THE CONTEST CORNER

No. 40

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

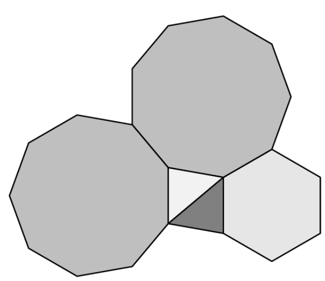
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

CC196. On dispose de neuf tuiles carrées dont les côtés ont des longueurs respectives de 1, 4, 7, 8, 9, 10, 14, 15 et 18 unités. Ces tuiles sont juxtaposées pour paver la surface d'un rectangle. Déterminer les longueurs des côtés du rectangle et montrer comment les tuiles doivent être placées.

CC197. On choisit au hasard deux entiers a et b, pas nécessairement distincts, parmi les entiers de 1 à 100. Quelle est la probabilité pour que le chiffre des unités du nombre $3^a + 7^b$ soit un 6 ?

CC198. La figure suivante est formée de cinq polygones, soit deux triangles, un hexagone régulier et deux ennéagones réguliers, placés de manière que certains polygones partagent un côté.



Démontrer que chacun des triangles est isocèle.

CC199. Étant donné un nombre réel u, soit $\{u\} = u - \lfloor u \rfloor$ où $\lfloor u \rfloor$ représente le plus grand entier inférieur ou égal à u. ($\{u\}$ est parfois appelé la partie fractionnaire de u.) Par exemple, $\{\pi\} = \pi - 3$ et $\{-2,4\} = -2,4 - (-3) = 0,6$. Déterminer tous les nombres réels x tels que $\{(x+1)^3\} = \{x^3\}$.

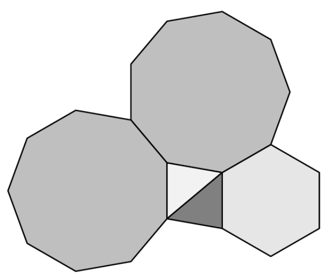
CC200. Déterminer tous les entiers positifs m et n tels que $m! + 76 = n^2$. (On rappelle que $m! = m \times (m-1) \times \cdots \times 2 \times 1$.)

. . . .

CC196. You are given nine square tiles, with sides of lengths 1, 4, 7, 8, 9, 10, 14, 15 and 18 units, respectively. They can be used to tile a rectangle without gaps or overlaps. Find the lengths of the sides of the rectangle, and show how to arrange the tiles.

CC197. Let a and b be two randomly chosen positive integers (not necessarily distinct) such that $a, b \leq 100$. What is the probability that the units digit of $3^a + 7^b$ is 6 ?

CC198. The diagram shows five polygons placed together edge to edge : two triangles, a regular hexagon and two regular nonagons.



Prove that each of the triangles is isosceles.

CC199. For any real number u, let $\{u\} = u - \lfloor u \rfloor$ denote the fractional part of u (here, $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u). For example, $\{\pi\} = \pi - 3$ and $\{-2.4\} = -2.4 - (-3) = 0.6$. Find all real numbers x such that $\{(x+1)^3\} = \{x^3\}$.

CC200. Find all positive integers m and n such that $m! + 76 = n^2$, where $m! = m \times (m-1) \times \cdots \times 2 \times 1$.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2014 : 40(10), p. 413–414. All the problems in this section are from Savin's tournament "Mathematics 6-8", as printed in Kvant 2014 (1).

CC146. Determine the number of integer solutions (x, y) to the equation

$$xy = x + y + 999,999,999.$$

Originally problem #13 by Galperin.

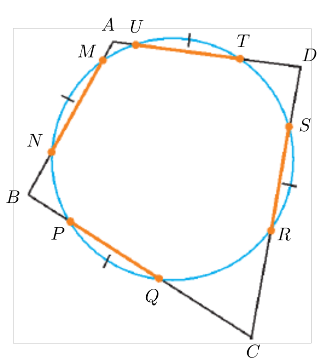
There were five correct solutions for this problem and two incorrect solutions. We present the solution by Joel Schlosberg.

The given equation is equivalent to

$$1,000,000,000 = xy - x - y + 1 = (x - 1)(y - 1),$$

so $x-1 \mid 10^9$ and $y = 10^9/(x-1)+1$. Conversely, if $d \mid 10^9$, $(x, y) = \big(d + 1, 10^9/d + 1\big)$ is a solution. Therefore, the number of integer solutions (x, y) equals the number of integer divisors, *not* necessarily positive, of $10^9 = (2\cdot 5)^9$. By unique factorization, those are of the form $\pm 2^a 5^b$ for $a, b \in \{0, 1, 2, \dots , 9\}$, so they number $2\cdot 10^2 = 200$.

CC147. A circle intersects every side of a quadrilateral in such a way that the sides of the quadrilateral cut away equal length arcs.



Show that you can inscribe a circle into this quadrilateral.

Originally problem #1 by Dvoryaninov.

We received three solutions, two of which were correct and complete. We present the solution by Titu Zvonaru.

We clearly have $MN = TU$. By the power of point A with respect to the circle, we have

$$\begin{aligned} AM \cdot AN &= AU \cdot AT \\ AM(AM + MN) &= AU(AU + AT) \\ 0 &= (AM - AU)(AM + AU) + MN(AM - AU) \\ 0 &= (AM - AU)(AM + AU + MN), \end{aligned}$$

and so $AM = AU$. Analogously, $DT = DS$, $CQ = CR$, $BN = BP$.

Therefore, we have

$$\begin{aligned} AB + CD &= MU + RS + AM + BN + DS + CR \\ &= TU + PQ + AU + DT + BQ + CQ \\ &= AD + BC \end{aligned}$$

and hence the quadrilateral has an incircle.

Editor's Comment. There is a simpler and more general solution that proceeds as follows. Let O be the center, R be the radius, and $2A$ the common length of the four chords. A chord of length $2A$ is tangent to a circle centered at O with radius $\sqrt{R^2 - A^2}$, so this is the incircle. This establishes concentricity and generalizes to polygons, simple or crossed, of any number of sides.

CC148. Using cubes of size $1 \times 1 \times 1$, Amanda puts together a rectangular brick of size $6 \times 10 \times 15$. How many little cubes does the main diagonal of the big brick cross?

Originally problem #9, folklore.

We received no solutions for this problem.

CC149. Find all positive numbers x, y, z such that for any triangle with side lengths a, b, c there exists a triangle with sides ax, by, cz.

Originally problem #8 by Dvoryaninov.

We received only one correct solution. We present the solution by Digby Smith.

There exists triangle ABC with side lengths a, b, c if and only if the following "triangle" inequalities all hold, $a + b > c$, $b + c > a$ and $c + a > b$.

Suppose a, b, c, k are positive real numbers with $a+b > c$, $b+c > a$, and $c+a > b$. It follows that $ka + kb > kc$, $kb + kc > ka$ and $kc + ka > kb$. That is given a triangle with side lengths a, b, c there exists a triangle with side lengths ka, kb, kc for all positive real numbers k. So if $x = y = z$ we are satisfied.

Let x, y, z be real numbers and suppose without loss of generality that $x > y$. Since $x > y$ it then follows that $x - y > 0$ and there exists $n \in \mathbb{N}$ such that $n(x - y) > z$ with $nx > ny + z$. Let $a = n$, $b = n$ and $c = 1$. There exists a triangle with side lengths a, b, c but there does not exist a triangle with side lengths ax, by, cz.

It follows that only when $x = y = z$ does there exist a triangle with side lengths ax, by, cz for all triangles with side lengths a, b, c.

CC150. Shane writes down all numbers from 1 to 2015 in red and blue pen. The largest blue number is equal to the number of blue numbers ; the smallest red number is equal to half the number of red numbers. How many red numbers did Shane write down ?

Originally problem #6 by Dvoryaninov.

We received five correct solutions. We present the solution given by Fernando Ballesta Yagüe.

The largest blue number is equal to the number of blue numbers. Therefore, if b is the number of blue numbers, then b is the largest blue number, and every number smaller than it is also blue (because, if not, then there would not be b blue numbers with b being the largest one). Therefore, the rest of numbers, that is, from $b + 1$ to 2015 are red.

$$\overbrace{1, 2, \ldots, b}^{b}, \overbrace{b + 1, b + 2, \ldots, 2015}^{r}$$

So the smallest red number is $b+1$. Call the number of red numbers $r = 2015 - b$. Then, as the smallest red number is equal to half the number of red numbers, we can deduce :

$$b + 1 = \tfrac{r}{2} \to b + 1 = \tfrac{2015-b}{2} \to 2b + 2 = 2015 - b \to$$

$$3b = 2013 \to b = 671 \to r = 2015 - 671 = 1344$$

There are 1344 red numbers.