

## OLYMPIAD SOLUTIONS

**OC91.** Prove that no integer consisting of one 2, one 1 and the rest of digits 0 can be written neither as the sum of two perfect squares nor the sum of two perfect cubes.

*Originally question 8 from the 2011 Estonian National Olympiad.*

*Solved by O. Geupel; D. E. Manes; and T. Zvonaru. We give the solution by Titu Zvonaru.*

Since the sum of the digits of  $n$  is 3, it follows that  $n$  is divisible by 3 but not divisible by 9.

Suppose by contradiction that there exists  $a, b$  such that  $n = a^2 + b^2$ . As the quadratic residues modulo 3 are 0 and 1 and  $a^2 + b^2 \equiv 0 \pmod{3}$  it follows that

$$a \equiv b \equiv 0 \pmod{3}.$$

Then  $a^2 + b^2$  is divisible by 9. But this is a contradiction.

Next assume by contradiction that there exist  $a, b$  so that  $n = a^3 + b^3$ . By Fermat Little theorem,  $x^3 \equiv x \pmod{3}$  for all integers  $x$ , and therefore

$$0 \equiv n \equiv a^3 + b^3 \equiv a + b \pmod{3}.$$

This implies that  $b = 3k - a$  for some integer  $k$ . Then we have

$$n = a^3 + b^3 = a^3 + (3k - a)^3 = a^3 + 27k^3 - 27k^2a + 9ka^2 - a^3 = 9(3k^3 - 3k^2a + ka^2).$$

This shows that  $9|n$  which is a contradiction.

**OC92.** Let  $ABCD$  be a convex quadrilateral. Let  $P$  be the intersection of external bisectors of  $\angle DAC$  and  $\angle DBC$ . Prove that  $\angle APD = \angle BPC$  if and only if  $AD + AC = BC + BD$ .

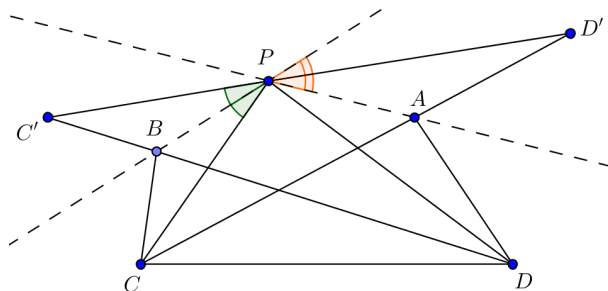
*Originally question 4 from the 2011 Italian National Olympiad.*

*Solved by Š. Arslanagić; O. Geupel; and J. G. Heuver. We give the solution of John G. Heuver.*

Suppose  $AD + AC = BC + BD$ . Let  $A, B$  be points on the ellipse with foci  $C$  and  $D$ . Then the external bisectors of  $\angle DAC$  and  $\angle DBC$  are known to be tangents to the ellipse at  $A$  and  $B$ .

Let  $C'$  be the reflection of  $C$  in  $PB$ , and  $D'$  be the reflection of  $D$  in  $PA$ . Then  $BC' = BC, AD' = AD$  and hence  $DC = DB + BC = CA + AD = CD$ .

The triangles  $CD'P$  and  $C'DP$  are congruent from which it follows that  $\angle C'PD = \angle CPD'$ . By subtracting  $\angle CPD$  from both, we get  $\angle C'PC = \angle CPD'$  and thus  $\angle APD = \angle BPC$  as required.



Conversely, assume  $\angle APD = \angle BPC$  and reflecting  $PC$  and  $PD$  in  $PB$  respectively  $PA$  we observe that triangles  $C'PD$  and  $CPD'$  are congruent by a rotation. Therefore,

$$C'D = C'B + BD = CD' = CA + AD'.$$

Since  $C'B = CB$  and  $AD' = AD$  we have

$$AD + DC = BC + BD,$$

which completes the proof.

**OC93.** For every positive integer  $n$ , determine the maximum number of edges a simple graph with  $n$  vertices can have if it contain no cycles of even length.

*Originally question 3 from Day 1 Romanian Team Selection Test, Day 1, 2011.*

*No solution to this problem was received.*

**OC94.** Let  $x_1, x_2, \dots, x_{25}$  be real numbers such that for all  $1 \leq i \leq 25$  we have  $0 \leq x_i \leq i$ . Find the maximum value of

$$x_1^3 + x_2^3 + \dots + x_{25}^3 - (x_1x_2x_3 + x_2x_3x_4 + \dots + x_{25}x_1x_2).$$

*Originally question 4 from the Korean National Olympiad 2011, Test 2.*

*There was one incorrect solution received to this problem.*

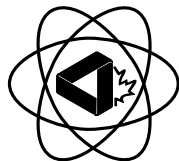
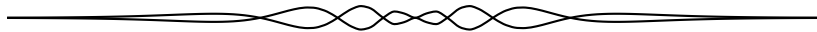
**OC95.** Can we find three relatively prime integers  $a, b, c$  so that the square of each number is divisible by the sum of the other two?

*Originally question 4 from Russia National Olympiad 2011, Grade 9, Day 1.*

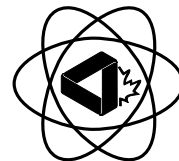
*Solved by David E. Manes.*

The answer is yes. Let  $p, q$  be two different odd primes. Then  $a = p, b = q, c = -(p + 1)$  works.

*Editor's Comment* : There was a typo in the problem ; the three integers were supposed to be positive.



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We also acknowledge with thanks the assistance from the staff of the Department of Mathematics and Statistics, especially Ros English, Wanda Heath, Menie Kavanagh and Leonce Morrissey, in the preparation of this material.

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