OLYMPIAD SOLUTIONS

OC71. Define a_n a sequence of positive integers by $a_1 = 1$ and a_{n+1} being the smallest integer so that

$$\mathrm{lcm}(a_1, .., a_{n+1}) > \mathrm{lcm}(a_1, .., a_n)\,.$$

Find the set $\{a_n | n \in \mathbb{Z}\}$.
(*Originally question 4 from the 2011 Austrian Mathematical Olympiad.*)

One incorrect solution was received to this problem.

As this problem is similar to Problem 2 from the fourth test of Romania IMO Selection Test 1995, which the editor wrote, we give a modified version of the Editor's solution to that problem.

Let $b_1 = 1, b_2 = 2, b_3 = 3, \ldots$ be the sequence of positive integers which are divisible by a most one prime, that is $b_1 = 1$ and $b_2, \ldots, b_n, \ldots$ are exactly the positive integers which are powers of primes.

Claim 1: Let n, k be so that $b_n \le k < b_{n+1}$. Then

$$\mathrm{lcm}\{1, 2, \ldots, k\} = \mathrm{lcm}\{b_1, b_2, \ldots, b_n\}\,.$$

Proof of Claim 1. As $\{b_1, b_2, \ldots, b_n\} \subset \{1, 2, \ldots, k\}$ we have

$$\mathrm{lcm}\{1, 2, \ldots, k\} \ge \mathrm{lcm}\{b_1, b_2, \ldots, b_n\}\,.$$

To complete the proof, we show that $\mathrm{lcm}\{b_1, b_2, \ldots, b_n\}$ is a common multiple of $\mathrm{lcm}\{1, 2, \ldots, k\}$.

Let $1 \le m \le k$, and let

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j}\,,$$

be the prime factorization of m. Then for each $1 \le i \le j$ there exists some k_i such that

$$p_i^{\alpha_i} = b_{k_i}\,.$$

As

$$1 \le p_i^{\alpha_i} \le m \le k < b_{n+1}\,,$$

we get $k_i \le n$. Therefore $p_i^{\alpha_i} = b_{k_i}$ divides $\mathrm{lcm}\{b_1, b_2, \ldots, b_n\}$.

As $p_1^{\alpha_1}$, $p_2^{\alpha_2}$, $\ldots$, $p_j^{\alpha_j}$ are pairwise relatively prime, and each one divides $\mathrm{lcm}\{b_1, b_2, \ldots, b_n\}$, it follows that their product m also divides $\mathrm{lcm}\{b_1, b_2, \ldots, b_n\}$.

Therefore $\mathrm{lcm}\{b_1, b_2, \ldots, b_n\}$ is divisible by all $1 \le m \le k$, which completes the proof of Claim 1.

Claim 2: For all $n \ge 1$ we have

$$\mathrm{lcm}\{b_1, b_2, \ldots, b_n\} < \mathrm{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,.$$

Proof of Claim 2. As $\{b_1, b_2, \ldots, b_n\} \subset \{b_1, b_2, \ldots, b_n, b_{n+1}\}$ we have

$$\text{lcm}\{b_1, b_2, \ldots, b_n\} \le \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,.$$

To prove our claim, we only need to show that

$$\text{lcm}\{b_1, b_2, \ldots, b_n\} \ne \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,.$$

Let $b_{n+1} = p^k$ for some prime p and some $k \ge 1$. Then, $p^k \mid \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}$.

If $k = 1$, then b_{n+1} is prime and none of $b_1, b_2, \ldots, b_n$ can be divisible by p, as $b_n < b_{n+1}$. Therefore

$$p \nmid \text{lcm}\{b_1, b_2, \ldots, b_n\}\,;\, p \mid \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,,$$

which shows that they cannot be equal.

If $k \ge 2$, then

$$p^k \mid \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,.$$

We claim that

$$p^k \nmid \text{lcm}\{b_1, b_2, \ldots, b_n\}\,.$$

Indeed, each of $b_1, b_2, \ldots, b_n$ is either relatively prime to p, or of the form p^i where $1 \le i \le k-1$. It follows immediately from here that the power of p in $\text{lcm}\{b_1, b_2, \ldots, b_n\}$ is exactly $k-1$.

Claim 3: For all $n \ge 1$ we have

$$a_n = b_n\,.$$

Proof of Claim 3. We prove this result by strong induction on n.

The initial step is obvious, as $b_1 = 1 = a_1$.

We now prove the inductive step. We know that $a_1 = b_1$, $a_2 = b_2$, $\ldots$, $a_n = b_n$ and we need to show that $a_{n+1} = b_{n+1}$.

Let $b_n \le k < b_{n+1}$. then by *Claim 1* we have

$$\text{lcm}\{a_1, a_2, \ldots, a_n\} = \text{lcm}\{b_1, b_2, \ldots, b_n\} = \text{lcm}\{1, 2, \ldots, k\} \ge \text{lcm}\{a_1, a_2, \ldots, a_n, k\}\,.$$

Therefore

$$\text{lcm}\{a_1, a_2, \ldots, a_n, k\} \le \text{lcm}\{a_1, a_2, \ldots, a_n\}\,,$$

for all $k < b_{n+1}$.

Moreover, by *Claim 2*, we have

$$\text{lcm}\{a_1, a_2, \ldots, a_n, b_{n+1}\} = \text{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\} > \text{lcm}\{b_1, b_2, \ldots, b_n\} = \text{lcm}\{a_1, a_2, \ldots, a_n\}\,.$$

Hence, the definition of a_{n+1} implies

$$a_{n+1} = b_{n+1}\,,$$

which proves the inductive step.

OC72. Prove that there are infinitely many positive integers so that the arithmetic and geometric mean of their divisors are integers.
(*Originally question 4 from the 2011 Kazahstan Math Olympiad, Grade 10.*)

Solved by Daniel Văcaru, Piteşti, Romania; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Zelator.

We show that if p is a prime of the form $3k+1$, then p^2 has the desired properties. As there are infinitely many primes of the form $3k+1$, this solves the problem.

Indeed, the divisors of p^2 are $1, p$ respectively p^2. Their arithmetic mean is

$$\frac{1+p+p^2}{3}.$$

As

$$1 \equiv p \equiv p^2 \pmod 3,$$

we have $3 \mid p^2+p+1$, and hence

$$\frac{1+p+p^2}{3} \in \mathbb{Z}$$

as claimed.

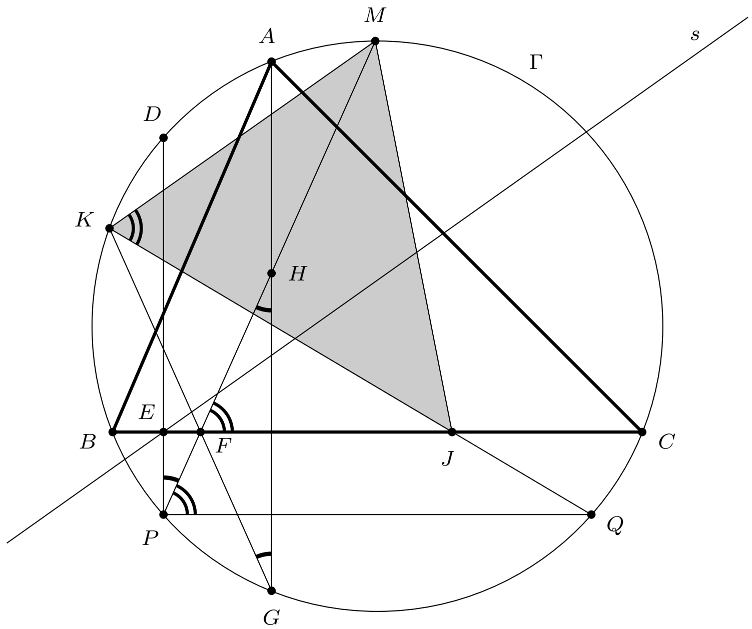
As geometric mean of $1, p, p^2$ is $p \in \mathbb{Z}$, we therefore see that p^2 has the desired property, as claimed.

OC73. Find all non-decreasing sequences $a_1, a_2, a_3, \ldots$ of natural numbers such that for each $i, j \in \mathbb{N}$, $i+j$ and $a_i + a_j$ have the same number of divisors. (a non-decreasing sequence is a sequence such that for all $i < j$, we have $a_i \leq a_j$.)
(*Originally question 3 from the 2011 Iran National Olympiad, Day 2.*)

No solution to this problem was received.

OC74. Let H be the orthocenter of an acute triangle ABC with circumcircle Γ. Let P be a point on the arc BC (not containing A) of Γ, and let M be a point on the arc CA (not containing B) of Γ such that H lies on the segment PM. Let K be another point on Γ such that KM is parallel to the Simson line of P with respect to triangle ABC. Let Q be another point on Γ such that $PQ \parallel BC$. Segments BC and KQ intersect at a point J. Prove that $\triangle KJM$ is an isosceles triangle.
(*Originally question 1 from the 2011 China team selection test, Day 2.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.



Let the perpendicular from P onto BC intersect Γ and BC at points D respectively E, where $D \neq P$. Let s be the Simson line of P. Let F be the intersection of BC and MP, and let the line AH meet Γ at point $G \neq A$. By a well-known property of the Simson line (see [1] § 288), we have $AD \parallel s$. Hence, $AD \parallel KM$ and $\widehat{AK} = \widehat{MD}$. As the quadrilateral $ABGC$ is cyclic and because the sides of $\angle BAC$ are perpendicular to the sides of $\angle BHC$, we have $\angle CGB = 180^\circ - \angle BAC = \angle BHC$. Thus, BC is the perpendicular bisector of GH. In other words, the reflection of the orthocentre in the side of a triangle is a point of the circumcircle; this is, in fact, Theorem 178 of [1].

It follows that $\angle AGK = \angle MPD = \angle FPE = \angle FHG = \angle AGF$. Thus, F, G, and K are collinear. Whence, $\angle JKM = \angle QKM = \angle QPM = \angle JFM$, that is, the points F, J, M, and K are concyclic. Therefore,

$$\begin{aligned}\angle KMJ &= 180^\circ - \angle JKM - \angle MJK = 180^\circ - \angle JFM - \angle MFK \\ &= 180^\circ - \angle JFK = \angle KFB = \angle GFJ = \angle JFH = \angle JFM = \angle JKM.\end{aligned}$$

Consequently, $JK = JM$, and $\triangle KJM$ is isosceles.

References

[1] N. Altshiller-Court, *College Geometry*, 2nd ed., New York, 1952

OC75. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $Q(x) = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_0$ be two polynomials with integral coefficients so that $a_n - b_n$ is a prime, $a_nb_0 - a_0b_n \neq 0$, and $a_{n-1} = b_{n-1}$. Suppose that there exists a rational number r such that $P(r) = Q(r) = 0$. Prove that $r \in \mathbb{Z}$.
(*Originally question 3 from the 2011 India Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, Franceand Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Bataille, modified by the editor.

Let $a_n - b_n = p$.

If $n = 1$ then $P(x) - Q(x) = px$ and hence $r = 0 \in \mathbb{Z}$.

Now assume $n \geq 2$. Since $a_{n-1} = b_{n-1}$, we have

$$P(x) - Q(x) = px^n - c_{n-2}x^{n-2} - \cdots - c_0\,,$$

with $c_j = b_j - a_j$, $0 \leq j \leq n-2$. As $P(r) = Q(r) = 0$ we have

$$pr^n = c_{n-2}r^{n-2} + \cdots + c_1r + c_0\,. \tag{1}$$

If $r = 0$ we are done. Otherwise, we have $r = \dfrac{l}{m}$ for some co-prime integers l, m with $l \neq 0$ and $m \geq 1$. Multiplying 1 by m^{n-2} we get

$$\frac{pl^n}{m^2} = c_{n-2}l^{n-2} + c_{n-3}l^{n-3}m + \cdots + c_1lm^{n-3} + c_0m^{n-2}\,.$$

Therefore $\dfrac{pl^n}{m^2} \in \mathbb{Z}$, which implies $m^2 \mid pl^n$. As m, n are co-prime, m^2 and l^n are also co-prime, and hence m^2 must divide p. Since p is prime, it follows that $m^2 = 1$, and hence $m = 1$. This completes the proof.

Note, the condition that $a_nb_0 - a_0b_n \neq 0$ is not needed.