of f look like the derivatives of the function g discussed at the beginning of the proof. Hence a critical point of f must satisfy $z_i = \sqrt{z_{i-1}z_{i+1}}$ for $i = 2, \ldots, n-1$; that is, the sequence $\{z_1, \ldots, z_n\}$ is a geometric progression.

Now suppose $x_1, \ldots, x_n$ are as given in the question, with $x_1 > \cdots > x_n > 0$ and $x_1 \cdot \ldots \cdot x_n = 1$. Let $y_1, \ldots, y_n$ be the geometric progression with $y_1 = x_1$, $y_n = x_n$. Denote the common ratio $\sqrt[n-1]{\frac{x_1}{x_n}}$ by r; that is, $r > 1$ and $r = y_{i-1}/y_i$ for $i = 2, \ldots, n$. Let $f : \mathbb{R}^{n-2} \to \mathbb{R}$ be as described above, with $z_1 = x_1$ and $z_n = x_n$. Then, since $x_1 \ldots x_n = 1$,

$$\begin{aligned}\frac{x_1^2+x_2^2}{x_1-x_2} \cdot \ldots \cdot \frac{x_{n-1}^2+x_n^2}{x_{n-1}-x_n} \cdot \frac{x_1^2+x_n^2}{x_1-x_n} &= \frac{1}{x_1 \cdot \ldots \cdot x_n} \cdot \frac{x_1^2+x_2^2}{x_1-x_2} \cdot \ldots \cdot \frac{x_{n-1}^2+x_n^2}{x_{n-1}-x_n} \cdot \frac{x_1^2+x_n^2}{x_1-x_n} \\ &= f(x_2, \ldots, x_{n-1}) \\ &\geq f(y_2, \ldots, y_{n-1}).\end{aligned}$$

Using the fact that $y_{i-1} = ry_i$ for $i = 2, \ldots, n$, and $x_1 = y_1$, $x_n = y_n$, we have

$$\begin{aligned}f(y_2, \ldots, y_{n-1}) &= \frac{1}{y_1 \cdots y_n} \cdot \frac{y_1^2 + \frac{y_1^2}{r^2}}{y_1 - \frac{y_1}{r}} \cdot \ldots \cdot \frac{y_{n-1}^2 + \frac{y_{n-1}^2}{r^2}}{y_{n-1} - \frac{y_{n-1}}{r}} \cdot \frac{(r^{n-1}y_n)^2 - y_n^2}{r^{n-1}y_n - y_n} \\ &= \frac{1}{y_1 \cdots y_n} \cdot \frac{y_1\left(1+\frac{1}{r^2}\right)}{1-\frac{1}{r}} \cdot \ldots \cdot \frac{y_{n-1}\left(1+\frac{1}{r^2}\right)}{1-\frac{1}{r}} \cdot \frac{y_n(r^{2(n-1)}-1)}{r^{n-1}-1} \\ &= \left(\frac{r^2+1}{r(r-1)}\right)^{n-1} \cdot \frac{r^{2(n-1)}+1}{r^{n-1}-1}.\end{aligned}$$

We now find a lower bound for this formula in r; this bound will not be sharp. Replace r^2+1 by r^2-1, and use Mahler's inequality (which is that for positive $u_1, \ldots, u_n$, $v_1, \ldots, v_n$, $\sqrt[n]{\prod_{i=1}^n (u_i + v_i)} \geq \sqrt[n]{\prod_{i=1}^n u_i} + \sqrt[n]{\prod_{i=1}^n v_i}$). That is,

$$\begin{aligned}\left(\frac{r^2+1}{r(r-1)}\right)^{n-1} \cdot \frac{r^{2(n-1)}+1}{r^{n-1}-1} &> \left(\frac{r^2-1}{r(r-1)}\right)^{n-1} \cdot \frac{r^{2(n-1)}-1}{r^{n-1}-1} \\ &= \underbrace{\left(1+r^{-1}\right) \ldots \left(1+r^{-1}\right)}_{n-1 \text{ factors}}(1+r^{n-1}) \\ &\geq \left(\sqrt[n]{1^{n-1}1} + \sqrt[n]{(r^{-1})^{n-1}r^{n-1}}\right)^n = 2^n > 2^{3/n}.\end{aligned}$$

This proves the inequality with the sharper right-hand side bound 2^n.

Editor's comments. Instead of approximating as shown, the second proof could be finished similarly to the first, obtaining the improved bound $(2+2\sqrt{2})^{n-1}$.