

THE OLYMPIAD CORNER

No. 317

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The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to crux-olympiad@cms.math.ca or mail them to the address inside the back cover. Electronic submissions are preferable.

Submissions of solutions. *Each solution should be contained in a separate file named using the convention LastName_FirstName_OCProblemNumber (example Doe_Jane_OC1234.tex). It is preferred that readers submit a \LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

*To facilitate their consideration, solutions should be received by the editor by **1 March 2015**, although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, for translations of the problems.



OC151. Let ABC be a triangle. The tangent at A to the circumcircle intersects the line BC at P . Let Q and R be the symmetrical of P with respect to the lines AB and AC , respectively. Prove that $BC \perp QR$.

OC152. Find all non-constant polynomials $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ with integer coefficients whose roots are exactly the numbers a_0, a_1, \dots, a_{n-1} each with multiplicity 1.

OC153. Find all non-decreasing functions from the set of real numbers to itself such that for all real numbers x, y we have

$$f(f(x^2) + y + f(y)) = x^2 + 2f(y).$$

OC154. For $n \in \mathbb{Z}^+$ we denote

$$x_n := \binom{2n}{n}.$$

Prove there exist infinitely many finite sets A, B of positive integers, such that $A \cap B = \emptyset$, and

$$\frac{\prod_{i \in A} x_i}{\prod_{j \in B} x_j} = 2012.$$

OC155. There are 42 students taking part in the Team Selection Test. It is known that every student knows exactly 20 other students. Show that we can divide the students into 2 groups or 21 groups such that the number of students in each group is equal and every two students in the same group know each other.

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OC151. Soit ABC un triangle et soit P le point d'intersection de la ligne BC et de la tangente du cercle circonscrit au point A . Soit Q et R symétriques à P par rapport aux lignes AB et AC respectivement. Démontrer que $BC \perp QR$.

OC152. Déterminer tous les polynômes non constants $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ avec coefficients entiers dont les racines sont exactement les nombres a_0, a_1, \dots, a_{n-1} avec les mêmes multiplicités.

OC153. Déterminer toutes les fonctions non décroissantes des nombres réels aux nombres réels telles que pour tout x, y réels on a

$$f(f(x^2) + y + f(y)) = x^2 + 2f(y).$$

OC154. Pour $n \in \mathbb{Z}^+$, dénotons

$$x_n := \binom{2n}{n}.$$

Démontrer qu'il existe un nombre infini d'ensembles finis d'entiers positifs A et B , tels que $A \cap B = \emptyset$ et

$$\frac{\prod_{i \in A} x_i}{\prod_{j \in B} x_j} = 2012.$$

OC155. Soit 42 étudiants, où on sait que tout étudiant connaît exactement 20 autres étudiants. Démontrer qu'il est possible de répartir l'ensemble des étudiants en 2 sous-ensembles ou en 21 sous-ensembles de façon à ce que chaque sous-ensemble ait le même nombre d'étudiants et que tous les étudiants dans un sous-ensemble se connaissent.



OLYMPIAD SOLUTIONS

OC91. Prove that no integer consisting of one 2, one 1 and the rest of digits 0 can be written neither as the sum of two perfect squares nor the sum of two perfect cubes.

Originally question 8 from the 2011 Estonian National Olympiad.

Solved by O. Geupel; D. E. Manes; and T. Zvonaru. We give the solution by Titu Zvonaru.

Since the sum of the digits of n is 3, it follows that n is divisible by 3 but not divisible by 9.

Suppose by contradiction that there exists a, b such that $n = a^2 + b^2$. As the quadratic residues modulo 3 are 0 and 1 and $a^2 + b^2 \equiv 0 \pmod{3}$ it follows that

$$a \equiv b \equiv 0 \pmod{3}.$$

Then $a^2 + b^2$ is divisible by 9. But this is a contradiction.

Next assume by contradiction that there exist a, b so that $n = a^3 + b^3$. By Fermat Little theorem, $x^3 \equiv x \pmod{3}$ for all integers x , and therefore

$$0 \equiv n \equiv a^3 + b^3 \equiv a + b \pmod{3}.$$

This implies that $b = 3k - a$ for some integer k . Then we have

$$n = a^3 + b^3 = a^3 + (3k - a)^3 = a^3 + 27k^3 - 27k^2a + 9ka^2 - a^3 = 9(3k^3 - 3k^2a + ka^2).$$

This shows that $9|n$ which is a contradiction.

OC92. Let $ABCD$ be a convex quadrilateral. Let P be the intersection of external bisectors of $\angle DAC$ and $\angle DBC$. Prove that $\angle APD = \angle BPC$ if and only if $AD + AC = BC + BD$.

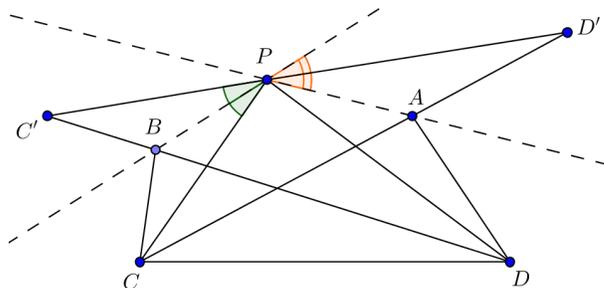
Originally question 4 from the 2011 Italian National Olympiad.

Solved by Š. Arslanagić; O. Geupel; and J. G. Heuver. We give the solution of John G. Heuver.

Suppose $AD + AC = BC + BD$. Let A, B be points on the ellipse with foci C and D . Then the external bisectors of $\angle DAC$ and $\angle DBC$ are known to be tangents to the ellipse at A and B .

Let C' be the reflection of C in PB , and D' be the reflection of D in PA . Then $BC' = BC, AD' = AD$ and hence $DC = DB + BC = CA + AD = CD$.

The triangles $CD'P$ and $C'DP$ are congruent from which it follows that $\angle C'PD = \angle CPD'$. By subtracting $\angle CPD$ from both, we get $\angle C'PC = \angle CPD'$ and thus $\angle APD = \angle BPC$ as required.



Conversely, assume $\angle APD = \angle BPC$ and reflecting PC and PD in PB respectively PA we observe that triangles $C'PD$ and CPD' are congruent by a rotation. Therefore,

$$C'D = C'B + BD = CD' = CA + AD'.$$

Since $C'B = CB$ and $AD' = AD$ we have

$$AD + DC = BC + BD,$$

which completes the proof.

OC93. For every positive integer n , determine the maximum number of edges a simple graph with n vertices can have if it contain no cycles of even length.

Originally question 3 from Day 1 Romanian Team Selection Test, Day 1, 2011.

No solution to this problem was received.

OC94. Let x_1, x_2, \dots, x_{25} be real numbers such that for all $1 \leq i \leq 25$ we have $0 \leq x_i \leq i$. Find the maximum value of

$$x_1^3 + x_2^3 + \dots + x_{25}^3 - (x_1x_2x_3 + x_2x_3x_4 + \dots + x_{25}x_1x_2).$$

Originally question 4 from the Korean National Olympiad 2011, Test 2.

There was one incorrect solution received to this problem.

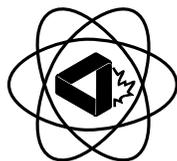
OC95. Can we find three relatively prime integers a, b, c so that the square of each number is divisible by the sum of the other two?

Originally question 4 from Russia National Olympiad 2011, Grade 9, Day 1.

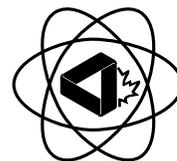
Solved by David E. Manes.

The answer is yes. Let p, q be two different odd primes. Then $a = p, b = q, c = -(p + 1)$ works.

Editor's Comment : There was a typo in the problem ; the three integers were supposed to be positive.



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This volume contains a selection of some of the problems that have been used in the Newfoundland and Labrador Senior Mathematics League, which is sponsored the the Newfoundland and Labrador Teachers Association Mathematics Special Interest Council. The support of many teachers and schools is gratefully acknowledged.

We also acknowledge with thanks the assistance from the staff of the Department of Mathematics and Statistics, especially Ros English, Wanda Heath, Menie Kavanagh and Leonce Morrissey, in the preparation of this material.

Many of the problems in the booklet admit several approaches. As opposed to our earlier 1995 book of problems, *Shaking Hands in Corner Brook*, available from the Waterloo Mathematics Foundation, this booklet contains no solutions, only answers. Also, the problems are arranged in the form in which we use them in games. We hope that this will be of use to other groups running Mathematics Competitions.

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