PROBLEM OF THE MONTH

No. 6

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This column is dedicated to the memory of former ***CRUX with MAYHEM*** *Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students, with readers of CRUX with MAYHEM, and, through his work on mathematics contests and outreach programs, with many others. The "Problem of the Month" features a problem and solution that we know Jim would have liked.*

A Test of Time

As I began working on this problem, I glanced at my clock. As I finished solving the problem, I glanced again at the clock. To my surprise, the hour and minute hands of the clock had switched places. How long did it take me to solve this problem?

Although solvable with basic algebra, it is always interesting to tackle problems from various angles. In doing so, this has led me to consider the roots of the generalized linear polynomial

$$ax + b\lfloor cx + d\rfloor + e, \tag{1}$$

where $\lfloor\cdot\rfloor$ is the floor function.

In searching the literature, no general formula for the roots of (1) was to be found. Moreover, the Internet revealed but a trail of inquiries relating to (1) but no substantial answers. I was therefore left to work out my own formula, presented below. Based on this general formula, a solution to the puzzle is then given.

General formula

Let $abc \neq 0$. Replacing $cx + d$ by X, dividing by b, rearranging, setting $A = \frac{a}{bc}$ and $E = \frac{ec-ad}{bc}$, leads to solving for X in the equivalent equation

$$AX + \lfloor X\rfloor + E = 0. \tag{2}$$

Let X be a root. Therefore there exists $m \in \mathbb{Z}$ such that $X \in [m, m+1)$. Equation (2) becomes $AX + m + E = 0$ which yields $X = -\frac{E+m}{A}$, meaning that there is at most one root per range $[m, m+1)$. Next, the requirement $m \le X < m+1$ becomes $m \le -\frac{E+m}{A} < m+1$ implying that $m \in \left(-\frac{E+A}{1+A}, -\frac{E}{1+A}\right]$. Each m in this range generates a root.

Note 1. If X is an integer root, then (2) becomes $X = -\frac{E}{1+A}$. Since this is a unique number ($A \neq -1$), there can be at most one integer root; other roots will be nonintegers.

Note 2. If $A=-1$, then $X=E+m$ for all $m \in \mathbb{Z}$ provided $m \le -\frac{E+m}{A} < m+1$, which becomes $0 \le E < 1$. If this is not satisfied while $A=-1$ then there are no solutions.

Expressing these results with the original parameters gives:

General formula: *For $a \neq 0$, the roots of $ax + b\lfloor cx+d \rfloor + e$ are $x_m = -\frac{e+bm}{a}$, such that*

- *if $a+bc \neq 0$ then $m \in \left(\frac{ad-ce-a}{a+bc}, \frac{ad-ce}{a+bc}\right]$*
- *if $a+bc=0$ then $m \in \mathbb{Z}$ provided that $0 \le d + \frac{e}{b} < 1$, without which there are no solutions.*

Cases with $a=0$ lead to trivial solutions : if $bc \neq 0$ then for all $x_0 \in \left[-\frac{e}{b}, 1-\frac{e}{b}\right)$, $x = \frac{x_0-d}{c}$ is a solution provided $\frac{e}{b} \in \mathbb{Z}$. If $bc=0$ then then there are no variables in the equation to solve for.

Solution to the puzzle

Let the clock be gauged from 0 to 1. Note that the time indicated by a clock can be characterized solely by the position of the hour hand. Let s_1 be the initial time and s_2 the final time. In other words, s_1 is the initial position of the hour hand and s_2 the initial position of the minute hand. Similarly, s_2 is the final position of the hour hand and s_1 the final position of the minute hand. Recall that we are working with values between 0 and 1 and not 0 to 12.

When the hour hand points to s_1, the minute hand points to $s_2 = 12s_1 - \lfloor 12s_1 \rfloor$. Similarly, when the hour hand points to s_2 the minute hand points to $s_1 = 12s_2 - \lfloor 12s_2 \rfloor$. Combining these last two equalities and simplifying gives $143s_1 - \lfloor 144s_1 \rfloor = 0$. From the general formula above, the roots of this equation are $s_1 = \frac{n}{143}$ for $n = 0, 1, \ldots, 142$. There are thus 143 distinct beginning times such that a perfect switch occurs (s_1 belonging to the first turn, "midnight to noon"). For instance, with $n=1$ we get $s_1 = \frac{1}{143}$ and $s_2 = \frac{12}{143}$. Converting to HH:MM:SS gives beginning time $s_1 =$ 0:05:02 and end time $s_2 =$ 1:00:25. This is a 0:55:23 time interval (times rounded to the nearest second).

But we are not interested in when these occur but the time interval they generate, as per requested by the puzzle. The time interval is $\Delta s = s_2 - s_1 = (12s_1 - \lfloor 12s_1 \rfloor) - s_1$ which, after simplification equals $\frac{n}{13} - \left\lfloor \frac{n}{13} + \frac{n}{143} \right\rfloor$, $n = 0, 1, \ldots, 142$. Enumerating, or doing a little algebra, we find that this has a 13 cycle giving the 13 answers $\Delta s = \frac{(0,1,2,3,4,5,6,7,8,9,10,11,-1)}{13}$. For instance, $\Delta s = \frac{1}{13}$ is the 0:55:23 time interval and $\Delta s = \frac{11}{13}$ is a 10:09:14 time interval. The $\Delta s = \frac{-1}{13}$ needs to be addressed. It means that $s_2 < s_1$ or equivalently that the minute hand is before the hour hand during "midnight to noon". Therefore to switch, the hour hand must pass over "noon" and is the only answer that has this property. In other words, this particular switch can not be done in the same half–day. Alternatively, we may take s_2 as the start time and s_1 as the end time, giving $\Delta s = \frac{12}{13} =$ 11:04:37 switch time. This is the switched $\frac{1}{13} =$ 0:55:23; the $\Delta s = \frac{1}{13}$ is simply the "missing" $\frac{12}{13}$. This switch starts at 1:00:25, the hour hand crossing over the "12" and ends at 0:05:02, a time interval of 11:04:37.

There are thus a class of 13 answers to the riddle: $\Delta s = \frac{m}{13}$, $m = 0, 1, \ldots, 12$. Converting gives the 13 answers: $\Delta t = \frac{12m}{13}$ hours, $m = 0, 1, \ldots, 12$. Other solutions are found with $m > 12$ and represent adding multiples of 12 h to the previous answers. The table below displays the first thirteen answers.

Note. Only the first thirteen time intervals have been explicated. These are not the times at which these intervals occur. During a day, each time interval can occur several times. For instance, the time interval of $\frac{1}{13}$ = 0:55:23 can begin at s_1 =0:05:02 AM and end at s_2 = 1:00:25 AM, or begin at 0:05:02 PM and end at 1:00:25 PM, or begin at 1:10:29 AM and end at 2:05:52 AM, and there are many more. All these values may be found by applying $s_1 = \frac{n}{143}$ to two turns (a whole day).

Time to switch

m	**hours**	**hh:mm:ss**
0	0	0
1	0.9231	0:55:23
2	1.8462	1:50:46
3	2.7692	2:46:09
4	3.6923	3:41:32
5	4.6154	4:36:55
6	5.5385	5:32:18
7	6.4615	6:27:43
8	7.3846	7:23:05
9	8.3077	8:18:28
10	9.2308	9:13:51
11	10.1538	10:09:14
12	11.0769	11:04:37

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