CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2014 : 40(10), p. 413–414. All the problems in this section are from Savin's tournament "Mathematics 6-8", as printed in Kvant 2014 (1).

CC146. Determine the number of integer solutions (x, y) to the equation

$$xy = x + y + 999,999,999.$$

Originally problem #13 by Galperin.

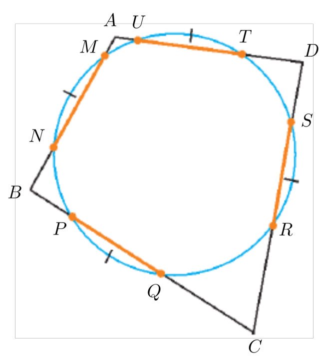
There were five correct solutions for this problem and two incorrect solutions. We present the solution by Joel Schlosberg.

The given equation is equivalent to

$$1,000,000,000 = xy - x - y + 1 = (x - 1)(y - 1),$$

so $x-1 \mid 10^9$ and $y = 10^9/(x-1)+1$. Conversely, if $d \mid 10^9$, $(x, y) = \big(d + 1, 10^9/d + 1\big)$ is a solution. Therefore, the number of integer solutions (x, y) equals the number of integer divisors, *not* necessarily positive, of $10^9 = (2 \cdot 5)^9$. By unique factorization, those are of the form $\pm 2^a 5^b$ for $a, b \in \{0, 1, 2, \dots, 9\}$, so they number $2 \cdot 10^2 = 200$.

CC147. A circle intersects every side of a quadrilateral in such a way that the sides of the quadrilateral cut away equal length arcs.



Show that you can inscribe a circle into this quadrilateral.

Originally problem #1 by Dvoryaninov.

We received three solutions, two of which were correct and complete. We present the solution by Titu Zvonaru.

We clearly have $MN = TU$. By the power of point A with respect to the circle, we have

$$\begin{aligned} AM \cdot AN &= AU \cdot AT \\ AM(AM + MN) &= AU(AU + AT) \\ 0 &= (AM - AU)(AM + AU) + MN(AM - AU) \\ 0 &= (AM - AU)(AM + AU + MN), \end{aligned}$$

and so $AM = AU$. Analogously, $DT = DS$, $CQ = CR$, $BN = BP$.

Therefore, we have

$$\begin{aligned} AB + CD &= MU + RS + AM + BN + DS + CR \\ &= TU + PQ + AU + DT + BQ + CQ \\ &= AD + BC \end{aligned}$$

and hence the quadrilateral has an incircle.

Editor's Comment. There is a simpler and more general solution that proceeds as follows. Let O be the center, R be the radius, and $2A$ the common length of the four chords. A chord of length $2A$ is tangent to a circle centered at O with radius $\sqrt{R^2 - A^2}$, so this is the incircle. This establishes concentricity and generalizes to polygons, simple or crossed, of any number of sides.

CC148. Using cubes of size $1 \times 1 \times 1$, Amanda puts together a rectangular brick of size $6 \times 10 \times 15$. How many little cubes does the main diagonal of the big brick cross?

Originally problem #9, folklore.

We received no solutions for this problem.

CC149. Find all positive numbers x, y, z such that for any triangle with side lengths a, b, c there exists a triangle with sides ax, by, cz.

Originally problem #8 by Dvoryaninov.

We received only one correct solution. We present the solution by Digby Smith.

There exists triangle ABC with side lengths a, b, c if and only if the following "triangle" inequalities all hold, $a + b > c$, $b + c > a$ and $c + a > b$.

Suppose a, b, c, k are positive real numbers with $a+b > c$, $b+c > a$, and $c+a > b$. It follows that $ka + kb > kc$, $kb + kc > ka$ and $kc + ka > kb$. That is given a triangle with side lengths a, b, c there exists a triangle with side lengths ka, kb, kc for all positive real numbers k. So if $x = y = z$ we are satisfied.

Let x, y, z be real numbers and suppose without loss of generality that $x > y$. Since $x > y$ it then follows that $x - y > 0$ and there exists $n \in \mathbb{N}$ such that $n(x - y) > z$ with $nx > ny + z$. Let $a = n$, $b = n$ and $c = 1$. There exists a triangle with side lengths a, b, c but there does not exist a triangle with side lengths ax, by, cz.

It follows that only when $x = y = z$ does there exist a triangle with side lengths ax, by, cz for all triangles with side lengths a, b, c.

CC150. Shane writes down all numbers from 1 to 2015 in red and blue pen. The largest blue number is equal to the number of blue numbers ; the smallest red number is equal to half the number of red numbers. How many red numbers did Shane write down ?

Originally problem #6 by Dvoryaninov.

We received five correct solutions. We present the solution given by Fernando Ballesta Yagüe.

The largest blue number is equal to the number of blue numbers. Therefore, if b is the number of blue numbers, then b is the largest blue number, and every number smaller than it is also blue (because, if not, then there would not be b blue numbers with b being the largest one). Therefore, the rest of numbers, that is, from $b+1$ to 2015 are red.

$$\overbrace{1, 2, \ldots, b}^{b}, \overbrace{b+1, b+2, \ldots, 2015}^{r}$$

So the smallest red number is $b+1$. Call the number of red numbers $r = 2015 - b$. Then, as the smallest red number is equal to half the number of red numbers, we can deduce :

$$b+1 = \tfrac{r}{2} \to b+1 = \tfrac{2015-b}{2} \to 2b+2 = 2015 - b \to$$
$$3b = 2013 \to b = 671 \to r = 2015 - 671 = 1344$$

There are 1344 red numbers.