FOCUS ON ...
No. 7

Michel Bataille

Decomposition into Partial Fractions

Introduction

The decomposition of a rational fraction into partial fractions is primarily taught because of its applications to integration. The rather heavy calculations involved often lead to some resentment against this technique. The few examples selected here only require a small amount of calculations and might, if need be, reconcile one to this useful algebraic tool.

Using the polynomial part

Let F be any field and $A(x) = \dfrac{N(x)}{D(x)}$ where $N(x)$, $D(x)$ are coprime elements of $F[x]$. The polynomial part of $A(x)$ is the quotient in the division of $N(x)$ by $D(x)$: if $N(x) = D(x)Q(x) + R(x)$ with $\text{degree}(R(x)) < \text{degree}(D(x))$, this polynomial part, which we will denote $pp(A(x))$, is just $Q(x)$. Then we can write $A(x) = pp(A(x)) + A_1(x)$ where $A_1(x) = \dfrac{R(x)}{D(x)}$. We have achieved the very first step of the decomposition of the fraction $A(x)$; the rest follows from the general theorems which apply to fractions with a numerator of degree less than the denominator (to review these theorems, see for example [1]). Of course, $pp(A(x)) = 0$ if $\text{degree}(N(x)) < \text{degree}(D(x))$, and $pp(A(x)) = 1$ if $N(x)$, $D(x)$ are both monic polynomials of the same positive degree. Another useful example is $A(x) = \dfrac{x^m}{x-w}$ where m is a positive integer and w a nonzero element of F: writing $x^m = x^m - w^m + w^m$, we readily see that

$$pp(A(x)) = x^{m-1} + wx^{m-2} + \cdots + w^{m-2}x + w^{m-1}.$$

An important, not often explicated property, is the linearity of the function pp:

$$pp(\alpha A(x) + \beta B(x)) = \alpha pp(A(x)) + \beta pp(B(x))$$

whenever α, β are in F and $A(x), B(x)$ are rational fractions on F. As an application, consider n distinct elements $w_1, w_2, \ldots, w_n$ of F and the associated sums

$$S_n(m) = \sum_{i=1}^{n} \frac{w_i^m}{\displaystyle\prod_{k=1, k\neq i}^{n} (w_i - w_k)}$$

where m is a nonnegative integer. We revisit the calculation of these sums that have been introduced in several Crux problems (compare with solution *III* of problem **2487** [2000 : 512] or the solution to **KLAMKIN–10** [2006 : 323]). Let $D(x) = \prod_{i=1}^{n}(x - w_i)$ so that $\prod_{k=1,k\neq i}^{n}(w_i - w_k)$ is just $D'(w_i)$ and

$$\frac{1}{D(x)} = \sum_{i=1}^{n} \frac{1}{D'(w_i)} \cdot \frac{1}{x - w_i} \tag{1}$$

($D'(x)$ is the derivative of $D(x)$). Now, multiply each side of (1) by x^n and equate the polynomial parts to obtain

$$\begin{aligned} 1 &= \sum_{i=1}^{n} \frac{x^{n-1} + w_i x^{n-2} + \cdots + w_i^{n-2}x + w_i^{n-1}}{D'(w_i)} \\ &= x^{n-1}S_n(0) + x^{n-2}S_n(1) + \cdots + xS_n(n-2) + S_n(n-1) \end{aligned}$$

and, in one sweep,

$$\sum_{i=1}^{n} \frac{w_i^m}{\prod_{k=1,k\neq i}^{n}(w_i - w_k)} = 0, \text{ for } m = 0, 1, \ldots, n-2,$$

$$\text{and} \quad \sum_{i=1}^{n} \frac{w_i^{n-1}}{\prod_{k=1,k\neq i}^{n}(w_i - w_k)} = 1. \tag{2}$$

What if $m > n - 1$? Well, the method still applies but becomes more and more involved as m increases. Indeed, as above, multiply (1) by x^m. If we set $m = n + s$ and take (2) into account, the polynomial part of the right-hand side is

$$x^s + S_n(n)x^{s-1} + \cdots + S_n(n+s-2)x + S_n(n+s-1).$$

It remains to compare this to the polynomial part of the left-hand side $\frac{x^{n+s}}{D(x)}$ (directly obtained by division). For example, it is easily checked that

$$S_n(n) = w_1 + w_2 + \cdots + w_n, \quad S_n(n+1) = (w_1 + w_2 + \cdots + w_n)^2 - \sum_{i<j} w_i w_j.$$

[The readers familiar with Faa di Bruno's formula will derive a general expression of $S_n(n + s - 1)$ as a polynomial in the elementary symmetric functions of $w_1, w_2, \ldots, w_n$.]

The decomposition of $\dfrac{1}{x^n-1}$

Here we take $F = \mathbb{C}$, the field of complex numbers. The decomposition of $\dfrac{1}{x^n-1}$ is easy to obtain and deserves to be well-known. Let $\omega = \exp(-2\pi i/n)$. Here,

$$D(x) = x^n - 1 = \prod_{k=0}^{n-1}(x - \omega^{-k}) \qquad \text{and} \qquad D'(\omega^{-k}) = n\omega^{-k(n-1)} = n\omega^k.$$

It follows that

$$\frac{1}{x^n-1} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\omega^k x - 1}. \tag{3}$$

To see this decomposition at work, consider the following trigonometric identity: Let α, β be real numbers with $\alpha, \beta \notin \pi\mathbb{Z}$ and n a positive integer. Then

$$\sum_{k=0}^{n-1}\left(\cot\left(\frac{\alpha - k\pi}{n}\right) - \cot\left(\frac{\beta - k\pi}{n}\right)\right) = n(\cot\alpha - \cot\beta).$$

Proof. From $\cot(\alpha) = i + \dfrac{2i}{e^{2i\alpha}-1}$ (easily checked), we deduce

$$n(\cot\alpha - \cot\beta) = 2ni\left(\frac{1}{e^{2i\alpha}-1} - \frac{1}{e^{2i\beta}-1}\right).$$

Now, from (3), we have

$$\frac{1}{e^{2i\alpha}-1} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\omega^k e^{2\alpha i/n}-1} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{e^{2i(\alpha-k\pi)/n}-1}$$

hence

$$\begin{aligned} n(\cot\alpha - \cot\beta) &= \sum_{k=0}^{n-1}\left(\frac{2i}{e^{2i(\alpha-k\pi)/n}-1} - \frac{2i}{e^{2i(\beta-k\pi)/n}-1}\right) \\ &= \sum_{k=0}^{n-1}\left(\cot\left(\frac{\alpha-k\pi}{n}\right) - \cot\left(\frac{\beta-k\pi}{n}\right)\right). \end{aligned}$$

An unexpected application

Consider the following problem: Let $a_1, a_2, \ldots, a_n$ be n distinct positive real numbers, where $n \geq 2$. For $k = 1, 2, \ldots, n$, let $p_k = \prod_{j\neq k}(a_j - a_k)$. Show that $\displaystyle\sum_{k=1}^{n}\frac{1}{p_k\sqrt{a_k}} > 0$.

Rather surprisingly, a simple and elegant solution follows from the decomposition of a rational fraction. We introduce

$$A(x) = \frac{1}{(x^2+a_1)(x^2+a_2)\cdots(x^2+a_n)},$$

whose decomposition in $\mathbb{R}(x)$ is of the form

$$A(x) = \sum_{k=1}^{n} \frac{\alpha_k x + \beta_k}{x^2 + a_k} \tag{4}$$

for some real numbers α_k, β_k, where $k = 1, 2, \ldots, n$. Multiplying both sides of (4) by $x^2 + a_k$ and taking $x = i\sqrt{a_k}$ in the deduced equality immediately show that $\alpha_k = 0$, $\beta_k = \frac{1}{p_k}$, hence

$$A(x) = \sum_{k=1}^{n} \frac{1}{p_k} \cdot \frac{1}{x^2 + a_k}. \tag{5}$$

We will use (5) to compute an integral (at last!). Since $\int_0^\infty \frac{dx}{x^2 + a_k} = \frac{\pi}{2\sqrt{a_k}}$, we see that

$$\int_0^\infty A(x)\,dx = \frac{\pi}{2} \sum_{k=1}^{n} \frac{1}{p_k\sqrt{a_k}}.$$

The desired inequality follows since this integral is positive (recall that $A(x) > 0$ for $x \in [0, \infty)$).

Exercises

Here are some examples that are close to those treated above.

(a) Consider the sums $S_n(m)$ again and suppose $w_i \neq 0$ $(i = 1, 2, \ldots, n)$. Calculate $S_n(-1)$ and $S_n(-2)$ (hint: for the latter, first compute the derivative of $\frac{1}{D(x)}$).

(b) Using the decomposition of $\frac{1}{x^n - 1}$, rework problem **2657** [2001 : 336 ; 2002 : 401]: Prove that

$$\sum_{n=0}^{2k-1} \tan\left(\frac{(4n-1)\pi + (-1)^n 4\theta}{8k}\right) = \frac{2k}{1 + (-1)^{k+1}\sqrt{2}\sin\theta}.$$

(c) Problem **3140** ([2006 : 238,240; 2007 : 243]) required a proof of the inequality $\prod_{k=1}^{n} a_k^{\frac{1}{p_k}} < 1$ (with the notations of the last paragraph). Find an alternative to Walther Janous's featured proof.

References

[1] Algebra, J. W. Archbold, Pitman, 1964, pp. 134-151.