OLYMPIAD SOLUTIONS

OC91. Prove that no integer consisting of one 2, one 1 and the rest of digits 0 can be written neither as the sum of two perfect squares nor the sum of two perfect cubes.

Originally question 8 from the 2011 Estonian National Olympiad.

Solved by O. Geupel; D. E. Manes; and T. Zvonaru. We give the solution by Titu Zvonaru.

Since the sum of the digits of n is 3, it follows that n is divisible by 3 but not divisible by 9.

Suppose by contradiction that there exists a, b such that $n = a^2 + b^2$. As the quadratic residues modulo 3 are 0 and 1 and $a^2 + b^2 \equiv 0 \pmod 3$ it follows that

$$a \equiv b \equiv 0 \pmod 3 .$$

Then $a^2 + b^2$ is divisible by 9. But this is a contradiction.

Next assume by contradiction that there exist a, b so that $n = a^3 + b^3$. By Fermat Little theorem, $x^3 \equiv x \pmod 3$ for all integers x, and therefore

$$0 \equiv n \equiv a^3 + b^3 \equiv a + b \pmod 3 .$$

This implies that $b = 3k - a$ for some integer k. Then we have

$$n = a^3 + b^3 = a^3 + (3k - a)^3 = a^3 + 27k^3 - 27k^2 a + 9ka^2 - a^3 = 9(3k^3 - 3k^2 a + ka^2).$$

This shows that $9|n$ which is a contradiction.

OC92. Let $ABCD$ be a convex quadrilateral. Let P be the intersection of external bisectors of $\angle DAC$ and $\angle DBC$. Prove that $\angle APD = \angle BPC$ if and only if $AD + AC = BC + BD$.

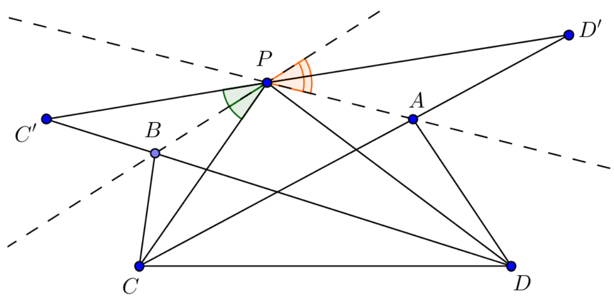
Originally question 4 from the 2011 Italian National Olympiad.

Solved by Š. Arslanagić; O. Geupel; and J. G. Heuver. We give the solution of John G. Heuver.

Suppose $AD + AC = BC + BD$. Let A, B be points on the ellipse with foci C and D. Then the external bisectors of $\angle DAC$ and $\angle DBC$ are known to be tangents to the ellipse at A and B.

Let C' be the reflection of C in PB, and D' be the reflection of D in PA. Then $BC' = BC, AD' = AD$ and hence $DC = DB + BC = CA + AD = CD$.

The triangles $CD'P$ and $C'DP$ are congruent from which it follows that $\angle C'PD = \angle CPD'$. By subtracting $\angle CPD$ from both, we get $\angle C'PC = \angle CPD'$ and thus $\angle APD = \angle BPC$ as required.



Conversely, assume $\angle APD = \angle BPC$ and reflecting PC and PD in PB respectively PA we observe that triangles $C'PD$ and CPD' are congruent by a rotation. Therefore,

$$C'D = C'B + BD = CD' = CA + AD'.$$

Since $C'B = CB$ and $AD' = AD$ we have

$$AD + DC = BC + BD\,,$$

which completes the proof.

OC93. For every positive integer n, determine the maximum number of edges a simple graph with n vertices can have if it contain no cycles of even length.

Originally question 3 from Day 1 Romanian Team Selection Test, Day 1, 2011.

No solution to this problem was received.

OC94. Let $x_1, x_2, \cdots, x_{25}$ be real numbers such that for all $1 \le i \le 25$ we have $0 \le x_i \le i$. Find the maximum value of

$$x_1^3 + x_2^3 + \cdots + x_{25}^3 - (x_1x_2x_3 + x_2x_3x_4 + \cdots x_{25}x_1x_2)\,.$$

Originally question 4 from the Koreean National Olympiad 2011, Test 2.

There was one incorrect solution received to this problem.

OC95. Can we find three relatively prime integers a, b, c so that the square of each number is divisible by the sum of the other two ?

Originally question 4 from Russia National Olympiad 2011, Grade 9, Day 1.

Solved by David E. Manes.

The answer is yes. Let p, q be two different odd primes. Then $a = p, b = q, c = -(p+1)$ works.

Editor's Comment : There was a typo in the problem ; the three integers were supposed to be positive.