THE CONTEST CORNER

No. 15

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The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention LastName_FirstName_CCProblemNumber *(example Doe_Jane_CC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at* crux-contest@cms.math.ca. *Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page.*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 September 2014***, although solutions received after this date will also be considered until the time when a solution is published.*

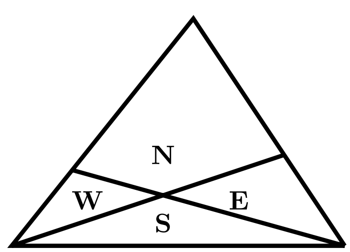
Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the Solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC71. A bag is filled with red and blue balls. Before drawing a ball, there is a $\frac{1}{4}$ chance of drawing a blue ball. After drawing out a ball, there is now a $\frac{1}{5}$ chance of drawing a blue ball. How many red balls are in the bag?

CC72. From the set of natural numbers $1, 2, 3, \ldots, n$, four consecutive even numbers are removed. The remaining numbers have an average value of $51\frac{9}{16}$. Determine all sets of four consecutive even numbers whose removal creates this situation.

CC73. A farmer owns a triangular field, as shown. He reckons 5 sheep can graze in the west field, 10 sheep can graze in the south field, and 8 can graze in the east field. (All sheep eat the same amount of grass.) How many sheep can graze in the north field?



CC74. Let $1000 \leq n = ABCD_{10} \leq 9999$ be a positive integer whose digits $ABCD$ satisfy the divisibility condition:

$$1111 \mid (ABCD + AB \times CD).$$

Determine the smallest possible value of n.

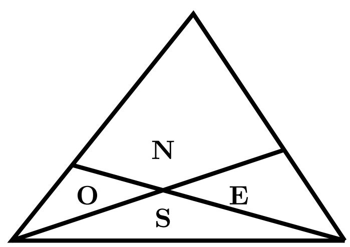
CC75. Let P be a point inside the triangle ABC such that $\angle PAC = 10°$, $\angle PCA = 20°$, $\angle PAB = 30°$ and $\angle ABC = 40°$. Determine $\angle BPC$.

. . . .

CC71. Un sac contient des boules rouges et des boules bleues. Si on pige au hasard une boule du sac, la probabilité de choisir une boule bleue est de $\frac{1}{4}$. Après avoir pigé une boule, la probabilité de choisir une boule bleue est maintenant de $\frac{1}{5}$. Combien y a-t-il de boules rouges dans le sac ?

CC72. On enlève quatre entiers pairs consécutifs de l'ensemble contenant les entiers positifs $1, 2, 3, \ldots, n$. Les nombres qui restent ont une moyenne de $51\frac{9}{16}$. Déterminer tous les ensembles de quatre entiers pairs consécutifs que l'on aurait pu enlever.

CC73. Un fermier possède un champ de forme triangulaire, comme dans la figure ci-dessous. Il calcule que 5 brebis peuvent brouter dans le champ ouest, 10 brebis peuvent brouter dans le champ sud et 8 brebis peuvent brouter dans le champ est. (Toutes les brebis mangent la même quantité d'herbe.) Combien de brebis peuvent brouter dans le champ nord ?



CC74. Soit n $(1000 \leq n = ABCD_{10} \leq 9999)$ un entier positif dont les chiffres $ABCD$ vérifient la condition de divisibilité suivante :

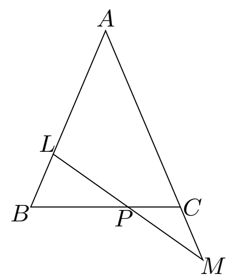
$$1111 \mid (ABCD + AB \times CD).$$

Déterminer la plus petite valeur possible de n.

CC75. Soit P un point à l'intérieur du triangle ABC de manière que $\angle PAC = 10°$, $\angle PCA = 20°$, $\angle PAB = 30°$ et $\angle ABC = 40°$. Déterminer la mesure de l'angle BPC.

CONTEST CORNER SOLUTIONS

CC21. In the diagram ΔABC is isosceles with $AB = AC$. Prove that if $LP = PM$, then $LB = CM$.
(*Originally question # 10 from the 2008 Manitoba Mathematical Competition.*)



Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; Matei Coiculescu, East Lyme High School, East Lyme, CT, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany; Richard I. Hess, Rancho Palos Verdes, CA, USA; David Jonathan, Palembang, Indonesia; Mihaï-Ioan Stoënescu, Bischwiller, France; Daniel Văcaru, Piteşti, Romania; Jacques Vernin, Marseille, France; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We present the solution of Heuver.

Let $LX \parallel AM$ meet BC in X. Then triangles LPX and MPC are congruent. Furthermore triangle BLX is isosceles with $BL = LX$. It follows that $BL = LX = CM$ as required.

CC22. Points $A_1, A_2, \ldots, A_{2k}$ are equally spaced around the circumference of a circle and $k \geq 2$. Three of these points are selected at random and a triangle is formed using these points as its vertices. Determine the probability that the triangle is acute.
(*Originally question # 10 b) from the 2006 Euclid Competition.*)

One incorrect solution was received.

CC23. The three-term geometric progression $(2, 10, 50)$ is such that

$$(2 + 10 + 50) \times (2 - 10 + 50) = 2^2 + 10^2 + 50^2.$$

(a) Generalize this (with proof) to other three-term geometric progressions.

(b) Generalize this (with proof) to geometric progressions of length n.

(*Originally question #5 from the 2000 APICS Competition.*)

Solved by Norvald Midttun, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; and Titu Zvonaru, Comăneşti, Romania. Partial solutions by Matei Coiculescu, East Lyme High School, East Lyme, CT, USA; Greg Cook, Angelo State University, San Angelo, TX, USA; Jacques Vernin, Marseille, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Zvonaru modified by the editor.

a) Suppose a geometric progression has first term a and common ratio r, so that its terms are a, ar and ar^2. We shall assume the geometric progression is non-trivial, so $r \neq \pm 1$. We want to show

$$(a + ar + ar^2)(a + a(-r) + a(-r)^2) = a^2 + (ar)^2 + (ar^2)^2.$$

Evaluating the left side, we have

$$\begin{aligned}(a + ar + ar^2)(a + a(-r) + a(-r)^2) &= a\left(\frac{r^3-1}{r-1}\right) \cdot a\left(\frac{(-r)^3-1}{-r-1}\right) \\ &= a^2\frac{(r^3-1)(r^3+1)}{(r-1)(r+1)} \\ &= a^2\frac{r^6-1}{r^2-1} \\ &= a^2(1 + r^2 + r^4) \\ &= a^2 + (ar)^2 + (ar^2)^2,\end{aligned}$$

as desired.

b) To generalize part a), we establish a similar identity but split the cases when n is even or odd. First consider when n is odd. We will establish that for any a, r with $r \neq \pm 1$,

$$\begin{aligned}&(a + ar + ar^2 + \cdots + ar^{n-1})(a + a(-r) + a(-r)^2 + \cdots + a(-r)^{n-1}) \\ &= a^2 + (ar)^2 + (ar^2)^2 + \cdots + (ar^{n-1})^2.\end{aligned}$$

Similar to part a), we have

$$\begin{aligned}&(a + ar + ar^2 + \cdots + ar^{n-1})(a + a(-r) + a(-r)^2 + \cdots + a(-r)^{n-1}) \\ &= a^2\frac{(r^n-1)((-r)^n-1)}{(r-1)(-r-1)} \\ &= a^2\frac{r^{2n}-1}{r^2-1} \\ &= a^2(1 + r^2 + r^4 + \cdots + r^{2n-2}) \\ &= a^2 + (ar)^2 + (ar^2)^2 + \cdots + (ar^{n-1})^2,\end{aligned}$$

as desired.

Now consider when n is even, say $n = 2k+2$ where $k \geq 0$. Then the terms of the geometric progression are $a, ar, ar^2, \cdots, ar^{2k+1}$. We will prove that

$$\begin{aligned}&(a + ar + ar^2 + \cdots + ar^{2k+1})(a + a(-r) + a(-r)^2 + \cdots + a(-r)^{2k+1})\\&= a^2 + (ar)^2 + (ar^2)^2 + \cdots + (ar^k)^2 - (ar^{k+1})^2 - \cdots - (ar^{2k+1})^2.\end{aligned}$$

Indeed we have

$$\begin{aligned}&a^2 + (ar)^2 + (ar^2)^2 + \cdots + (ar^k)^2 - (ar^{k+1})^2 - \cdots - (ar^{2k+1})^2\\&= (a^2 + (ar)^2 + \cdots + (ar^k)^2)(1 - ar^{2k+2})\\&= \frac{a^2r^{2k+2} - 1}{r^2 - 1}(1 - ar^{2k+2})\\&= -\frac{(a^2r^{2k+2} - 1)^2}{r^2 - 1}\end{aligned}$$

whereas

$$\begin{aligned}&(a + ar + ar^2 + \cdots + ar^{2k+1})(a + a(-r) + a(-r)^2 + \cdots + a(-r)^{2k+1})\\&= \frac{ar^{2k+2} - 1}{r - 1} \cdot \frac{a(-r)^{2k+2} - 1}{-r - 1}\\&= -\frac{(a^2r^{2k+2} - 1)^2}{r^2 - 1}.\end{aligned}$$

exactly as desired.

CC24. Given the equation

$$x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 = 24.$$

(a) Prove that the equation has no integer solutions.
(b) Does this equation have rational solutions? If yes, give an example. If no, prove it.
(*Originally question #2 from the 2009 Memorial University of Newfoundland Undergraduate Mathematics Competition.*)

Solved by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany; Billy Jin, Waterloo Collegiate Institute and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We present Curtis' solution.

We can rearrange the equation,

$$x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 = 24 \qquad (1)$$

into the factored form,

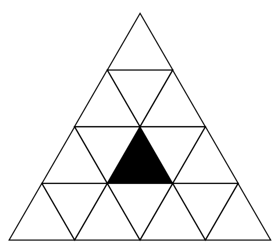
$$-(x + y + z)(y + z - x)(z + x - y)(x + y - z) = 24 \qquad (2)$$

(a) If x, y and z are integers, then all four factors have the same parity since, for example, $x+y+z$ and $y+z-x$ differ by the even number $2x$. But the product of four odd numbers is odd, so our four factors must be even. There do not exist four even numbers whose product is -24, as any such product must be divisible by 16, which -24 is not. Hence the equation (1) has no integer solutions.

(b) One the other hand, $(-\frac{5}{2}, -\frac{1}{2}, -1)$ is a solution to (2) and hence (1) in rational numbers since

$$\begin{aligned}\left[\left(-\frac{5}{2}\right)+\left(-\frac{1}{2}\right)+(-1)\right]&\left[\left(-\frac{1}{2}\right)+(-1)-\left(-\frac{5}{2}\right)\right]\\ &\times\left[(-1)+\left(-\frac{5}{2}\right)-\left(-\frac{1}{2}\right)\right]\left[\left(-\frac{5}{2}\right)+\left(-\frac{1}{2}\right)-(-1)\right]\\ &=(-4)(1)(-3)(-2)=24\end{aligned}$$

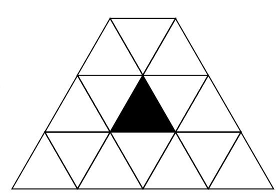
CC25. Alphonse and Beryl are playing a game, starting with the geometric shape shown. Alphonse begins the game by cutting the original shape into two pieces along one of the lines. He then passes the piece containing the black triangle to Beryl, and discards the other piece. Beryl repeats these steps with the piece she receives; that is to say, she cuts along the length of a line, passes the piece containing the black triangle back to Alphonse, and discards the other piece. This process continues, with the winner being the player who, at the beginning of his or her turn, receives only the black triangle. Is there a strategy that Alphonse can use to be guaranteed that he will win?
(*Originally question B3 b) from the 2000 Canadian Open Mathematics Challenge.*)

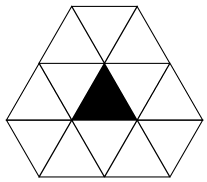


Solved by Richard I. Hess, Rancho Palos Verdes, CA, USA.

We present his solution below (with a little correction from the editor).

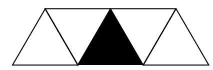
Alphonse's strategy is as follows. On his first move, he should remove one of the three corners, leaving the shape on the right. He then passes things off to Beryl, who has one of two possible actions.

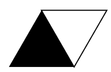




If Beryl snips another small corner, Alphonse responds by snipping off the remaining corner to give the shape on the left. Then Beryl must snip either a row of 3 triangles or a row of 5 triangles. Alphonse will then make an appropriate parallel cut and hand the shape below to Beryl.

At this point, Beryl has two options, snip off a single triangle, or snip off two triangles. Whatever she does, Alphonse performs the opposite cut (cutting off two triangles if she snips one, and snipping off one if she snips off two triangles). This leaves Beryl with the final figure, which she must cut and then hand only the black piece to Alphonse.





Alternatively, she could snip off more than a single triangle. Alphonse's response is to make a cut parallel to Beryl's to produce the 5-triangle winning position shown above, and proceed as discussed.

[*Ed.: The submitted solution was correct up until the row of five triangles, where their solution indicated a mirroring strategy would work. This would work for a simple misinterpretation of the rules, where the winner is the one who hands the black triangle away at the end of their turn. We give Mr. Hess the benefit of the doubt and present his otherwise fine solution.*]