# THE OLYMPIAD CORNER

## No. 313

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*The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention* LastName_FirstName_OCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file* <u>and</u> *a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at* `crux-olympiad@cms.math.ca`*. Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on* <u>each</u> *solution, and each solution should start on a separate page.*

*To facilitate their consideration, solutions to the problems should be received by the editor by* **1 September 2014***, although solutions received after this date will also be considered until the time when a solution is published.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.*

*The editor thanks Rolland Gaudet, of l'Université Saint-Boniface in Winnipeg, for translations of the problems.*

**OC131**. Find all $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$g(f(x) - y) = f(g(y)) + x,$$

for all $x, y \in \mathbb{R}$.

**OC132**. Find all primes $p$ and $q$ such that

$$(p+q)^p = (q-p)^{(2q-1)}.$$

**OC133**. Let $f(x) = (x+a)(x+b)$ where $a, b > 0$. Find the maximum of

$$F = \sum_{1 \leqslant i < j \leqslant n} \min \{f(x_i), f(x_j)\},$$

where $x_1, x_2, \ldots, x_n \geqslant 0$ are real numbers satisfying $x_1 + x_2 + \ldots + x_n = 1$.

**OC134**. Let $ABC$ be an acute-angled triangle with $AB \neq AC$. Let $\Gamma$ be the circumcircle of $ABC$, $H$ the orthocentre of $ABC$ and $O$ the centre of $\Gamma$. Let $M$ be the midpoint of $BC$. The line $AM$ meets $\Gamma$ again at $N$ and the circle with diameter $AM$ crosses $\Gamma$ again at $P$. Prove that the lines $AP, BC$ and $OH$ are concurrent if and only if $AH = HN$.

**OC135**. Prove that for each $n \in \mathbb{N}$ there exist natural numbers $a_1 < a_2 < \cdots < a_n$ such that $\phi(a_1) > \phi(a_2) > \cdots > \phi(a_n)$ where $\phi$ denotes the Euler $\phi$ function.

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**OC131**. Déterminer toutes $f, g : \mathbb{R} \to \mathbb{R}$ telles que

$$g(f(x) - y) = f(g(y)) + x$$

pour tous $x, y \in \mathbb{R}$.

**OC132**. Déterminer tous les nombres premiers $p$ et $q$ tels que

$$(p+q)^p = (q-p)^{(2q-1)} .$$

**OC133**. Soit $f(x) = (x+a)(x+b)$ où $a, b > 0$. Déterminer le maximum de

$$F = \sum_{1 \leqslant i < j \leqslant n} \min \{f(x_i), f(x_j)\} ,$$

où $x_1, x_2, \ldots, x_n \geqslant 0$ sont des nombres réels satisfaisant $x_1 + x_2 + \ldots + x_n = 1$

**OC134**. Soit $ABC$ un triangle à angles aigus tel que $AB \neq AC$. Soit $\Gamma$ le cercle circonscrit de $ABC$, $H$ l'orthocentre de $ABC$, $O$ le centre de $\Gamma$, et $M$ le mipoint de $BC$. La ligne $AM$ rencontre $\Gamma$ de nouveau à $N$. Le cercle avec diamètre $AM$ croise $\Gamma$ de nouveau à $P$. Démontrer que les lignes $AP$, $BC$ et $OH$ sont concourantes si et seulement si $AH = HN$.

**OC135**. Démontrer que pour tout $n \in \mathbb{N}$ il existe des nombres naturels $a_1 < a_2 < \cdots < a_n$ an tels que $\phi(a_1) > \phi(a_2) > \cdots > \phi(a_n)$ où $\phi$ dénote la fonction $\phi$ d'Euler.

# OLYMPIAD SOLUTIONS

**OC71**. Define $a_n$ a sequence of positive integers by $a_1 = 1$ and $a_{n+1}$ being the smallest integer so that

$$\operatorname{lcm}(a_1, .., a_{n+1}) > \operatorname{lcm}(a_1, .., a_n) .$$

Find the set $\{a_n | n \in \mathbb{Z}\}$.
(*Originally question 4 from the 2011 Austrian Mathematical Olympiad.*)

*One incorrect solution was received to this problem.*

As this problem is similar to Problem 2 from the fourth test of Romania IMO Selection Test 1995, which the editor wrote, we give a modified version of the Editor's solution to that problem.

Let $b_1 = 1, b_2 = 2, b_3 = 3, \dots$ be the sequence of positive integers which are divisible by a most one prime, that is $b_1 = 1$ and $b_2, \dots, b_n, \dots$ are exactly the positive integers which are powers of primes.

*Claim 1*: Let $n, k$ be so that $b_n \le k < b_{n+1}$. Then

$$\operatorname{lcm}\{1, 2, \dots, k\} = \operatorname{lcm}\{b_1, b_2, \dots, b_n\} .$$

*Proof of Claim 1.* As $\{b_1, b_2, \dots, b_n\} \subset \{1, 2, \dots, k\}$ we have

$$\operatorname{lcm}\{1, 2, \dots, k\} \ge \operatorname{lcm}\{b_1, b_2, \dots, b_n\} .$$

To complete the proof, we show that $\operatorname{lcm}\{b_1, b_2, \dots, b_n\}$ is a common multiple of $\operatorname{lcm}\{1, 2, \dots, k\}$.

Let $1 \le m \le k$, and let

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j} ,$$

be the prime factorization of $m$. Then for each $1 \le i \le j$ there exists some $k_i$ such that

$$p_i^{\alpha_i} = b_{k_i} .$$

As

$$1 \le p_i^{\alpha_i} \le m \le k < b_{n+1} ,$$

we get $k_i \le n$. Therefore $p_i^{\alpha_i} = b_{k_i}$ divides $\operatorname{lcm}\{b_1, b_2, \dots, b_n\}$.

As $p_1^{\alpha_1}$, $p_2^{\alpha_2}$, $\dots$, $p_j^{\alpha_j}$ are pairwise relatively prime, and each one divides $\operatorname{lcm}\{b_1, b_2, \dots, b_n\}$, it follows that their product $m$ also divides $\operatorname{lcm}\{b_1, b_2, \dots, b_n\}$.

Therefore $\operatorname{lcm}\{b_1, b_2, \dots, b_n\}$ is divisible by all $1 \le m \le k$, which completes the proof of Claim 1.

*Claim 2*: For all $n \ge 1$ we have

$$\operatorname{lcm}\{b_1, b_2, \dots, b_n\} < \operatorname{lcm}\{b_1, b_2, \dots, b_n, b_{n+1}\} .$$

*Proof of Claim 2.* As $\{b_1, b_2, \ldots, b_n\} \subset \{b_1, b_2, \ldots, b_n, b_{n+1}\}$ we have

$$\text{lcm}\{b_1, b_2, \ldots, b_n\} \le \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,.$$

To prove our claim, we only need to show that

$$\text{lcm}\{b_1, b_2, \ldots, b_n\} \neq \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,.$$

Let $b_{n+1} = p^k$ for some prime $p$ and some $k \ge 1$. Then, $p^k \mid \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}$.

If $k = 1$, then $b_{n+1}$ is prime and none of $b_1, b_2, \ldots, b_n$ can be divisible by $p$, as $b_n < b_{n+1}$. Therefore

$$p \nmid \text{lcm}\{b_1, b_2, \ldots, b_n\}\,;\, p \mid \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,,$$

which shows that they cannot be equal.

If $k \ge 2$, then

$$p^k \mid \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,.$$

We claim that

$$p^k \nmid \text{lcm}\{b_1, b_2, \ldots, b_n\}\,.$$

Indeed, each of $b_1, b_2, \ldots, b_n$ is either relatively prime to $p$, or of the form $p^i$ where $1 \le i \le k-1$. It follows immediately from here that the power of $p$ in $\text{lcm}\{b_1, b_2, \ldots, b_n\}$ is exactly $k-1$.

*Claim 3*: For all $n \ge 1$ we have

$$a_n = b_n\,.$$

*Proof of Claim 3.* We prove this result by strong induction on $n$.

The initial step is obvious, as $b_1 = 1 = a_1$.

We now prove the inductive step. We know that $a_1 = b_1$, $a_2 = b_2$, $\ldots$, $a_n = b_n$ and we need to show that $a_{n+1} = b_{n+1}$.

Let $b_n \le k < b_{n+1}$. then by *Claim 1* we have

$$\text{lcm}\{a_1, a_2, \ldots, a_n\} = \text{lcm}\{b_1, b_2, \ldots, b_n\} = \text{lcm}\{1, 2, \ldots, k\} \ge \text{lcm}\{a_1, a_2, \ldots, a_n, k\}\,.$$

Therefore

$$\text{lcm}\{a_1, a_2, \ldots, a_n, k\} \le \text{lcm}\{a_1, a_2, \ldots, a_n\}\,,$$

for all $k < b_{n+1}$.

Moreover, by *Claim 2*, we have

$$\text{lcm}\{a_1, a_2, \ldots, a_n, b_{n+1}\} = \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\} > \text{lcm}\{b_1, b_2, \ldots, b_n\} = \text{lcm}\{a_1, a_2, \ldots, a_n\}\,.$$

Hence, the definition of $a_{n+1}$ implies

$$a_{n+1} = b_{n+1}\,,$$

which proves the inductive step.

**OC72**. Prove that there are infinitely many positive integers so that the arithmetic and geometric mean of their divisors are integers.
(*Originally question 4 from the 2011 Kazahstan Math Olympiad, Grade 10.*)

*Solved by Daniel Văcaru, Piteşti, Romania; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Zelator.*

We show that if $p$ is a prime of the form $3k+1$, then $p^2$ has the desired properties. As there are infinitely many primes of the form $3k+1$, this solves the problem.

Indeed, the divisors of $p^2$ are $1, p$ respectively $p^2$. Their arithmetic mean is

$$\frac{1+p+p^2}{3}.$$

As

$$1 \equiv p \equiv p^2 \pmod 3 ,$$

we have $3 \mid p^2+p+1$, and hence

$$\frac{1+p+p^2}{3} \in \mathbb{Z}$$

as claimed.

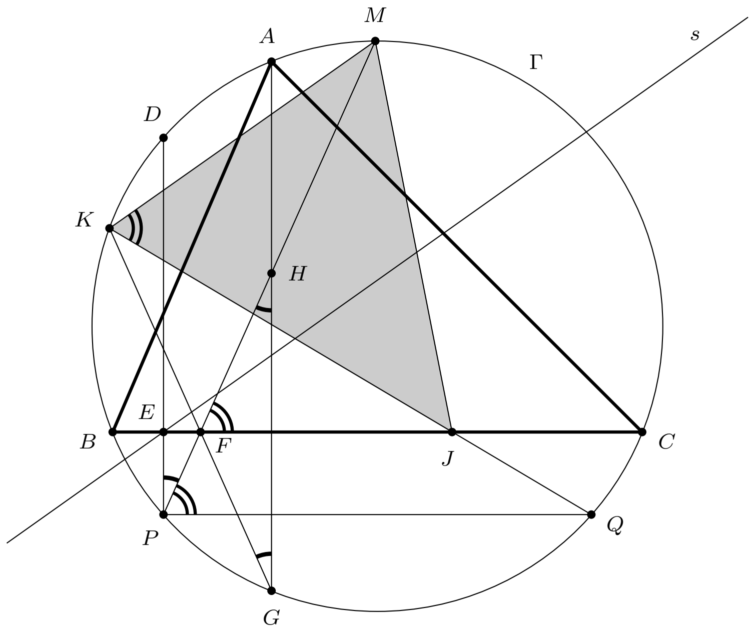
As geometric mean of $1, p, p^2$ is $p \in \mathbb{Z}$, we therefore see that $p^2$ has the desired property, as claimed.

**OC73**. Find all non-decreasing sequences $a_1, a_2, a_3, \ldots$ of natural numbers such that for each $i, j \in \mathbb{N}$, $i+j$ and $a_i + a_j$ have the same number of divisors. (a non-decreasing sequence is a sequence such that for all $i < j$, we have $a_i \le a_j$.)
(*Originally question 3 from the 2011 Iran National Olympiad, Day 2.*)

*No solution to this problem was received.*

**OC74**. Let $H$ be the orthocenter of an acute triangle $ABC$ with circumcircle $\Gamma$. Let $P$ be a point on the arc $BC$ (not containing $A$) of $\Gamma$, and let $M$ be a point on the arc $CA$ (not containing $B$) of $\Gamma$ such that $H$ lies on the segment $PM$. Let $K$ be another point on $\Gamma$ such that $KM$ is parallel to the Simson line of $P$ with respect to triangle $ABC$. Let $Q$ be another point on $\Gamma$ such that $PQ \parallel BC$. Segments $BC$ and $KQ$ intersect at a point $J$. Prove that $\triangle KJM$ is an isosceles triangle.
(*Originally question 1 from the 2011 China team selection test, Day 2.*)

*Solved by Oliver Geupel, Brühl, NRW, Germany.*




Let the perpendicular from $P$ onto $BC$ intersect $\Gamma$ and $BC$ at points $D$ respectively $E$, where $D \neq P$. Let $s$ be the Simson line of $P$. Let $F$ be the intersection of $BC$ and $MP$, and let the line $AH$ meet $\Gamma$ at point $G \neq A$. By a well-known property of the Simson line (see [1] § 288), we have $AD \parallel s$. Hence, $AD \parallel KM$ and $\widehat{AK}=\widehat{MD}$. As the quadrilateral $ABGC$ is cyclic and because the sides of $\angle BAC$ are perpendicular to the sides of $\angle BHC$, we have $\angle CGB = 180^\circ - \angle BAC = \angle BHC$. Thus, $BC$ is the perpendicular bisector of $GH$. In other words, the reflection of the orthocentre in the side of a triangle is a point of the circumcircle; this is, in fact, Theorem 178 of [1].

It follows that $\angle AGK = \angle MPD = \angle FPE = \angle FHG = \angle AGF$. Thus, $F$, $G$, and $K$ are collinear. Whence, $\angle JKM = \angle QKM = \angle QPM = \angle JFM$, that is, the points $F$, $J$, $M$, and $K$ are concyclic. Therefore,

$$\begin{aligned}\angle KMJ &= 180^\circ - \angle JKM - \angle MJK = 180^\circ - \angle JFM - \angle MFK \\ &= 180^\circ - \angle JFK = \angle KFB = \angle GFJ = \angle JFH = \angle JFM = \angle JKM.\end{aligned}$$

Consequently, $JK = JM$, and $\triangle KJM$ is isosceles.

## References

[1] N. Altshiller-Court, *College Geometry*, 2nd ed., New York, 1952

**OC75**. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $Q(x) = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_0$ be two polynomials with integral coefficients so that $a_n - b_n$ is a prime, $a_nb_0 - a_0b_n \neq 0$, and $a_{n-1} = b_{n-1}$. Suppose that there exists a rational number $r$ such that $P(r) = Q(r) = 0$. Prove that $r \in \mathbb{Z}$.
(*Originally question 3 from the 2011 India Mathematical Olympiad.*)

*Solved by Michel Bataille, Rouen, Franceand Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Bataille, modified by the editor.*

Let $a_n - b_n = p$.

If $n = 1$ then $P(x) - Q(x) = px$ and hence $r = 0 \in \mathbb{Z}$.

Now assume $n \geq 2$. Since $a_{n-1} = b_{n-1}$, we have

$$P(x) - Q(x) = px^n - c_{n-2}x^{n-2} - \cdots - c_0\,,$$

with $c_j = b_j - a_j$, $0 \leq j \leq n-2$. As $P(r) = Q(r) = 0$ we have

$$pr^n = c_{n-2}r^{n-2} + \cdots + c_1r + c_0\,. \tag{1}$$

If $r = 0$ we are done. Otherwise, we have $r = \dfrac{l}{m}$ for some co-prime integers $l, m$ with $l \neq 0$ and $m \geq 1$. Multiplying 1 by $m^{n-2}$ we get

$$\frac{pl^n}{m^2} = c_{n-2}l^{n-2} + c_{n-3}l^{n-3}m + \cdots + c_1lm^{n-3} + c_0m^{n-2}\,.$$

Therefore $\dfrac{pl^n}{m^2} \in \mathbb{Z}$, which implies $m^2 \mid pl^n$. As $m, n$ are co-prime, $m^2$ and $l^n$ are also co-prime, and hence $m^2$ must divide $p$. Since $p$ is prime, it follows that $m^2 = 1$, and hence $m = 1$. This completes the proof.

Note, the condition that $a_nb_0 - a_0b_n \neq 0$ is not needed.