SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3781. [2012 : 378, 380] *Proposed by Marcel Chiriţiţă.*

Solve the equation
$$3^{1-x} + 3^{\sqrt{3x-2x^2}} = 4\,.$$

Solved by M. Bataille ; D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru ; B. D. Beasley ; P. Deiermann ; N. Hodžić and S. Malikić ; O. Kouba ; D. Koukakis ; C. R. Pranesachar ; D. Smith ; and the proposer. There were five solutions that were either incorrect or incomplete. We present the solution composed from solutions by several solvers.

Any solution must satisfy $0 \le x \le 3/2$. Two solutions are $x = 0$ and $x = 1$. Note that when $1 < x < 3/2$, then $1 - (3x - 2x^2) = (x - 1)(2x - 1) > 0$, so that
$$3^{1-x} + 3^{\sqrt{3x-2x^2}} < 1 + 3 = 4.$$

Suppose that $4/19 < x < 1$. Then $9(3x - 2x^2) - (2 + x)^2 = (19x - 4)(1 - x) > 0$ and so
$$1 - x + 3(\sqrt{3x - 2x^2} - 1) = 3\sqrt{3x - 2x^2} - (2 + x) > 0.$$
Hence, by the arithmetic-geometric means inequality, we have
$$\begin{aligned} 3^{1-x} + 3^{\sqrt{3x-2x^2}} &= 3^{1-x} + 3^{\sqrt{3x-2x^2}-1} + 3^{\sqrt{3x-2x^2}-1} + 3^{\sqrt{3x-2x^2}-1} \\ &\ge 4\left(3^{1-x+3(\sqrt{3x-2x^2}-1)}\right)^{1/4} > 4. \end{aligned}$$

Now, suppose that $0 < x < 3/11$. Then $(3x - 2x^2) - 9x^2 = x(3 - 11x) > 0$, so that
$$\begin{aligned} 3^{1-x} + 3^{\sqrt{3x-2x^2}} &= 3^{-x} + 3^{-x} + 3^{-x} + 3^{\sqrt{3x-2x^2}} \\ &\ge 4\left(3^{-3x+\sqrt{3x-2x^2}}\right)^{1/4} > 4. \end{aligned}$$

Since $4/19 < 3/11$, we conclude that the equation has no solution in the set $(0, 1) \cup (1, 3/2)$, and so $x = 0$ and $x = 1$ are the only solutions.

Editor's Comments. There are quick arguments for some parts of the domain. Since $3x - 2x^2 > 1$ for $1/2 < x < 1$, it is easy to see that the left side of the equation exceeds 4 on this interval. Since $3^{1-x} + 3^{\sqrt{3x-2x^2}}$ strictly decreases for $3/4 \le x \le 3/2$, the only solution in this interval is $x = 1$.

Deiermann showed more generally that $x = 0$ and $x = 1$ are the only solutions of

$$b^{\sqrt{b-a}(1-x)} + b^{\sqrt{bx-ax^2}} = 1 + b^{\sqrt{b-a}}$$

when $0 < a < b \le 2a$, $1 < b$ and $c - c^{-1} \le b(b-a)^{-1}$, where $c = b^{\sqrt{b-a}}$. He did this by recasting the equation as $h(x) = g(x)$, with $h(x) = \sqrt{bx - ax^2}\ln b$ and $g(x) = \ln(1 + b^{\sqrt{b-a}} - b^{\sqrt{b-a}(1-x)})$. Noting that $h(0) = g(0) = h(b/a) = 0$ and $h(1) = g(1) = \sqrt{b-a}\ln b$, he analyzed the graphs of these two functions to show that they crossed only when $x = 0$ and $x = 1$. The above problem is the case $(a, b) = (2, 3)$.

3782. [2012 : 378, 380] *Proposed by Edward T. H. Wang and Billy Jin.*

For $n \in \mathbb{N}$, let $S = \{1, 2, 3, \ldots, n\}$. For each nonempty $T \subseteq S$ define the "drop" of T by $d(T) = f(T) - g(T)$ where $f(T)$ and $g(T)$ denote the maximum and minimum elements of T, respectively. (e.g., $d(\{2\}) = 0$, $d(\{2, 3, 7\}) = 5$) Evaluate $D_n = \sum d(T)$, the total of the drops of S, where the summation is over all non-empty subsets T of S.

Solved by AN-anduud Problem Solving Group; M. Bataille; D. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru; P. De; O. Kouba; K. Lau; S. Malikić; Missouri State University Problem Solving Group; C. R. Pranesachar; D. Smith; E. Suppa; I. Uchiha; and the proposers. We present the solution by Itachi Uchiha slightly expanded by the editor.

Note first that each $i \in S$ is the maximum element of 2^{i-1} subsets of S and the minimum element of 2^{n-i} subsets of S. Hence,

$$\begin{aligned} D_n = \sum d(T) &= \sum f(T) - \sum g(T) = \sum_{i=1}^{n} i2^{i-1} - \sum_{i=1}^{n} i2^{n-i} \\ &= \sum_{i=1}^{n} i2^{i-1} - \sum_{i=0}^{n-1} (n-i)2^i = 3\sum_{i=1}^{n} i2^{i-1} - n\sum_{i=0}^{n} 2^i. \end{aligned}$$

For $x \ne 1$, define

$$h(x) = \frac{1 - x^{n+1}}{1 - x} = 1 + x + x^2 + \cdots + x^n. \tag{1}$$

Now we can write

$$D_n = 3h'(2) - nh(2). \tag{2}$$

Since $h'(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}$ and $xh'(x) = x + 2x^2 + 3x^3 + \cdots + nx^n$, we have

$$(1-x)h'(x) = 1 + x + x^2 + \cdots + (n-1)x^{n-1} - nx^n.$$

Or, in other words,

$$h'(x) = \frac{1 - x^n}{(1-x)^2} - \frac{nx^n}{1-x}. \tag{3}$$

Now, by substituting (1) and (3) into (2), we conclude that

$$\begin{aligned} D_n &= 3(1-2^n+n2^n)+n(1-2^{n+1}) \\ &= (n-3)2^n+n+3. \end{aligned}$$

Editor's Comment. Using a counting argument, Lau established the formula $D_n = \sum_{k=1}^{n-1} k(n-k)2^{k-1}$. *AN-anduud Problem Solving Group obtained the recurrence formula* $D_n = D_{n-1} + (n-2)2^{n-1} + 1$. *Prithwijit gave the recurrence* $D_n = D_{n-1} + \sum_{k=1}^{n-1} k2^{k-1}$ *without proof.*

3783. [2012 : 378, 380] *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers. Prove that

$$(3a^2+2)\frac{a^3+b^3}{a^2+ab+b^2} + (3b^2+2)\frac{b^3+c^3}{b^2+bc+c^2} + (3c^2+2)\frac{c^3+a^3}{c^2+ca+a^2} \geq 10abc\,.$$

Solved by A. Alt; AN-anduud Problem Solving Group; Š. Arslanagić; D. Bailey, E. Campbell, and C. Diminnie; M. Bataille; D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru (2 solutions); P. De; C. M. Quang; N. Hodžić; O. Kouba; S. Malikić; P. Perfetti; C. R. Pranesachar; D. Smith; D. Văcaru; S. Wagon; and the proposer. We present the solution by AN-anduud Problem Solving Group.

Let L denote the left side of the given inequality. Since

$$3(x^2-xy+y^2)-(x^2+xy+y^2) = 2(x^2-2xy+y^2) = 2(x-y)^2 \geq 0,$$

we have

$$\frac{x^2-xy+y^2}{x^2+xy+y^2} \geq \frac{1}{3}. \tag{1}$$

Using (1), the condition that $ab+bc+ca=3$ and the AM-GM Inequality, we have that

$$\begin{aligned} L &= \sum_{\text{cyclic}} (3a^2+2)(a+b)\left(\frac{a^2-ab+b^2}{a^2+ab+b^2}\right) \\ &\geq \frac{1}{3}\sum_{\text{cyclic}} (3a^2+2)(a+b) \\ &= \sum_{\text{cyclic}} a^3 + \sum_{\text{cyclic}} a^2b + \frac{4}{3}\left(\sum_{\text{cyclic}} a\right)\left(\frac{ab+bc+ca}{3}\right) \\ &\geq 3\sqrt[3]{a^3b^3c^3} + 3\sqrt[3]{(a^2b)(b^2c)(c^2a)} + 4\sqrt[3]{abc}\cdot\sqrt[3]{(ab)(bc)(ca)} \\ &= 3abc+3abc+4abc \\ &= 10abc. \end{aligned}$$

Editor's Comment. Almost all the submitted solutions are similar to the one featured above. Both Alt and Arslanagić gave the counterexample $a=b=c=2$ *to* *disprove the original incorrect version and both gave a variant, with proof, of the original inequality by replacing the left side with*

$$\sum(3a^2+2b^2)\left(\frac{a^3+b^3}{a^2-ab+b^2}\right).$$

Wagon's proof was based on using Mathematica's FindInstance.

Arslanagić, Bailey, Campbell, Diminnie, Bătineţu-Giurgiu, Stanciu, and Zvonaru all gave another variant, with proof, in which the condition is $a+b+c=3$. In addition, Kouba gave a variant, with proof, in which the condition is $abc=1$. Bătineţu-Giurgiu, Stanciu, and Zvonaru gave the following two generalizations :

1. *If $a,b,c,m,n \in (0,\infty)$ such that $ab+bc+ca=3$, then*

$$\sum(ma^2+n)\left(\frac{a^3+b^3}{a^2+ab+b^2}\right) \geq 2(m+n)abc.$$

2. *If $a,b,c,m,n,k \in (0,\infty)$ such that $ab+bc+ca \leq k$, then*

$$\sum(ma^2+n)\left(\frac{a^3+b^3}{a^2+ab+b^2}\right) \geq \left(\frac{2(mk+3n)}{k}\right)abc.$$

3784. [2012 : 378, 380] *Proposed by Constantin Mateescu.*

Let ABC be a triangle with circumradius R, inradius r and semiperimeter s for which we denote $Q=\sum_{\text{cyclic}}\cos\left(\frac{A}{2}\right)$. Prove that

$$s=2Q\left(\sqrt{R^2Q^2-Rr}-2R\right) .$$

Solved by A. Alt ; M. Bataille ; K. Lau ; S. Malikić ; C. R. Pranesachar ; P. Y. Woo ; T. Zvonaru ; and the proposer. We present the solution by Kee-Wai Lau, modified slightly by the editor.

The following well-known identities can be found as entries 56, 57, and 58 in *Recent Advances in Geometric Inequalities* by D.S. Mitrinović, J.E. Pečarić, and V. Volenec (Kluwer Academic Publishers, The Netherlands, 1989) :

$$\sum_{\text{cyclic}}\cos^2\left(\frac{A}{2}\right)=\frac{4R+r}{2R},$$

$$\sum_{\text{cyclic}}\cos^2\left(\frac{A}{2}\right)\cos^2\left(\frac{B}{2}\right)=\frac{s^2+(4R+r)^2}{16R^2},$$

$$\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)=\frac{s}{4R}.$$

Hence,

$$Q^2 = \left(\sum_{\text{cyclic}} \cos\left(\frac{A}{2}\right) \right)^2 = \frac{4R+2}{2R} + 2 \sum_{\text{cyclic}} \cos\left(\frac{A}{2}\right) \cos\left(\frac{B}{2}\right) \qquad (1)$$

and

$$\begin{aligned}(2RQ^2 - (4R+r))^2 &= \left(4R \sum_{\text{cyclic}} \cos\left(\frac{A}{2}\right) \cos\left(\frac{B}{2}\right) \right)^2 \\ &= 16R^2 \left(\sum_{\text{cyclic}} \cos\left(\frac{A}{2}\right) \cos\left(\frac{B}{2}\right) \right)^2 \\ &= 16R^2 \left(\frac{s^2 + (4R+r)^2}{16R^2} + 2\left(\frac{s}{4R}\right) Q \right) \\ &= s^2 + (4R+r)^2 + 8RQs \qquad (2)\end{aligned}$$

Simplifying (2), we obtain

$$\begin{aligned}&4R^2Q^4 - 4RQ^2(4R+r) = s^2 + 8RQs \quad \text{or} \\ &s^2 + 8RQs + 4RQ^2(4R+r) - 4R^2Q^4 = 0. \qquad (3)\end{aligned}$$

The discriminant of the quadratic equation $f(s) = 0$ in (3) is

$$\begin{aligned}D &= 64R^2Q^2 - 16RQ^2(4R+r) + 16R^2Q^4 \\ &= 16R^2Q^4 - 16rRQ^2 \\ &= 16RQ^2(RQ^2 - r) \qquad (4)\end{aligned}$$

From (1) it is clear that $Q^2 > 2$. Furthermore, $R \geq 2r$ by Euler's formula. Hence, from (4), $RQ^2 - r > 2R - r > 0$, so $D > 0$. Therefore, $f(s)$ has two real roots given by

$$s = \frac{1}{2}\left(-8RQ \pm \sqrt{16RQ^2(RQ^2 - r)}\right) = 2Q\left(\pm\sqrt{R^2Q^2 - Rr} - 2R\right).$$

Rejecting the negative root, we finally have $s = 2Q\left(\sqrt{R^2Q^2 - Rr} - 2R\right)$, which completes the proof.

3785. [2012 : 378, 380] *Proposed by Václav Koneçný.*

Consider an ellipse $\mathcal{E}$ given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$. Find the coordinates, in the first quadrant, of the point P on $\mathcal{E}$ such that the acute angle θ between the tangent t to $\mathcal{E}$ at P and the line OP is minimized.

Solved by A. Alt ; G. Apostolopoulos ; Š. Arslanagić ; M. Bataille ; C. Curtis ; O. Geupel ; O. Kouba ; S. Malikić ; Missouri State University Problem Solving Group ; D. Smith ; I. Uchiha ; and the proposer. In addition, three submissions were incorrect and one was incomplete. We present a composite of solutions by Salem Malikić and Itachi Uchiha.

Let (x, y) be the coordinates of a variable point P of the ellipse in the first quadrant ; that is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x, y > 0$. The line OP has slope $\frac{y}{x}$ while the tangent to the ellipse at P has slope -$\frac{b^2x}{a^2y}$, so the acute angle between these lines satisfies (with the help of the AM-GM inequality)

$$\tan\theta = \frac{\frac{y}{x} - \left(-\frac{b^2x}{a^2y}\right)}{1 - \frac{b^2}{a^2}} = \frac{1}{a^2 - b^2}\left(\frac{a^2y}{x} + \frac{b^2x}{y}\right) \geq \frac{2ab}{a^2 - b^2}.$$

Equality holds if and only if $\frac{a^2y}{x} = \frac{b^2x}{y}$; that is, if and only if $\frac{x^2}{a^2} = \frac{y^2}{b^2}$. But in the equation of the ellipse these equal fractions sum to 1, so they must each equal $\frac{1}{2}$. Since the tangent function is strictly increasing, $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ is the point where the acute angle θ achieves its minimum.

3786. [2012 : 379, 380] *Proposed by Mehmet Şahin.*

Let ABC be a triangle with medians m_a, m_b and m_c, circumradius R and inradius r. Let P be the point of intersection of the bisector of $\angle A$ and the median from B, Q be the point of intersection of the bisector of $\angle B$ and the median from C, and R be the point of intersection of the bisector of $\angle C$ and the median from A. If $\angle APB = \alpha$, $\angle BQC = \beta$ and $\angle CRA = \gamma$, prove that

$$\frac{m_a m_b m_c \sin\alpha\sin\beta\sin\gamma}{(a + 2b)(b + 2c)(c + 2a)} = \frac{r}{32R} \ .$$

Solved by A. Alt ; AN-anduud Problem Solving Group ; G. Apostolopoulos ; M. Bataille ; C. Curtis ; J. G. Heuver ; D. Koukakis ; S. Malikić ; C. R. Pranesachar ; D. Văcaru ; P. Y. Woo ; T. Zvonaru ; and the proposer. We present a composite solution.

Let M be the midpoint of AC. Because AP bisects $\angle A$ in ΔABM, we have $\frac{BP}{PM} = \frac{AB}{AM} = \frac{c}{b/2}$, whence

$$\frac{BP}{2c} = \frac{PM}{b} = \frac{BP + PM}{2c + b} = \frac{m_b}{b + 2c}.$$

From the sine law applied to ΔABP, $\frac{BP}{c} = \frac{\sin A/2}{\sin\alpha}$, which yields

$$\frac{m_b \sin\alpha}{b + 2c} = \frac{1}{2}\sin\frac{A}{2}.$$

Similarly,

$$\frac{m_c \sin\beta}{c+2a} = \frac{1}{2}\sin\frac{B}{2} \quad \text{and} \quad \frac{m_a \sin\gamma}{a+2b} = \frac{1}{2}\sin\frac{C}{2}.$$

By multiplication it follows that

$$\frac{m_a m_b m_c \sin\alpha\sin\beta\sin\gamma}{(a+2b)(b+2c)(c+2a)} = \frac{1}{8}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

The desired result follows immediately from the identity

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R},$$

which holds for all triangles ABC.

3787. [2012 : 379, 381] *Proposed by Michel Bataille.*

Let S be a finite set with cardinality $|S| = n \geq 1$ and let k be a positive integer. Calculate

$$\sum_{(A)} |A(1) \cap A(2) \cap \cdots \cap A(k)| \text{ and } \sum_{(A)} |A(1) \cup A(2) \cup \cdots \cup A(k)|,$$

where the summation $\sum_{(A)}$ is over all mappings A from $\{1, 2, \ldots, k\}$ to the power set $\mathcal{P}(S)$.

Solved by AN-anduud Problem Solving Group; O. Geupel; O. Kouba; Missouri State University Problem Solving Group; C. R. Pranesachar; and the proposer. Skidmore College Problem Group provided correct solution but without proof. We present the solution by Oliver Geupel.

Let us introduce convenient notation :

$$f(n,k) = \sum_{(A)} |A(1) \cap A(2) \cap \cdots \cap A(k)|,$$
$$g(n,k) = \sum_{(A)} |A(1) \cup A(2) \cup \cdots \cup A(k)|.$$

We prove that

$$f(n,k) = n \cdot 2^{k(n-1)}, \tag{1}$$
$$g(n,k) = n \cdot 2^{k(n-1)}(2^k - 1). \tag{2}$$

Without loss of generality, we may assume that $S = \{1, 2, \ldots, n\}$. We let $\mathcal{F}$ denote the set of all mappings $A : \{1, 2, \ldots, k\} \to \mathcal{P}(S)$. Then for any $A \in \mathcal{F}$, we define a $k \times n$ matrix $M(A) = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 0 & \text{if } j \notin A(i), \\ 1 & \text{if } j \in A(i). \end{cases}$$

Then it is easy to check that this correspondence defines a bijection between $\mathcal{F}$ and the set of all $k \times n$ $(0,1)$-matrices.

Note that for any $j \in S$, we have $j \in A(1) \cap A(2) \cap \cdots \cap A(k)$ if and only if the j-th column of the matrix $M(A)$ is $(1,1,\ldots,1)^T$. This holds for $2^{k(n-1)}$ $(0,1)$-matrices since the entries outside of the j-th column can be chosen arbitrarily from $\{0,1\}$. Hence, every element of S contributes a portion $2^{k(n-1)}$ to $f(n,k)$ and (1) follows.

Next, for any $j \in S$, we have $j \in A(1) \cup A(2) \cup \cdots \cup A(k)$ if and only if the j-th column of $M(A)$ is distinct from $(0,0,\ldots,0)^T$. This holds for $2^{k(n-1)}(2^k - 1)$ $(0,1)$-matrices since the entries outside of the j-th column can be chosen arbitrarily from $\{0,1\}$ and exactly one of the 2^k choices for the jth column, namely $(0,0,\ldots,0)^T$, is forbidden. Thus, every element of S contributes a portion $2^{k(n-1)}(2^k - 1)$ to $g(n,k)$ and (2) follows.

3788. [2012 : 379, 381] *Proposed by Panagiote Ligouras.*

Let a, b and c be the sides of an acute-angled triangle ABC. Let H be the orthocentre, and let d_a, d_b and d_c be the distances from H to the sides BC, CA and AB, respectively. Prove that

$$\sum_{\text{cyclic}} \sqrt{\frac{1}{a^2b^2} + \frac{1}{b^2c^2} - \frac{1}{c^2a^2}} \leq \frac{9}{4(d_a + d_b + d_c)^2} \ .$$

Solved by A. Alt; G. Apostolopoulos; Š. Arslanagić; M. Bataille; J. G. Heuver; N. Hodžić and Malikić; and the proposer. We present the solution by John G. Heuver modified by the editor.

Let s, r, and R denote the semiperimeter, the inradius, and the circumradius of $\triangle ABC$, respectively. Furthermore, let L denote the summation of the left side of the given inequality, and set $\alpha = \angle A$, $\beta = \angle B$, and $\gamma = \angle C$ for notational convenience.

We shall make use of the following identities or inequalities all of which are either well-known or easy to show :

(a) $c \cos\beta + b\cos\gamma = a$

(b) $abc = 4rsR$

(c) $d_c = 2R\cos\beta\cos\gamma$

(d) $\sum \cos\beta\cos\gamma = \dfrac{s^2 + r^2 - 4R^2}{4R^2}$

(e) $s^2 \leq 4R^2 + 4rR + 3r^2$ (Gerretsen's Inequality)

(f) $2r \leq R$ (Euler's Inequality)

By the Root-Mean-Square Inequality, together with (a) and (b), we have

$$\begin{aligned}
L &= \sum_{\text{cyclic}} \sqrt{\frac{c^2+a^2-b^2}{a^2b^2c^2}} = \frac{1}{abc}\sum_{\text{cyclic}} \sqrt{2ca\cos\beta} \\
&= \frac{1}{abc}\sum_{\text{cyclic}} \frac{\sqrt{2ca\cos\beta}+\sqrt{2ab\cos\gamma}}{2} \\
&\le \frac{1}{abc}\sqrt{a(c\cos\beta+b\cos\gamma)} = \frac{1}{abc}\sum_{\text{cyclic}}\sqrt{a^2} \\
&= \frac{2s}{abc} = \frac{2s}{4rsR} = \frac{1}{2rR}. \qquad (1)
\end{aligned}$$

Next, by (c) and (d), we have

$$\sum_{\text{cyclic}} d_c = 2R\sum_{\text{cyclic}}\cos\beta\cos\gamma = 2R\left(\frac{s^2+r^2-4R^2}{4R^2}\right) = \frac{s^2+r^2-4R^2}{2R}. \qquad (2)$$

We now show that

$$\frac{1}{2rR} \le \frac{9R^2}{(s^2+r^2-4R^2)^2} \qquad (3)$$

or equivalently that

$$(s^2+r^2-4R^2)^2 \le 18rR^3. \qquad (4)$$

By (e) and (f), we have

$$(s^2+r^2-4R^2)^2 \le (r^2+4rR+3r^2)^2 = 16r^2(r+R)^2 \le 16r\left(\frac{R}{2}\right)\left(\frac{3R}{2}\right)^2 = 18rR^3.$$

So (4) holds and (3) follows.

Finally, from (1), (2), and (3), we have

$$L \le \frac{9R^2}{(s^2+r^2-4R^2)^2} = \frac{9}{4}\left(\frac{2R}{s^2+r^2-4R^2}\right)^2 = \frac{9}{4(d_e+d_b+d_c)},$$

and our proof is complete.

3789. [2012 : 379, 381] *Proposed by Michel Bataille.*

Let triangle ABC be inscribed in a circle with centre O and radius R and P be any point in its plane. Let P' be such that $\Delta PBP'$ is directly similar to ΔCOA and P'' be the reflection of P in AC. Prove that

$$P'P'' \ge \frac{2F}{R}$$

where F is the area of ΔABC. For which P does equality hold?

Solved by O. Geupel ; L. Giugiuc ; O. Kouba ; C. R. Pranesachar ; and the proposer. We present four different solutions.

Solution 1 by the proposer.

Let ρ_{MN} denote the reflection whose axis is the line MN. Since $BP = BP'$ and $\angle PBP' = \angle COA = 2\angle CBA$, it follows that $P = \rho_{BC} \circ \rho_{AB}(P')$. As a result, $P'' = g(P')$ where g is the glide reflection $\rho_{AC} \circ \rho_{BC} \circ \rho_{AB}$. Let us determine g.

Let H_1, H_2, H_3 be the feet of the altitudes from A, B, C, respectively and let $C_1 = \rho_{AB}(C)$, $B_1 = \rho_{AC}(B)$, $U = \rho_{AB}(H_1)$, $V = \rho_{AC}(H_1)$. Since $g(C_1) = C$ and $g(B) = B_1$, the axis of g passes through the respective midpoints H_3 and H_2 of C_1C and BB_1. Thus, the axis is the line H_3H_2 when $\angle BAC \neq 90°$. The homothety with centre A that transforms H_1 into the orthocentre H of triangle ABC also transforms the midpoints K, L of H_1U, H_1V to H_3, H_2 respectively. Thus, $KL \| H_3H_2$; since $UV \| KL$, we have that $UV \| H_3H_2$. But $g(U) = V$, so the midpoint of UV is on the axis H_3H_2 of G. Therefore U, V, H_2, H_3 are collinear and so $g = \rho_{UV} \circ \tau_{UV} = \tau_{UV} \circ \rho_{UV}$, where τ_{UV} denotes the translation that takes U to V. (This decomposition remains valid when $\angle BAC = 90°$.)

Since $P'' = g(P')$, we see $P'P'' \geq UV$ with equality if and only if P' and P'' are on the axis UV of g. The calculation of UV is straightforward. Since A, K, H_1, L are on the circle with diameter AH_1, we have $KL = AH_1 \cdot \sin A$, and so

$$UV = 2KL = 2\sin A \cdot \frac{2F}{BC} = \frac{2F}{R}.$$

The desired inequality follows, with equality if and only if P lies on the line H_1H_2, the reflection of UV in AC. Note that the line through H_1 is perpendicular to OC.

Solution 2 by Leonard Giugiuc.

We set the situation in the complex plane with the affixes $A(2a)$, $B(2i)$, $C(-2c)$ with a and c real and $a + c > 0$. Let $D(0)$, the foot of the altitude from A, be at the origin. The centre O of the circle is located at $(a - c) + (1 - ac)i$, as can be seen by computing its distances from A, B and C. Let $P(z)$, with $z = 2x + 2yi$, and $P'(w)$ be two vertices of triangle PBP'. Since the triangles COA and PBP' are directly similar

$$\frac{A - O}{C - O} = \frac{P' - B}{P - B},$$

so that

$$\frac{a + c + (ac - 1)i}{-(a + c) + (ac - 1)i} = \frac{w - 2i}{2[x + (y - 1)i]}.$$

Let $u = a + c$ and $v = ac - 1$. Then $u^2 + v^2 = R^2$, $2F = 4u$ and

$$w = 2i - 2[x + (y - 1)i](u + vi)^2(u^2 + v^2)^{-1}.$$

Noting that the affix of P'' is $\bar{z}$, we compute

$$\begin{aligned}\bar{z} - w &= 2[x - (y + 1)i] + 2[(u^2 + v^2)^{-1}(u + vi)^2(x + (y - 1)i)] \\ &= 4(u^2 + v^2)^{-1}[xu^2 - (y - 1)uv + (-yv^2 - u^2 + xuv)i].\end{aligned}$$

The desired inequality is equivalent to $(u^2+v^2)|\bar{z}-w|^2 \geq (4u)^2$ or

$$[x^2u^4+(y-1)^2u^2v^2-2x(y-1)u^3v]+[y^2v^4+u^4+x^2u^2v^2+2yu^2v^2-2xu^3v-2xyuv^3]$$
$$\geq u^4+u^2v^2,$$

which reduces to $(xu-yv)^2(u^2+v^2)\geq 0$. Thus, the inequality holds with equality occurring if and only if $xu=yv$, *i.e.*, P is on the line that contains D and is perpendicular to CO.

Solution 3 by Omran Kouba.

We situate the problem in the complex plane with O at the origin, $R=1$ and the vertices A, B, C at the respective points a, b, 1.

Suppose that z, z' and z'' are the respective affixes of P, P' and P''. Then $z'=b+a(z-b)$ and $z''=1+a-a\bar{z}$. (Note that $|z''-1|=|z-1|$ and $|z''-a|=|z-a|$, so that AC right bisects PP''.)

Since

$$\begin{aligned} z''-z' &= 1+a-b+ab-a(z+\bar{z}) \\ &= a(\bar{a}+1-\bar{a}b+b-2\text{Re } z), \end{aligned}$$

we have that

$$P'P''=2|v-\text{Re } z|=2\sqrt{(\text{Im } v)^2+|\text{Re } v-\text{Re } z|^2},$$

where $v=\frac{1}{2}(1+\bar{a}+b-\bar{a}b)$. Thus

$$P'P''\geq 2|\text{Im } v|$$

with equality if and only if $\text{Re } z=\text{Re } v$.

We now interpret $\text{Im } v$ and $\text{Re } v$. Recalling the formula $\frac{1}{2}\text{Im } (\bar{z}_1z_2+\bar{z}_2z_3+\bar{z}_3z_1)$ for the area of a triangle with vertices at z_1, z_2, z_3, we find that

$$F=\frac{1}{2}|\text{Im } (\bar{a}+a\bar{b}+b)|=\frac{1}{2}|\text{Im } (\bar{a}-\bar{a}b+b)|=|\text{Im } v|.$$

The image of the point B under reflection in AC has the affix $b''=1+a-a\bar{b}$, so that $\frac{1}{2}(b''+b)=\bar{v}+\frac{1}{2}(b-\bar{b})$ is the affix of the midpoint between B and its reflection AC, *i.e.*, the foot D of the perpendicular from B to AC. Thus the equation $\text{Re } z=\text{Re } v=\text{Re } \bar{v}$ represents the line perpendicular to OC that passes through D. (One way to find the reflected image of B is to perform the rotation $z\longrightarrow 1+(z-1)e^{i\theta}$ where $a=-e^{-2i\theta}$; this rotates CA onto the real axis. Reflect the image of B in the real axis and rotate back.)

In conclusion, we have proved that $P'P''\geq 2F$ with equality if and only if P belongs to the line through the foot of the altitude to AC that is orthogonal to OC. Since $R=1$, this solves the problem.

Solution 4 by Oliver Geupel.

We use Cartesian coordinates. Wolog, let the circumcircle of triangle ABC be the unit circle and let AC be parallel to the $y-$axis, so that we have $A = (\cos\alpha, \sin\alpha)$, $B = (\cos\beta, \sin\beta)$ and $C = (\cos\alpha, -\sin\alpha)$. The area F of the triangle is equal to $|\sin\alpha(\cos\alpha - \cos\beta)|$.

Let D, Q and S be, respectively, the midpoint of AC, the midpoint of PP' and the foot of the perpendicular from P to AC. Then $P'P'' = 2QS$. The triangle PBQ is obtained from the triangle COD by the following successive transformations :

(i) Scale triangle COD by a factor r to obtain triangle C_1OD_1 ;

(ii) Rotate triangle C_1OD_1 around O by an angle ϕ to obtain triangle C_2OD_2 ;

(iii) Translate C_2OD_2 by the vector $\overrightarrow{OB} = (\cos\beta, \sin\beta)$ to obtain triangle PBQ.

We compute the coordinates of various points step by step :

$D = (\cos\alpha, 0)$,

$C_1 = (r\cos\alpha, -r\sin\alpha)$,

$D_1 = (r\cos\alpha, 0)$,

$C_2 = (r\cos\alpha\cos\phi + r\sin\alpha\sin\phi, r\cos\alpha\sin\phi - r\sin\alpha\cos\phi)$,

$D_2 = (r\cos\alpha\cos\phi, r\cos\alpha\sin\phi)$,

$P = (r(\cos\alpha\cos\phi + \sin\alpha\sin\phi) + \cos\beta, r(\cos\alpha\sin\phi - \sin\alpha\cos\phi) + \sin\beta)$,

$Q = (r\cos\alpha\cos\phi + \cos\beta, r\cos\alpha\sin\phi + \sin\beta)$,

$S = (\cos\alpha, r(\cos\alpha\sin\phi - \sin\alpha\cos\phi) + \sin\beta)$.

Therefore

$$\begin{aligned}
&(P'P'')^2 = 4QS^2 \\
&= 4(r\cos\phi - \cos\alpha(\cos\alpha - \cos\beta))^2 + 4\sin^2\alpha(\cos\alpha - \cos\beta)^2 \\
&= 4(r\cos\phi - \cos\alpha(\cos\alpha - \cos\beta))^2 + \left(\frac{2F}{R}\right)^2 \geq \left(\frac{2F}{R}\right)^2.
\end{aligned}$$

This proves the required inequality. Equality occurs if and only if

$$r\cos\phi = \cos\alpha(\cos\alpha - \cos\beta),$$

that is when P lies on a line perpendicular to OC.

Editor's Comment. One solver erroneously claimed, without explanation, that the result failed to hold when $A = P = P''$. However, if we have an equilateral triangle inscribed in a unit circle, then $2F = 3 \times (\sqrt{3}/2) < 3 = P'P''$.

3790. [2012 : 379, 381] *Proposed by Ovidiu Furdui.*

Let $a, \alpha \geq 0$ be nonnegative real numbers and let β be a positive number. Determine the limit

$$L(\alpha, \beta) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^\alpha}{(n^2 + kn + a)^\beta} \ .$$

Solved by M. Bataille ; O. Geupel ; R. Hess ; A. Kotronis ; O. Kouba ; P. Perfetti ; and the proposer. We present the solution by Oliver Geupel.

We prove that

$$L(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha + 1 < 2\beta, \\ +\infty & \text{if } \alpha + 1 > 2\beta, \\ \int_0^1 \frac{x^{2\beta-1}}{(1+x)^\beta} dx & \text{if } \alpha + 1 = 2\beta. \end{cases}$$

Note that

$$\sum_{k=1}^{n} \frac{k^\alpha}{(n^2 + kn + a)^\beta} = n^{\alpha+1-2\beta} \left[\frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{\left(1 + \frac{k}{n} + \frac{a}{n^2}\right)^\beta} \right].$$

For positive a and ϵ and sufficiently large n,

$$\frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{\left(1 + \frac{k}{n} + \epsilon\right)^\beta} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{\left(1 + \frac{k}{n} + \frac{a}{n^2}\right)^\beta} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{\left(1 + \frac{k}{n}\right)^\beta}.$$

Considering Riemann sums, we have, for all $\epsilon > 0$, that

$$\int_0^1 \frac{x^\alpha dx}{(1 + x + \epsilon)^\beta} \leq \liminf \frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{\left(1 + \frac{k}{n} + \frac{a}{n^2}\right)^\beta}$$
$$\leq \limsup \frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{\left(1 + \frac{k}{n} + \frac{a}{n^2}\right)^\beta} \leq \int_0^1 \frac{x^\alpha dx}{(1 + x)^\beta}.$$

Therefore

$$L(\alpha, \beta) = \lim_{n\to\infty} (n^{\alpha+1-2\beta}) \int_0^1 \frac{x^\alpha dx}{(1 + x)^\beta}$$

and the result follows.