SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3587★. [2010 : 461,463; 2011 : 480] *Proposed by Ignotus, Colegio Manablanca, Facatativá, Colombia.*

Define the *prime graph* of a set of positive integers as the graph obtained by letting the numbers be the vertices, two of which are joined by an edge if and only if their sum is prime.

(a) Prove that given any tree T on n vertices, there is a set of positive integers whose prime graph is isomorphic to T.

(b) For each positive integer n, determine $t(n)$, the smallest number such that for any tree T on n vertices, there is a set of n positive integers each not greater than $t(n)$ whose prime graph is isomorphic to T.

Solutions to part a) by Oliver Geupel, Brühl, NRW, Germany.

We proceed by mathematical induction on n. It is easy to verify the result holds when $n \in \{1, 2\}$.

Assume the result holds for $n = k \geq 2$, and consider a tree T with $k+1$ vertices. The tree T has a vertex v of degree 1. By the induction hypothesis, the tree $T' = T - \{v\}$ is isomorphic to the prime graph of a set $A = \{a, a_1, a_2, \ldots, a_{k-1}\}$ of positive integers. We label the vertices of T', each with its corresponding number. Let a be the label of the vertex adjacent to v in T. We must find an appropriate label for v, that is a positive integer $b \notin A$ such that $a + b$ is prime, while $a_i + b$ is composite for each $i \in \{i, \ldots, k-1\}$.

Let p_i be distinct primes such that $\gcd(a - a_1, p_i) = 1$ for $i \in \{1, \ldots, k-1\}$. By the Chinese Remainder Theorem, the simultaneous congruences

$$x \equiv a - a_i \pmod{p_i} \qquad (i \in \{1, \ldots, k-1\}$$

are equivalent to the single congruence

$$x \equiv c \pmod{p_1 p_2 \cdots p_{k-1}}\,.$$

By Dirichlet's Theorem, infinitely many primes p satisfy this congruence. Choose such a prime $q > 2a + \max\{p_1, p_2, \ldots p_{k-1}\} + \max\{a_1, a_2, \ldots a_{k-1}\}$ and let $b = p - a$. Since b is greater than each element of A, b is not a member of A. The number $a + b = p$ is prime. Also, $a_i + b = p - (a - a_i) \equiv 0 \pmod{p_1}$ and $a_i + b > p_i$, so the numbers $a_i + b$ are composite for $i \in \{1, \ldots, k-1\}$, completing the induction.

Part a) was also solved by MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA. They established, with the aid of a computer, that $t(1) = 1$, $t(2) = 2$, $t(3) = 3$, $t(4) = 5$, $t(5) = 9$, $t(6) = 13$, *and* $t(7) = 16$.

3741. [2012 : 194, 196] *Proposed by Péter Ivády, Budapest, Hungary.*

Find the largest value of a and the smallest value of b for which the inequalities

$$\frac{ax}{a+x^2} < \sin x < \frac{bx}{b+x^2},$$

hold for all $0 < x < \frac{\pi}{2}$.

Composite of similar solutions by Arkady Alt, San Jose, CA, USA; and Kee-Wai Lau, Hong Kong, China.

We show that $a = \dfrac{\pi^2}{2(\pi-2)}$ and $b = 6$.

By simple computations, it is easy to show that the given inequalities are equivalent to

$$a < \frac{x^2 \sin x}{x - \sin x} < b. \tag{1}$$

To find the largest value of a and the smallest value of b for which (1) holds for $0 < x < \dfrac{\pi}{2}$, we let

$$f(x) = \frac{x^2 \sin x}{x - \sin x}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Then

$$\begin{aligned} f'(x) &= \frac{1}{(x-\sin x)^2}\Big((x - \sin x)(2x\sin x + x^2 \cos x) - (x^2 \sin x)(1-\cos x)\Big) \\ &= \frac{1}{(x-\sin x)^2}(x^2 \sin x + x^3 \cos x - 2x \sin^2 x) = \frac{xg(x)}{(x-\sin x)^2} \end{aligned} \tag{2}$$

where $g(x) = x\sin x + x^2 \cos x - 2\sin^2 x$.

Since

$$0 < x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}$$

and

$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24},$$

we have

$$\begin{aligned} g(x) &< x\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) + x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) - 2\left(x - \frac{x^3}{6}\right)^2 \\ &= \left(2x^2 - \frac{2}{3}x^4 + \frac{1}{20}x^6\right) - \left(2x^2 - \frac{2}{3}x^4 + \frac{1}{18}x^6\right) = -\frac{1}{180}x^6 < 0. \end{aligned} \tag{3}$$

From (2) and (3) we have $f'(x) < 0$ so $f(x)$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ which implies

$$\lim_{x\to\frac{\pi}{2}^-} f(x) < f(x) < \lim_{x\to 0^+} f(x). \tag{4}$$

Since

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \frac{\left(\frac{\pi}{2}\right)^2}{\frac{\pi}{2}-1} = \frac{\pi^2}{2(\pi-2)}$$

and

$$\begin{aligned}\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{x\sin x}{1-\frac{\sin x}{x}} &= \lim_{x\to 0^+}\frac{x\left(x-\frac{x^3}{3!}+\cdots\right)}{1-\left(1-\frac{x^2}{3!}+\cdots\right)}\\ &= \lim_{x\to 0^+}\frac{x^2+o(x^3)}{\frac{x^2}{6}+o(x^3)} = 6,\end{aligned}$$

we have from (4) that $\dfrac{\pi^2}{2(\pi-2)} < f(x) < 6$ which completes the proof.

Also solved by JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; and the proposer. There was also an incorrect solution.

3742. [2012 : 194, 196] *Proposed by Michel Bataille, Rouen, France.*

In a scalene triangle ABC, let K, L, M be the feet of the altitudes from A, B, C, and P, Q, R be the midpoints of BC, CA, AB, respectively. Let LM and QR intersect at X, MK and RP at Y, KL and PQ at Z. Show that AX, BY, CZ are parallel.

Solution by Ricardo Barroso Campos, University of Seville, Seville, Spain.

Since $BMLC$ is a cyclic quadrilateral, $\angle MLA = \angle ABC = \angle ARQ$. Therefore $LQRM$ is cyclic and $XQ \cdot XR = XL \cdot XM$.

Let Γ be the circumcircle of ARQ. Since ABC maps to ARQ by a dilatation with factor $\frac{1}{2}$, its centre is the midpoint S of the segment AO, where O is the circumcentre of triangle ABC. Let AX intersect this circle at $U \neq A$. Then

$$XQ \cdot XR = XU \cdot XA.$$

Let Δ be the circle whose diameter is AH. Since AH subtends right angles at L and M, this circle is the circumcircle of ALM with diameter AH. Accordingly, its centre is the midpoint T of AH. Let AX intersect this circle at $V \neq A$. Then

$$XL \cdot XM = XV \cdot XA.$$

Since $XU \cdot XA = XQ \cdot XR = XL \cdot XM = XV \cdot XA$, $U = V$. Thus, AU is a chord of both Γ and Δ, so its right bisector contains both the centres S and T. But ST is the image of OH under a dilatation with centre A, so $ST \| OH$. It follows that $AX \perp OH$. Similarly it can be shown that both BY and CZ are perpendicular to OH, from which the desired result follows.

II. Solution by Oliver Geupel, Brühl, NRW, Germany.

We add the hypothesis that ABC is not a right triangle. Consider position vectors with the origin at point A. Then

$$\mathbf{K} = \frac{(\tan B)\mathbf{B} + (\tan C)\mathbf{C}}{\tan B + \tan C}, \quad \mathbf{M} = \frac{(\tan B)\mathbf{B}}{\tan A + \tan B}, \quad \mathbf{P} = \frac{\mathbf{B} + \mathbf{C}}{2}, \quad \mathbf{R} = \frac{\mathbf{B}}{2}.$$

The point Y is the intersection of the lines RP and MK. Hence, there are real numbers s and t for which

$$\mathbf{Y} = s\mathbf{P} + (1-s)\mathbf{R} = \frac{1}{2}\mathbf{B} + \frac{s}{2}\mathbf{C},$$

$$\mathbf{Y} = t\mathbf{K} + (1-t)\mathbf{M} = \left(\frac{t\tan B}{\tan B + \tan C} + \frac{(1-t)\tan B}{\tan A + \tan B}\right)\mathbf{B} + \frac{(t\tan C)\mathbf{C}}{\tan B + \tan C}.$$

Comparing the coefficients yields that

$$t = \frac{(\tan A - \tan B)(\tan B + \tan C)}{2\tan B(\tan A - \tan C)}, \quad s = \frac{\tan C}{\tan B} \cdot \frac{\tan A - \tan B}{\tan A - \tan C}.$$

Therefore,

$$\mathbf{Y} - \mathbf{B} = -\frac{1}{2}\mathbf{B} + \frac{1}{2} \cdot \frac{\tan C}{\tan B} \cdot \frac{\tan A - \tan B}{\tan A - \tan C}\mathbf{C}.$$

Similarly, we find that

$$\mathbf{Z} - \mathbf{C} = -\frac{1}{2}\mathbf{C} + \frac{1}{2} \cdot \frac{\tan B}{\tan C} \cdot \frac{\tan A - \tan C}{\tan A - \tan B}\mathbf{B}.$$

Consequently,

$$\mathbf{Z} - \mathbf{C} = -\frac{\tan B}{\tan C} \cdot \frac{\tan A - \tan C}{\tan A - \tan B}(\mathbf{Y} - \mathbf{B}).$$

Thus, BY and CZ are parallel. Analogously, we obtain that CZ and AX are parallel. This completes the proof.

III. Solution by the proposer.

We show that AX is perpendicular to the Euler line through the circumcentre O and orthocentre H. A similar argument applies to BY and CZ, so that AX, BY and CZ are parallel.

Let I denote the inversion with centre A that interchanges L and Q. Since $AM \cdot AR = AQ \cdot AL$, the inversion interchanges M and R. But L, Q, M, R all lie on the nine-point circle $\mathbf{N}$, so it follows that $I(\mathbf{N}) = \mathbf{N}$.

The circle with diameter AH contains the points L and M, and is carried by the inversion I to a line through $Q = I(L)$ and $R = I(M)$. The point $J = I(H)$ is therefore on the intersection of AH and QR.

Since $AH \perp BC$ and $BC \| QR$, it follows that $AJ \perp QR$ and the circle $\mathbf{\Gamma}$ with diameter AX passes through J. The points A and X are conjugate with respect to $\mathbf{N}$, so that $\mathbf{\Gamma}$ and $\mathbf{N}$ are orthogonal. Thus $I(\mathbf{\Gamma})$ is a diameter of $I(\mathbf{N}) = \mathbf{N}$ that contains $I(J) = H$. Since the centre of $\mathbf{N}$ lies on OH, $I(\mathbf{\Gamma}) = OH$. Since AX is orthogonal to $\mathbf{\Gamma}$ and is carried to itself by I, $AX \perp OH$ and the desired result follows.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; EDMUND SWYLAN, Riga, Latvia; and PETER Y. WOO, Biola University, La Mirada, CA, USA. The first and third solutions may need slight adaptation depending on the configuration.

3743. [2012 : 194, 196] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Two equal circles are tangent to the parabola $y = x^2$ at the same point. One of the circles is also tangent to the x–axis, while the other is tangent to the y– axis. Find the radius of the circles. *This problem was inspired by problem* **3732** *[2012 : 149, 151].*

Solution by Michel Bataille, Rouen, France.

Let $\mathcal{P}$ be the given parabola and let $T(u, u^2)$ be a point of $\mathcal{P}$. The equal circles with centres A and B, radius $r > 0$, and touching $\mathcal{P}$ at T are suitable if and only if the following three conditions hold:

(1) $A(2u - r, r),\ B(r, 2u^2 - r)$

(T is the midpoint of AB, and the circles are tangent to the x-axis and y-axis, respectively.),

(2) $(r - u) + 2u(u^2 - r) = 0$

(AB is perpendicular to the tangent to $\mathcal{P}$ at T; that is, AB has slope $-\frac{1}{2u}$),

(3) $(r - u)^2 + (u^2 - r)^2 = r^2 \quad (AT = r)$.

We will assume that $u \geq 0$, even $u > 0$ (because of (3)), keeping in mind that for any suitable pair of circles, a second suitable pair is obtained by reflection in the y-axis.

From (3), for some real θ,

$$r - u = r\cos\theta \quad \text{and} \quad r - u^2 = r\sin\theta.$$

Note that $\cos\theta \neq \pm 1$ and $\sin\theta \neq 0$ (otherwise $u^2 = r$ and $r - u = \pm r$, contradicting (2)) and that $\cot\theta = 2u$ (from (2)).

Since $u = r(1 - \cos\theta),\ u^2 = r(1 - \sin\theta)$, we have $\cot\theta = \dfrac{2u^2}{u} = \dfrac{2(1 - \sin\theta)}{(1 - \cos\theta)}$; that is,

$$2\sin\theta = 2 + \cos\theta - 3\cos^2\theta.$$

Squaring both sides, we readily see that $(\cos\theta - 1)(9\cos^2\theta + 3\cos\theta - 4) = 0$, whence there are two possibilities for $\cos\theta$, namely

$$-\frac{\sqrt{17} + 1}{6}, \quad \text{and} \quad \frac{\sqrt{17} - 1}{6}.$$

Conversely, if $\cos\theta$ takes one of these values and $\sin\theta\cos\theta > 0$, then setting $u = \dfrac{\cot\theta}{2}$ and $r = \dfrac{\cos\theta}{2\sin\theta(1 - \cos\theta)}$, we can easily reverse the calculations and obtain (2) and (3). This yields

- if $\cos\theta = -\dfrac{1+\sqrt{17}}{6}$:

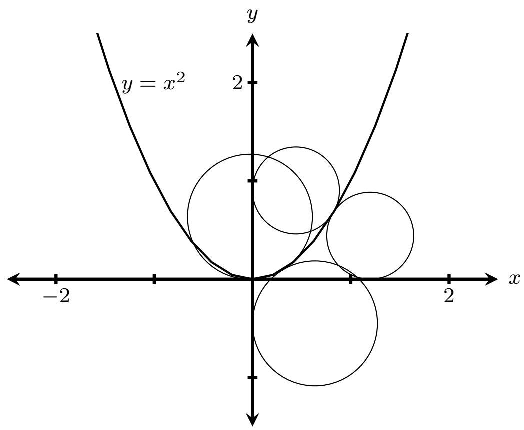
$$\sin\theta = \frac{1-\sqrt{17}}{6}, \quad u = \frac{9+\sqrt{17}}{16} \approx .8202, \quad r = \frac{69-3\sqrt{17}}{128} \approx .4424;$$

- if $\cos\theta = \dfrac{\sqrt{17}-1}{6}$:

$$\sin\theta = \frac{\sqrt{17}+1}{6}, \quad u = \frac{9-\sqrt{17}}{16} \approx .3048, \quad r = \frac{69+3\sqrt{17}}{128} \approx .6357.$$

In conclusion, the radius of suitable circles is

$$\text{either} \qquad \frac{69-3\sqrt{17}}{128} \qquad \text{or} \qquad \frac{69+3\sqrt{17}}{128}.$$



Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There were two incorrect submissions.

Only Bataille, Stallion, and Stoënescu solved the problem as it was stated. Everybody else did the problem that the proposer seems to have intended, namely to find the common radius when the outer circle is tangent to the x-axis and the inner circle to the y-axis, thereby rejecting the second solution as extraneous. This second solution reminds us that a circle which is tangent to a parabola always *intersects it in two further points, although those two points might be imaginary or coincident. The accompanying figure shows the two solutions that are tangent to the parabola at points in the first quadrant; note that the circle of the second solution that is tangent to the x-axis touches it slightly to the left of the origin (where $x \approx -.0261$). The four circles that are tangent to the parabola in the second quadrant are mirror images in the y-axis of the four that are shown.*

Sands observed that the radical $\sqrt{17}$ mysteriously pops up also in problem 3732 (concerning the contact point of the unit circle that is tangent to $y = x^2$) that inspired his problem. Might $\sqrt{17}$ one day replace the golden section as Nature's favorite irrational?

3744. [2012 : 194, 196] *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers with sum 4. Prove that

$$\frac{a^8+b^8}{(a^2+b^2)^2}+\frac{b^8+c^8}{(b^2+c^2)^2}+\frac{c^8+a^8}{(c^2+a^2)^2}+abc \geq a^3+b^3+c^3.$$

Solution by Arkady Alt, San Jose, CA, USA.

Since $a+b+c=4$, the given inequality is equivalent to

$$\frac{4(a^8+b^8)}{(a^2+b^2)^2}+\frac{4(b^8+c^8)}{(b^2+c^2)^2}+\frac{4(c^8+a^8)}{(c^2+a^2)^2}+abc(a+b+c) \geq (a+b+c)(a^3+b^3+c^3). \quad (1)$$

Using the trivial inequality $2(x^2+y^2) \geq (x+y)^2$ twice, we have for $x^2+y^2 \neq 0$,

$$\frac{4(x^8+y^8)}{(x^2+y^2)^2} \geq \frac{2(x^4+y^4)^2}{(x^2+y^2)^2} \geq \frac{2(x^4+y^4)^2}{2(x^4+y^4)} = x^4+y^4.$$

Therefore,

$$\frac{4(a^8+b^8)}{(a^2+b^2)^2}+\frac{4(b^8+c^8)}{(b^2+c^2)^2}+\frac{4(c^8+a^8)}{(c^2+a^2)^2} \geq 2(a^4+b^4+c^4). \quad (2)$$

Furthermore, we have, by Schur's Inequality

$$\begin{aligned} &2(a^4+b^4+c^4)+abc(a+b+c)-(a+b+c)(a^3+b^3+c^3) \\ &\qquad = a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b) \geq 0 \end{aligned}$$

so

$$2(a^4+b^4+c^4)+abc(a+b+c) \geq (a+b+c)(a^3+b^3+c^3). \quad (3)$$

Combining (2) and (3) we obtain (1) and the proof is complete.

*Also solved by *AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; *ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; *MICHEL BATAILLE, Rouen, France; *CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OLIVER GEUPEL, Brühl, NRW, Germany; *THANOS MAGKOS, Thessaloniki, Greece; *SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; *PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; STAN WAGON, Macalester College, St. Paul, MN, USA; *TITU ZVONARU, Comăneşti, Romania; and *the proposer. (A * indicates that the solution given by the solver also employed Schur's Inequality and is similar to the one featured above.)*

Cao, Geupel and Zvonaru pointed out that equality holds if and only if $a=b=c=4/3$. *Magkos remarked that the given inequality actually holds for all real* a, b, c *as long as no two of them are simultaneously equal to zero. As usual, Wagon's solution is based on an argument using Mathematica to show that there are no counterexamples to the negation of the given inequality. He also claimed, without proof, that the inequality holds true when the sum* $a+b+c=4$ *is replaced by* 5, 6, 7, 100, *or* 401/100, *and that in general, the result might even be true for all* a, b, *and* c *with* $a+b+c \geq 4$.

3745. [2012 : 195, 196] *Proposed by Abdilkadir Altintaş, mathematics teacher, Emirdağ, Turkey.*

In the square $ABCD$ the semicircle with diameter AD intersects the quarter circle with centre C and radius CD in the point P. Show that $PB = \sqrt{2}AP$.

I. Solution by Mihaï-Ioan Stoënescu, Bischwiller, France.

Sans réduire la généralité du problème, soit 2 la longueur du côté du carré $ABCD$. On considère le repère d'origine D et d'axes DC et DA. Ainsi on a $D(0,0), A(0,2), B(2,2), C(2,0), E(0,1)$ et finalement $P(a,b)$. En utilisant la formule pour la distance entre deux points, on a

$$a^2 + (b-1)^2 = 1, \tag{1}$$

car $PE^2 = 1$, et

$$(a-2)^2 + b^2 = 4, \tag{2}$$

car $PC^2 = 4$. Alors

$$\frac{PB^2}{PA^2} = \frac{(a-2)^2 + (b-2)^2}{a^2 + (b-2)^2} = \frac{4 - b^2 + (b-2)^2}{1 - (b-1)^2 + (b-2)^2},$$

en utilisant (1) et (2). En développant, on tire que $\frac{PB^2}{PA^2} = \frac{8-4b}{4-2b} = 2$. Ainsi $\frac{PB}{PA} = \sqrt{2}$, C.Q.F.D.

II. Composite of similar solutions by Dimitrios Koukakis, Kilkis, Greece, and by the proposer.

We know that for any point P in the plane of a rectangle $ABCD$

$$PA^2 + PC^2 = PB^2 + PD^2.$$

[It is easier to prove the equality than to find a reference: Simply apply the formula for the length of a median in terms of the sides to triangles PAC and PBD, while noting that these triangles share the median from P and have equal bases $AC = BD$.] Because triangle APD is inscribed in a semicircle, it has a right angle at P so that

$$PA^2 + PD^2 = AD^2.$$

Moreover, we are given $PC = DC = AD$. Plugging $PC^2 = AD^2 = PA^2 + PD^2$ into the first equation, we get

$$PA^2 + (PA^2 + PD^2) = PB^2 + PD^2,$$

which reduces $2PA^2 = PB^2$, or $PB = \sqrt{2}AP$, as desired.

III. Composite of similar solutions by Michel Bataille, Rouen, France; Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy; Cristóbal Sánchez–Rubio, I.B. Penyagolosa, Castellón, Spain; and Itachi Uchiha, Hong Kong, China.

Because the angle between a tangent to a circle and a chord through the point of contact equals the angle in the alternate segment, three applications of the alternate segment theorem tells us that $\angle PAB = \angle PDA = \angle PBD$ and $\angle PBA = \angle PDB$. Hence $\Delta APB \sim \Delta BPD$ (since corresponding angles are equal), so that

$$\frac{PB}{PA} = \frac{BD}{AB} = \sqrt{2},$$

and the result follows.

IV. Composite of similar solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by the AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

If the square has been labeled counterclockwise, a clockwise rotation through 90° about A takes D to B and P to a point P' outside the square. Then $AP = AP'$ and $\angle PAP' = 90^\circ$. Since $\Delta AP'B$ is the image of the right triangle ΔAPD (inscribed in the semicircle), $AP'B = 90^\circ$. Thus,

$$\angle PP'B = 90^\circ - 45^\circ = 45^\circ. \tag{3}$$

Because the angle between a tangent to a circle and a chord through the point of contact equals half the angle at the center that is subtended by the chord, $\angle PBA = \frac{1}{2}\angle PCB$ and $\angle ABP' = \angle ADP = \frac{1}{2}\angle DCP$. Therefore,

$$\angle PBP' = \angle PBA + \angle ABP' = \frac{1}{2}(\angle DCP + \angle PCB) = \frac{1}{2}\angle DCB = 45^\circ. \tag{4}$$

From equations (3) and (4) $\Delta PP'B$ is isosceles with $PB = PP'$. By the Pythagorean Theorem applied to $\Delta AP'P$,

$$PB^2 = PP'^2 = AP^2 + AP'^2 = 2AP^2;$$

consequently, $PB = \sqrt{2}AP$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GREG COOK, Student, Angelo State University, San Angelo, TX; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; NERMIN HODŽIĆ, Dobošnica, Bosnia and Herzegovina; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain (3 further solutions); CHASEN GRADY SMITH, Georgia Southern University, Statesboro, GA, USA; ALBERT STADLER, Herrliberg, Switzerland; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; ERCOLE SUPPA, Teramo, Italy; EDMUND SWYLAN, Riga, Latvia; DANIEL VĂCARU, Piteşti, Romania; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comăneşti, Romania.

The majority of the submissions used coordinates much like the first featured solution.

3746. [2012 : 195, 196] *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Let $Q(n)$ denote the sum of the digits of the positive integer n. Prove that there are infinitely many positive integers n such that

$$Q(n) + Q(n^2) + Q(n^3) = [Q(n)]^2.$$

This is an extension of problem **3506** *[2010 : 45, 47; 2011 : 57, 58].*

Composite of submitted solutions.

Let $n = a \cdot 10^k + b$. Then $n^2 = a^2 \cdot 10^{2k} + 2ab \cdot 10^k + b^2$ and $n^3 = a^3 \cdot 10^{3k} + 3a^2b \cdot 10^{2k} + 3ab^2 \cdot 10^k + b^3$. If k is at least as great as the number of digits in $3a^2b$, $3ab^2$ and b^3, then $Q(n) = Q(a)+Q(b)$, $Q(n^2) = Q(a^2)+Q(2ab)+Q(b^2)$ and $Q(n^3) = Q(a^3) + Q(3a^2b) + Q(3ab^2) + Q(b^3)$.

When $(a, b) = (2, 7), (7, 2), (1, 17), (17, 1), (3, 15), (15, 3)$, we find that $Q(n) = 9$, $Q(n^2) = 27$ and $Q(n^3) = 45$, when k is not less than 3, 3, 4, 4, 5, 5, respectively. Thus, we can obtain infinitely many examples.

In addition, there are individual values of n not comprised in the foregoing lists. The smallest such examples are 207, 414, 702, 1062, 1134, 1161, 1206, 1215, 1233, 1323, 1332, 1341, 1431, 1503, 2007, 2016, 2034, 2070, 2124. Other values are 3204 and 5301.

Multiplying any suitable value of n by a power of 10 provides other examples.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; DIGBY SMITH, Mount Royal University, Calgary, AB; DANIEL VĂCARU, Piteşti, Romania; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer.

The family of solutions provided by either of $(a, b) = (2, 7), (7, 2)$ was given by nine of the solvers. The validity of 207 or 702 was picked up by six solvers. The family provided by either of $(a, b) = (1, 17), (17, 1)$ was discovered by Bailey, Campbell and Diminnie, as well as by Barbara and Wagon. Barbara also identified the cases $(a, b) = (3, 15), (15, 3)$. Hess and Stone and Hawkins gave a list of the smallest values of n. The example 3204 is due to Curtis and 5301 to Hess and Manes.

Four solvers found their infinite family by determining one value of n and then taking its product with powers of 10, providing a solution that is technically correct but perhaps not in the spirit of the problem.

3747. [2012 : 195, 196] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, c be real numbers with $a + b + c = 0$ and $c \geq 1$. Prove that

$$a^4 + b^4 + c^4 - 3abc \geq \frac{3}{8}.$$

Composite of similar solutions by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; George Apostolopoulos, Messolonghi, Greece; and Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Since $(x+y)^2 \le 2(x^2+y^2)$ for all x and y we have

$$(a+b)^4 \le 4(a^2+b^2)^2 \le 8(a^4+b^4)$$

which together with $ab \le \frac{1}{4}(a+b)^2$ then yield

$$\begin{aligned} a^4+b^4+c^4-3abc &\ge \frac{1}{8}(a+b)^4+c^4-\frac{3}{4}(a+b)^2c \\ &= \frac{1}{8}c^4+c^4-\frac{3}{4}c^3 = \frac{3}{4}c^3\left(\frac{3}{2}c-1\right) \ge \frac{3}{4}\left(\frac{3}{2}-1\right) = \frac{3}{8}. \end{aligned}$$

Clearly, equality holds if and only if $a=b$ and $c=1$, that is, $a=b=-\frac{1}{2}$ and $c=1$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2 more solutions); DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; NERMIN HODŽIĆ, Dobošnica, Bosnia and Herzegovina and SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC (3 solutions); DIMITRIOS KOUKAKIS, Kilkis, Greece; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, Thessaloniki, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; OCTAVIAN STROE, Piteşti, Romania; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; STAN WAGON, Macalester College, St. Paul, MN, USA; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer (2 solutions).

3748⋆. [2012 : 195, 197] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Given three mutually external circles in general position, there will exist six distinct lines that are common internal tangents to pairs of the circles. Prove that if three of those common tangents, one to each pair of the circles, are concurrent, then the other three common tangents are also concurrent.

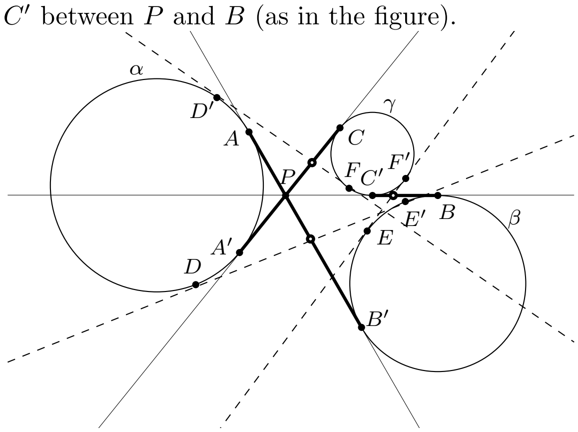
Comment by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

Here is a promising approach to the problem. We will see that when given three external circles situated so that three of the six common internal tangent lines pass through a point, then the length of one of the common tangents is the sum of the other two. It seems plausible that the converse also holds, hence

> **Conjecture**. *Given three mutually external circles in general position, there will be two internal common tangents (equal in length) between each pair of the circles, resulting in six distinct straight lines containing these six common tangents. These lines can be split into two subsets of three concurrent lines if and only if one of the lengths is the sum of the other two.*

We assume that we are given external circles α, β, and γ, and that three of their six distinct common internal tangent lines pass through a point P. By carefully labeling the figure, we can reduce the number of special cases to two. Circle α will be assigned the contact points A, A', D, D', while β's are B, B', E, E', and γ's are C, C', F, F'. They are labeled so that $AB' = DE'$ are the longest of the common internal tangents. Because there are only two tangent lines through P to any single circle, it is clear that for three of the common tangents to pass through P, each would necessarily belong to a different pair of circles. Thus we label our three lines through P to be AB', BC', and CA'. We wish to label the points where the tangents touch the circles so that $AB' > BC', CA'$ (and, consequently, $DE' > EF', FD'$). There will be two cases to consider:

- *P lies in the interior of the segment AB'.* The lines AB' and CA' divide the plane into four regions; because the lines are internal tangents, the circles β and γ must lie in the region vertically opposite the region enclosing α. The line $C'B$ will necessarily intersect α, and our decision to make $AB' > CA'$ will place C' between P and B (as in the figure).



- *B' lies between A and P.* In this case P is exterior to the three common tangents lying along the tangent lines through it. Because $PA = PA', PB = PB', PC = PC'$, our assumption that AB' be the longest internal common tangent forces B' to lie between A and P, B between C' and P, and C between A' and P (as in the figure).

With our notation in place we are ready to prove the claim that $AB' = BC' + CA'$ (and, consequently, $DE' = EF' + FD'$).

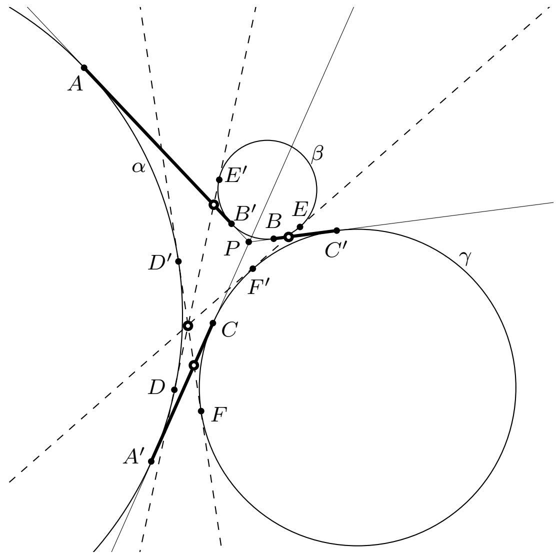
Case 1: P lies between A and B'.

$$\begin{aligned} AB' &= AP + PB' \\ &= A'P + PB \\ &= (A'C - PC) + (PC' + C'B) \\ &= A'C + C'B \end{aligned}$$

Case 2: B' lies between A and P.

$$\begin{aligned} AB' &= AP - PB' \\ &= A'P - PB \\ &= (A'C + PC) - (PC' - C'B) \\ &= A'C + C'B \end{aligned}$$

To see where difficulties arise with the converse, suppose that the lines



AB', BC', CA' all pass through P, so that $AB' = BC' + CA'$. Observe that because $CA' = FD'$, we also have $AB' = BC' + FD'$ even though the lines AB', BC', FD' are not concurrent. Somehow one must discover how to make use of the hypothesis that the lines AB', BC', CA' are concurrent (which implies that $AB' = BC' + CA'$ and, therefore, that $DE' = EF' + FD'$). The conjecture appears even harder to prove; for the conjecture one would be given only $AB' = BC' + CA'$ with some assignment of labels, and would have to describe how to correctly select a subset of three concurrent lines from the given set of six common internal tangent lines.

No solutions have been received; the problem remains open.

3749. [2012 : 195, 197] *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Let D and E be arbitrary points on the sides BC and AC of a triangle ABC. Prove that

$$\sqrt{[ADE]} + \sqrt{[BDE]} \le \sqrt{[ABC]},$$

where $[XYZ]$ denotes the area of triangle XYZ.

Solution by Arkady Alt, San Jose, CA, USA.

Let $t = \dfrac{BD}{BC}$, $s = \dfrac{CE}{CA}$, and $F = [ABC]$. Then $t, s \in [0,1]$,

$$[BDE] = Fst,$$

and

$$[ADE] = F(1-s)(1-t).$$

Hence the claimed inequality is equivalent to the inequality

$$\sqrt{(1-s)(1-t)} + \sqrt{st} \leq 1,$$

which follows immediately from the Cauchy-Schwarz inequality applied to the pairs $\left\langle \sqrt{1-s}, \sqrt{s} \right\rangle$ and $\left\langle \sqrt{1-t}, \sqrt{t} \right\rangle$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; NERMIN HODŽIĆ, Dobošnica, Bosnia and Herzegovina and SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; DIMITRIOS KOUKAKIS, Kilkis, Greece; KEE-WAI LAU, Hong Kong, China; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; ALBERT STADLER, Herrliberg, Switzerland; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comǎneşti, Romania; and the proposer.

3750. [2012 : 195, 197] *Proposed by Michel Bataille, Rouen, France.*

Let $T_k = 1 + 2 + \cdots + k$ be the k^{th} triangular number. Find all positive integers m, n such that $T_m = 2T_n$.

Composite of submitted solutions.

The equation is equivalent to $m(m+1) = 2n(n+1)$, which in turn becomes $x^2 - 2y^2 = -1$ when $x = 2m+1$ and $y = 2n+1$. The positive solutions $(x, y) = (x_k, y_k)$ of the Pellian equation $x^2 - 2y^2 = -1$ are all odd and given by

$$x_k + y_k\sqrt{2} = (1+\sqrt{2})(3+2\sqrt{2})^k$$

for $k = 0, 1, 2, \ldots$.

There are various ways of describing these solutions:

1. $(x_0, y_0) = (1, 1)$, and $(x_{k+1}, y_{k+1}) = (3x_k + 4y_k, 2x_k + 3y_k)$;
2. $(x_0, y_0) = (1, 1)$, $(x_1, y_1) = (7, 5)$, and
$$(x_{k+2}, y_{k+2}) = 6(x_{k+1}, y_{k+1}) - (x_k, y_k);$$
3. $$x_k = \frac{1}{2}\left[(1+\sqrt{2})^{2k+1} + (1-\sqrt{2})^{2k+1}\right],$$
$$y_k = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{2k+1} - (1-\sqrt{2})^{2k+1}\right];$$

4.

$$x_k = \sum_{i=0}^{k} \binom{2k+1}{2i} 2^i,$$
$$y_k = \sum_{i=0}^{k} \binom{2k+1}{2i+1} 2^i.$$

A complete set of solution pairs (m, n) can be derived from the pairs (x_k, y_k) when $k \geq 1$; the smallest pairs are

$$(3, 2), (20, 14), (119, 84), (696, 492), (4059, 2870), (23660, 16730), (137903, 97512).$$

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA and GEORGE MELKI, Lebanese University, Fanar, Lebanon; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; NERMIN HODZIĆ, Dobosnica, Bosnia and Herzegovina and SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Curtis rewrote the equation as $m^2 + (m+1)^2 = (2n+1)^2$, from which we see that the solution amounts to finding Pythagorean triplets whose smallest entries differ by 1. Stone and Hawkins pointed out that the pairs (m, n) satisfy the recurrence relation (with $(m_0, n_0) = (0, 0)$)

$$\begin{pmatrix} m_{k+1} \\ n_{k+1} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} m_k \\ n_k \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

They also wonder whether, given a triangular array of T_m dots, there is a transparent way to see that the bottom $m - n$ rows contain T_n dots.

Both Arslanagić and the pair Hodzić-Malakić pointed out that the problem appears in College Math. J. 25:3 (May, 1994), 241-243, as well as in the chapter that treats Pell's equation in the book Mathematical Olympiad Challenges by Titu Andreescu and Razvan Gelca. The theory of Pell's equation appealed to in the solution can be found in many elementary number theory textbooks.