

FOCUS ON...

No. 9

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Solutions to Exercises from Focus On... No. 2, 3, 4 and 5

Starting from Focus On... No. 2, this column included exercises for the reader's enjoyment and practice. In this number, we present solutions to problems proposed in Focus On... No. 2, 3, 4 and 5.

From Focus On... No. 2

Show that for any complex numbers a, b, c ,

$$|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)| \leq \frac{9}{16}(|a|^2 + |b|^2 + |c|^2)^2.$$

To bring "the geometry behind the scene" to light, we recall the given hint :

$$ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2) = (b - a)(a - c)(c - b)(a + b + c).$$

This identity immediately reduces the problem to proving the inequality

$$3OG \cdot AB \cdot BC \cdot CA \leq \frac{9}{16}(OA^2 + OB^2 + OC^2)^2, \quad (1)$$

where G denotes the isobarycentre of A, B, C .

Writing $OA^2 = (\overrightarrow{OG} + \overrightarrow{GA})^2 = OG^2 + GA^2 + 2\overrightarrow{OG} \cdot \overrightarrow{GA}$, etc. and taking $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$ into account, we readily obtain

$$OA^2 + OB^2 + OC^2 = 3OG^2 + GA^2 + GB^2 + GC^2.$$

Now, with $BC = a, AB = c, CA = b$ as usual, we have

$$GA^2 + GB^2 + GC^2 = \frac{1}{9}(2b^2 + 2c^2 - a^2 + 2c^2 + 2a^2 - b^2 + 2a^2 + 2b^2 - c^2) = \frac{1}{3}(a^2 + b^2 + c^2)$$

and so $OA^2 + OB^2 + OC^2 = 3OG^2 + \frac{1}{3}(AB^2 + BC^2 + CA^2)$. To conclude, the AM-GM inequality yields

$$OA^2 + OB^2 + OC^2 \geq 4 \left(3OG^2 \cdot \frac{AB^2}{3} \cdot \frac{BC^2}{3} \cdot \frac{CA^2}{3} \right)^{1/4} = 4\sqrt{\frac{OG \cdot AB \cdot BC \cdot CA}{3}}$$

and the inequality (1) follows by squaring.

From Focus On... No. 3

For integers m, n such that $0 \leq m \leq n$, prove that the following equality holds :

$$\sum_{j=0}^m \binom{n-j}{m-j} \binom{2n+1}{2j} = 2^{2m} \binom{m+n}{2m}.$$

As indicated in the column, we consider the identity $A_{2n}(1, -\frac{Y}{4}, 0, 1) = B_{2n}(1, -\frac{Y}{4}, 0, 1)$, which we write as

$$\sum_{k=0}^n \frac{1}{4^k} \binom{2n-k}{k} Y^k = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{2k+1} (1+Y)^k.$$

Expanding $(1+Y)^k$ with the binomial theorem transforms the right-hand side into

$$\frac{1}{2^{2n}} \sum_{k=0}^n Y^k \left(\sum_{j=k}^n \binom{j}{k} \binom{2n+1}{2j+1} \right).$$

A comparison with the left-hand side leads to

$$\sum_{j=k}^n \binom{j}{k} \binom{2n+1}{2j+1} = 2^{2(n-k)} \binom{2n-k}{k}$$

for $k = 0, 1, \dots, n$. This holds in particular for $k = n - m$, giving

$$\sum_{i=0}^m \binom{n-m+i}{n-m} \binom{2n+1}{2(n-m+i)+1} = 2^{2m} \binom{m+n}{n-m}.$$

Changing the index from i to $j = m - i$ and with the help of the formula $\binom{N}{p} = \binom{N}{N-p}$ for $0 \leq p \leq N$, we readily obtain the desired equality.

From Focus On... No. 4

(a) Let ℓ be a tangent to the circumcircle Γ of $\triangle ABC$ and let $BC = a, CA = b, AB = c, d_a = d(A, \ell), d_b = d(B, \ell), d_c = d(C, \ell)$. Then, one of the numbers $a\sqrt{d_a}, b\sqrt{d_b}, c\sqrt{d_c}$ is the sum of the other two.

A solution resting on barycentric equations is proposed in the column itself, with a call for a synthetic proof. Actually, a nice application of Ptolemy's Theorem provides such a proof. The key idea is to start with the case when ℓ is tangent to the circumcircle at a vertex of the triangle, say at A (see Figure 1, left).

Let B_1, C_1 be the orthogonal projections of B, C , respectively, on ℓ . Since the chord AB subtends both $\angle BAB_1$ and $\angle ACB$, we have $\sin C = \frac{BB_1}{c}$, hence $\frac{c}{2R} = \frac{BB_1}{c}$, where R is the circumradius. Similarly, $\frac{b}{2R} = \frac{CC_1}{b}$ and so $b\sqrt{d_b} = c\sqrt{d_c} (= \frac{bc}{\sqrt{2R}})$. Since $d_a = 0$, this implies the desired result.

Now, consider the general case where the point of tangency M is different from A, B, C , say on the arc BC not containing A (Figure 1, right). We apply the particular result just obtained to the triangles $\triangle MBC$ and $\triangle MAC$. Denoting the projections of A, B, C onto ℓ as A_1, B_1, C_1 , respectively, we find $MC \cdot \sqrt{BB_1} = MB \cdot \sqrt{CC_1}$ and $MC \cdot \sqrt{AA_1} = MA \cdot \sqrt{CC_1}$ so that

$$\frac{b\sqrt{d_b}}{b \cdot MB} = \frac{c\sqrt{d_c}}{c \cdot MC} = \frac{a\sqrt{d_a}}{a \cdot MA}.$$

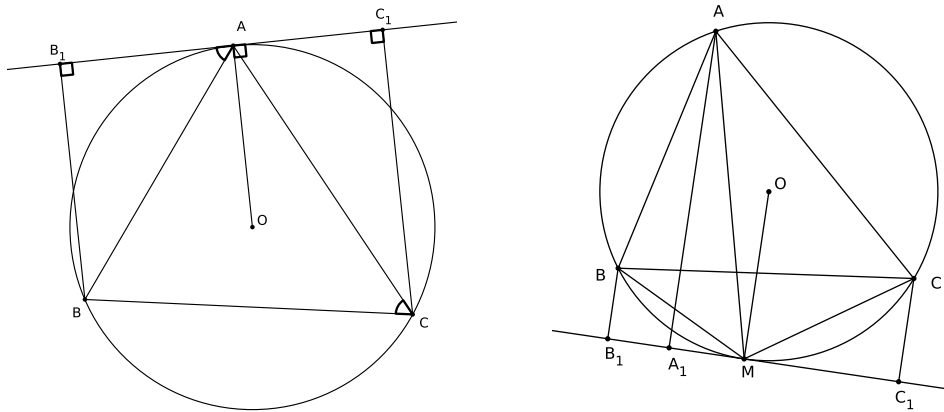
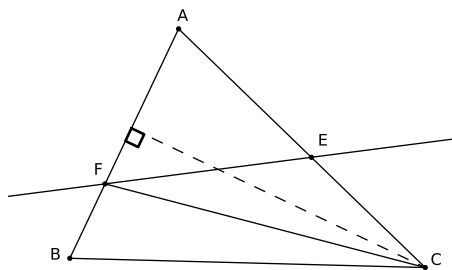


FIGURE 1: the point of tangency is at a vertex of the triangle or not.

Since $a \cdot MA = b \cdot MB + c \cdot MC$ (from Ptolemy's Theorem), we can conclude $a\sqrt{d_a} = b\sqrt{d_b} + c\sqrt{d_c}$.

(b) Let E and F be points on the sides AC and AB of ΔABC , respectively. Show that $[PBC]$ is the geometric mean of $[PAB]$ and $[PCA]$ for some point P on the line segment EF if and only if $AE \cdot AF \geq 4CE \cdot BF$.

Let $\beta = \frac{BF}{AF}$ and $\gamma = \frac{CE}{AE}$. Remarking that $\beta = \frac{BF \cdot d(C,BA)}{AF \cdot d(C,BA)} = \frac{[FBC]}{[FCA]}$, the barycentric coordinates of F relatively to (A, B, C) are $(\beta, 1, 0)$. In the same way, the coordinates of E are $(\gamma, 0, 1)$ and so the equation of the line EF is $x = \beta y + \gamma z$.



It follows that $[PBC] = \beta[PCA] + \gamma[PAB]$ for any point P on the line segment EF and therefore

$$\frac{[PBC]^2}{[PCA] \cdot [PAB]} = \beta^2 \rho + \gamma^2 \cdot \frac{1}{\rho} + 2\beta\gamma$$

where $\rho = \frac{[PCA]}{[PAB]}$.

A quick study of the function $f : x \mapsto f(x) = \beta^2 x + \gamma^2 \cdot \frac{1}{x} + 2\beta\gamma$ shows that f takes all values of the interval $[4\beta\gamma, \infty)$ when x varies in $(0, \infty)$, hence take the

value 1 if and only if $4\beta\gamma \leq 1$. The result follows since $[PBC]$ is the geometric mean of $[PAB]$ and $[PCA]$ for some point P of the segment EF if and only if $f(\rho) = 1$ for some positive ρ .

From Focus On... No. 5

(a) For $x, y, z > 0$, let $f(x, y, z) = (1-x)(1-y)(1-z)$ and

$$g(x, y, z) = 2 \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) - 4 - \left(\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} \right).$$

Show that $(a, b, c) = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ satisfies the constraint $g(x, y, z) = 0$ and $\partial_i f(a, b, c) = \lambda \partial_i g(a, b, c)$, $i = 1, 2, 3$ for some λ but $f(a, b, c)$ is not an extremum of f under the constraint.

It is readily seen that for $i = 1, 2, 3$, $\partial_i f(a, b, c) = -\frac{1}{16}$ and $\partial_i g(a, b, c) = -\frac{80}{9}$, hence $\partial_i f(a, b, c) = \lambda \partial_i g(a, b, c)$ with $\lambda = \frac{9}{16 \times 80}$. However, $f(a, b, c) = \frac{1}{64}$ while $g(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}) = 0$ and $f(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}) > \frac{1}{64}$, so that $f(a, b, c)$ is not a maximum of f under the constraint.

Also, if $w \in (0, \frac{1}{2})$, then $(w, w, 4w(1-w))$ satisfies the constraint and $f(w, w, 4w(1-w)) = (1-w)^2(1-2w)^2$. Since $\lim_{w \rightarrow 1/2} f(w, w, 4w(1-w)) = 0$, $\frac{1}{64}$ is not a minimum under the constraint either.

This example emphasizes the non-sufficiency of the existence of λ in the statement of the Lagrange multipliers theorem as given in the column.

(The interested reader can show that the sharp inequality $f(x, y, z) < 1$ holds for $x, y, z > 0$ satisfying $g(x, y, z) = 0$.)

(b) Consider the inequality

$$\frac{1}{1 - \left(\frac{x+y}{2}\right)^2} + \frac{1}{1 - \left(\frac{y+z}{2}\right)^2} + \frac{1}{1 - \left(\frac{z+x}{2}\right)^2} \leq \frac{11}{3}$$

when $x + y + z = 1$ and $x, y, z \geq 0$. Prove this inequality with the method of Lagrange multipliers.

Let $\phi(t) = \frac{1}{1-t^2}$, $h(t) = \frac{1}{3-t} + \frac{1}{1+t}$, $K = \{(x, y, z) | x, y, z \geq 0, x + y + z = 1\}$, and let $f(x, y, z)$ denote the left-hand side of the inequality. Whenever $(x, y, z) \in K$, we have

$$\phi\left(\frac{x+y}{2}\right) = \frac{4}{(2+x+y)(2-x-y)} = \frac{4}{(3-z)(1+z)} = \frac{1}{3-z} + \frac{1}{1+z} = h(z)$$

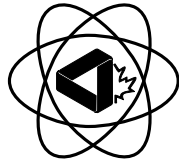
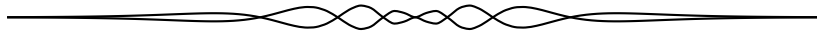
so that so that $f(x, y, z) = h(x) + h(y) + h(z)$.

Assume that f attained its maximum on K at (a, b, c) interior to K . Then we would have $\partial_i f(a, b, c) = \lambda$ for some λ , hence $h'(a) = h'(b) = h'(c)$. Since h' is strictly monotone on $[0, 1]$ (easily checked), we must have $a = b = c = \frac{1}{3}$ and $f(a, b, c) = \frac{27}{8}$. However, $\frac{27}{8}$ cannot be the maximum of f on K since $f(1, 0, 0) = \frac{11}{3} > \frac{27}{8}$ (actually,

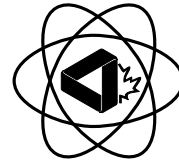
$\frac{27}{8}$ is the minimum of f on K , as it follows from the convexity of ϕ on $[0, \frac{1}{2}]$. Thus, the maximum of f on the compact K must be reached on the boundary of K . To conclude, we show that if $x = 0$ and $y, z \geq 0, y + z = 1$ (say), then $f(x, y, z) \leq \frac{11}{3}$. Indeed, in that case, $y^2 + z^2 = 1 - 2yz$ and so

$$f(x, y, z) = \frac{4}{3} + 4 \left(\frac{8 - (y^2 + z^2)}{4 - 4(y^2 + z^2) + y^2 z^2} \right) = \frac{4}{3} + 4 \left(\frac{7 + 2yz}{12 + 8yz + y^2 z^2} \right)$$

and the inequality follows from $\frac{7+2yz}{12+8yz+y^2z^2} \leq \frac{7}{12}$ (equivalent to the obvious $0 \leq 32yz + 7y^2z^2$).



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