THE CONTEST CORNER

No. 19

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The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to `crux-contest@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_CCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

To facilitate their consideration, solutions should be received by the editor by **1 March 2015***, although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC87. *Correction. In issue 8, we accidentally re-printed CC33 as CC87. This is the corrected version of CC87.*
Let $ABCDE$ be a regular pentagon with each side of length 1. The length of BE is θ and the angle FEA is α, where F is the intersection of AC and BE. Find θ and $\cos\alpha$.

CC91. A line segment of constant length 1 moves with one end on the x-axis and the other end on the y-axis. The region swept out (that is, the union of all possible placements) is R. Find the equation of the boundary of R.

CC92. Each of the positive integers 2013 and 3210 has the following three properties:

1. it is an integer between 1000 and 10000,
2. its four digits are consecutive integers, and
3. it is divisible by 3.

In total, how many positive integers have these three properties?

CC93. If $x, y, z > 0$ and $xyz = 1$, find the range of all possible values of

$$\frac{x^3 + y^3 + z^3 - x^{-3} - y^{-3} - z^{-3}}{x + y + z - x^{-1} - y^{-1} - z^{-1}}.$$

CC94. If $\log_2 x, (1 + \log_4 x)$ and $\log_8 4x$ are consecutive terms of a geometric sequence, determine the possible values of x.

CC95. Positive integers x, y, z satisfy $xy + z = 160$. Determine the smallest possible value of $x + yz$.

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CC87. *Correction. Dans le numéro 8 de la revue, on a présenté le problème CC33 à la place du problème CC87. Voici le vrai problème CC87.*
Soit $ABCDE$ un pentagone régulier ayant des côtés de longueur 1. Soit θ la longueur de BE, F le point d'intersection de AC et BE et α la mesure de l'angle FEA. Déterminer θ et $\cos\alpha$.

CC91. Un segment de droite de longueur 1 se déplace de manière qu'une de ses extrémités soit toujours sur l'axe des abscisses et l'autre sur l'axe des ordonnées. Soit R la région balayée par le segment (c'est-à-dire la réunion de tous les points sur les positions du segment à mesure qu'il se déplace). Déterminer l'équation de la frontière de R.

CC92. Chacun des entiers 2013 et 3210 satisfait aux trois propriétés suivantes :

1. il est un entier entre 1000 et 10000,
2. ses quatre chiffres sont des entiers consécutifs et
3. il est divisible par 3.

Combien y a-t-il d'entiers positifs qui satisfont à ces trois propriétés ?

CC93. Sachant que $x, y, z > 0$ et $xyz = 1$, déterminer l'étendue de toutes les valeurs possibles de l'expression

$$\frac{x^3 + y^3 + z^3 - x^{-3} - y^{-3} - z^{-3}}{x + y + z - x^{-1} - y^{-1} - z^{-1}}.$$

CC94. Sachant que $\log_2 x, (1 + \log_4 x)$ et $\log_8 4x$ sont des termes consécutifs d'une suite géométrique, déterminer toutes les valeurs possibles de x.

CC95. Les entiers strictement positifs x, y, z vérifient l'équation $xy + z = 160$. Déterminer la plus petite valeur possible de $x + yz$.

CONTEST CORNER SOLUTIONS

CC41. Ace runs with constant speed and Flash runs x times as fast, $x > 1$. Flash gives Ace a head start of y metres, and, at a given signal, they start off in the same direction. Find the distance Flash must run to catch Ace.

Originally problem 7 of 2005 W.J. Blundon Mathematics Contest.

Solved by R. I. Hess ; and D. Văcaru. We present the solution by Daniel Văcaru.

Let v be Ace's speed, then Flash has speed vx. Let t be the amount of time it takes for Flash to catch Ace. When Flash catches up to Ace, Ace is $vt + y$ metres from the start and Flash is xvt metres from the start.

Thus, $vt + y = xvt$. Solving for t, we get $t = \frac{y}{v(x-1)}$. At that time, Flash has run $\frac{xvy}{v(x-1)} = \frac{xy}{x-1}$ metres.

CC42. $\triangle ABC$ has its vertices on a circle of radius r. If the lengths of two of the medians of $\triangle ABC$ are equal to r, determine the side lengths of $\triangle ABC$.

Originally 2012 Canadian Senior Mathematics Contest, problem B3c.

Solved by M. Amengual Covas ; Š. Arslanagić ; M. Bataille ; M. Coiculescu ; R. Hess ; and D. Văcaru. We present the solution by Miguel Amengual Covas.

Let G be the centroid of $\triangle ABC$ and suppose that the two equal medians are the median AD to side BC and the median to side CA. Clearly, then, $\triangle ABC$ is isosceles with $BC = CA$. Thus the median CM to side AB lies along the perpendicular bisector of chord AB and it passes through the circumcentre O of $\triangle ABC$. Therefore, we have

$$AO^2 - OM^2 = AG^2 - GM^2. \tag{1}$$

Let $GM = x$. Since G trisects each median of $\triangle ABC$, we have $OM = OA - MC = r - 3x$ and $AG = \frac{2}{3}AD = \frac{2}{3}r$. When these are substituted into (1), we get $r^2 - (r - 3x)^2 = (\frac{2}{3}r)^2 - x^2$. Solving for x, we obtain $x = \frac{2}{3}r$ (which is not admissible) and $x = \frac{r}{12}$. Hence,

$$AB = 2 \cdot AM = 2\sqrt{r^2 - \left(\frac{3r}{4}\right)^2} = \frac{r\sqrt{7}}{2}$$

and

$$BC = CA = \sqrt{AM^2 + MC^2} = \sqrt{\left(\frac{r\sqrt{7}}{4}\right)^2 + \left(\frac{r}{4}\right)^2} = \frac{r\sqrt{2}}{2}.$$

CC43. A circle has diameter AB. P is a fixed point of AB lying between A and B. A point X, distinct from A and B, lies on the circumference of the circle. Prove that $\tan(\angle AXP) \div \tan(\angle XAP)$ is constant for all values of X.

Originally Question 6 of 2005 APICS Math Competition.

Solved by M. Amengual Covas; Š. Arslanagić; M. Bataille; R. I. Hess; J. G. Heuver; and T. Zvonaru. We present the solution of Michel Bataille modified by the editor.

For simplicity, let $\alpha = \angle XAP$ and $\beta = \angle AXP$. Since AB is a diameter, $\angle AXB = 90^\circ$ and hence $\angle BXP = 90^\circ - \beta$. Since triangle AXB is right-angled with right angle at X, $\angle PBX = 90^\circ - \alpha$. We now apply Law of Sines on $\triangle AXP$ and $\triangle PXB$. On $\triangle AXP$ we have $\frac{PA}{\sin\beta} = \frac{PX}{\sin\alpha}$, so

$$\frac{\sin\beta}{\sin\alpha} = \frac{PA}{PX}. \qquad (1)$$

On $\triangle PXB$,

$$\frac{PB}{\sin(90^\circ - \beta)} = \frac{PX}{\sin(90^\circ - \alpha)}.$$

Since $\sin(90^\circ - \beta) = \cos\beta$ and $\sin(90^\circ - \alpha) = \cos\alpha$, this implies

$$\frac{\cos\alpha}{\cos\beta} = \frac{PX}{PB}. \qquad (2)$$

Equations (1) and (2) imply

$$\frac{\tan(\angle AXP)}{\tan(\angle XAP)} = \frac{\tan\beta}{\tan\alpha} = \frac{\sin\beta}{\cos\beta} \cdot \frac{\cos\alpha}{\sin\alpha} = \frac{PA}{PX}\frac{PX}{PB} = \frac{PA}{PB},$$

which is constant for all values of X.

CC44. Let $a_0 = 1$ and for $n \geq 0$ let $a_{n+1} = a_n - \frac{1}{2}a_n^2$. Find $\lim_{n\to\infty} na_n$, if it exists.

Originally Question 6 on 2009 University of Waterloo Big E Contest.

Solved by M. Bataille; and D. Văcaru. We present Michel Bataille's solution.

We show that $\lim\limits_{n\to\infty} na_n = 2$.

Since $a_{n+1} - a_n = -\frac{1}{2}a_n^2 < 0$ for all $n \geq 0$, the sequence $\{a_n\}$ is decreasing. It follows that $a_n \leq a_0 = 1$ for all $n \geq 0$. From $a_{n+1} = \frac{a_n}{2}(2 - a_n)$, an easy induction shows that $a_n > 0$ for all $n \geq 0$. Being decreasing and bounded below, the sequence $\{a_n\}$ is convergent.

Let $\ell = \lim\limits_{n\to\infty} a_n$. Since ℓ is also the limit of $\{a_{n+1}\}$, we must have $\ell = \ell - \frac{1}{2}\ell^2$ and so $\ell = 0$. Because $\dfrac{a_{n+1}}{a_n} = 1 - \frac{1}{2}a_n$, we have $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1$. Now, we calculate

$$\frac{1}{a_{n+1}} - \frac{1}{a_n} = \frac{a_n - a_{n+1}}{a_n a_{n+1}} = \frac{\frac{1}{2}a_n^2}{a_n a_{n+1}} = \frac{1}{2} \cdot \frac{a_n}{a_{n+1}}$$

and so the sequence $\dfrac{1}{a_{n+1}} - \dfrac{1}{a_n}$ is convergent towards $\dfrac{1}{2}$. The same is true of its Cesàro mean $\{C_n\}$ defined by

$$C_n = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{a_n} - \frac{1}{a_{n-1}}\right).$$

But

$$C_n = \frac{1}{n}\left(\frac{1}{a_n} - 1\right) = \frac{1}{na_n} - \frac{1}{n}$$

and so $\lim\limits_{n\to\infty} na_n = \lim\limits_{n\to\infty} \dfrac{1}{C_n + \frac{1}{n}} = 2.$

CC45. The *baseball sum* of two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is defined to be $\frac{a+c}{b+d}$. Starting with the rational numbers $\frac{0}{1}$ and $\frac{1}{1}$ as Stage 0, the baseball sum of each consecutive pair of rational numbers in a stage is inserted between the pair to arrive at the next stage. The first few stages of this process are shown below :

STAGE 0 :	$\frac{0}{1}$								$\frac{1}{1}$
STAGE 1 :	$\frac{0}{1}$				$\frac{1}{2}$				$\frac{1}{1}$
STAGE 2 :	$\frac{0}{1}$		$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$		$\frac{1}{1}$
STAGE 3 :	$\frac{0}{1}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{1}$

Prove that :

i) no rational number will be inserted more than once,

ii) no inserted fraction is reducible, and

iii) every rational number between 0 and 1 will be inserted in the pattern at some stage.

Originally 2006 Canadian Open Mathematics Challenge, problem B4 b).

One incorrect solution was received.