# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

**3791**. [2012 : 420, 422] *Proposed by John Lander Leonard.*

The numbers 0 through 12 are randomly arranged around a circle.

(a) Show that there must exist a trio of three adjacent numbers which sum to at least 18.

(b) Determine the maximum $n$ such that there must exist a trio of three adjacent numbers which sum to at least $n$.

*Solved by E. G. Goodaire and M. Parmenter; R. Hess; S. Malikić; Skidmore College Problem Group; P. Y. Woo (generalization of part (a)); T. Zvonaru; and the proposer. We present a composite of similar solutions submitted by Edgar G. Goodaire and Michael Parmenter (together) and by Titu Zvonaru (independently).*

Set $a_0 = 0$ and let $a_1, a_2, \dots, a_{12}$ be a permutation of the integers from 1 to 12. Since

$$(a_1+a_2+a_3)+(a_4+a_5+a_6)+(a_7+a_8+a_9)+(a_{10}+a_{11}+a_{12}) = 1+2+\cdots+12 = 78,$$

we deduce that there exists an index $i \in \{1, 4, 7, 10\}$ such that

$$a_i + a_{i+1} + a_{i+2} \geq \left\lceil \frac{78}{4} \right\rceil = 20.$$

Taking $(a_0, a_1, a_2, \dots, a_{12}) = (0, 12, 7, 1, 11, 6, 3, 10, 5, 4, 9, 2, 8)$, we see that

$$\max\{a_0+a_1+a_2, a_1+a_2+a_3, \dots, a_{10}+a_{11}+a_{12}, a_{11}+a_{12}+a_0, a_{12}+a_0+a_1\} = 20.$$

Thus, $n = 20$ must be the maximum value of $n$ satisfying the requirement of part (b), namely, there must exist a trio of three adjacent numbers that sum to at least 20. Of course, this argument also provides a solution to part (a).

**3792**. [2012 : 420, 422; 2013 : 31, 34] *Proposed by Marcel Chiriţiţă.*

Solve the following system

$$2^x + 2^y = 12$$
$$3^x + 3^y = 36$$

for $x, y \in \mathbb{R}$.

*Solved by Š. Arslanagić (2 solutions); R. Barbara; M. Bataille; B. D. Beasley; R. Boukharfane; P. Deiermann; J. Hawkins and D. R. Stone; R. Hess; N. Hodžić and S. Malikić; V. Konečný; O. Kouba; J. Ling; D. E. Manes; P. Perfetti; V. Sadaphal; D. Smith; H. Wang and J. Wojdylo; and the proposer. We present the solution by Roy Barbara.*

One easily confirms that $x = 2, y = 3$ is a solution to the given system of equations, as is $x = 3, y = 2$. We shall see that these are the only solutions. The claim is based on the following observation: Fix positive constants $a$ and $b$ and let $k > 1$; then the equation in $t$,

$$t^k + (a-t)^k = b,\ 0 \le t \le a,$$

can have at most two solutions. Indeed, consider the real continuous function in $t$ (with $0 \le t \le a$), $f(t) = t^k + (a-t)^k - b$. Since $k - 1 > 0$, the derivative $f'(t) = k\left(t^{k-1} - (a-t)^{k-1}\right)$ is negative for $t < \frac{a}{2}$, it vanishes at $t = \frac{a}{2}$, and is positive for $t > \frac{a}{2}$. Hence, $f(t)$ strictly decreases form 0 to $\frac{a}{2}$, and strictly increases from $\frac{a}{2}$ to $a$. The observation follows.

Set $t = 2^x, u = 2^y, k = \frac{\ln 3}{\ln 2}$, and note that $k > 1$. The given system becomes

$$\begin{aligned} t + u &= 12 \\ t^k + u^k &= 36. \end{aligned}$$

Hence, $t^k + (12-t)^k = 36$. As $t$ and $u$ are positive, we have $0 < t < 12$. By the observation our equation in $t$ has at most two solutions, namely $t = 4, 8$. Hence $(t, u) = (4, 8), (8, 4)$ are all the solutions of our transformed system of equations, yielding (as $2^x = t, 2^y = u$) all the solutions $(x, y) = (2, 3), (3, 2)$ of the given system.

*Editors Comment.* Deiermann's submission dealt with the more general system of equations,

$$\begin{aligned} a^x + a^y &= a^2(a+1) \\ b^x + b^y &= b^2(b+1), \end{aligned}$$

for $1 < a < b$. Using arguments similar to our featured solution, he proved that the only solutions are $\{x, y\} = \{2, 3\}$. Hawkins and Stone provided a similar generalization.

**3793**. [2012 : 420, 422; ; 2013 : 89, 91] *Proposed by George Apostolopoulos.*

Let $a$, $b$, and $c$ be positive real numbers such that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} = 1007\sqrt{2}\ .$$

Find the minimum value of the expression

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}\ .$$

*Solved by A. Alt; AN-anduud Problem Solving Group; Š. Arslanagić; D. Bailey, E. Campbell, and C. Diminnie; M. Bataille; N. Evgenidis; O. Geupel; R. Hess; J. G. Heuver; O. Kouba; A. Li; S. Malikić; M. R. Modak; C. Mortici; P. Perfetti; R. Peiro; A. Plaza; C. M. Quang (found the maximum only); D. Smith; I. Stallion; E. Swylan; I. Uchiha; S. Wagon; H. Wang and J. Wojdylo; T. Zvonaru; and the proposer. We present 2 solutions.*

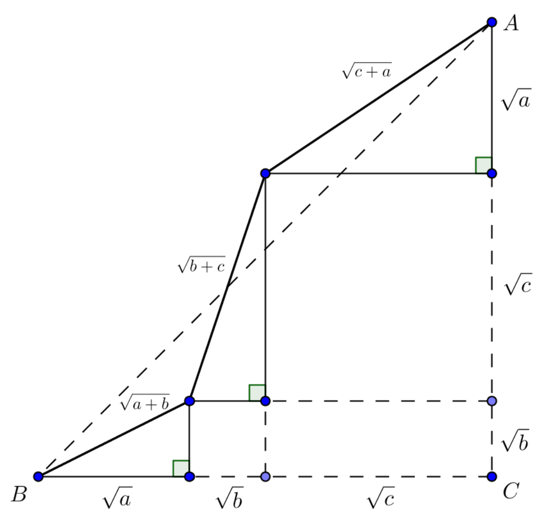
*Solution 1 by Ang Li.*

By using the root-mean-square inequality we find that

$$\begin{aligned}\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} &= \sqrt{2}\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}\right)\\ &= \sqrt{2}\left(\sqrt{\frac{(\sqrt{a})^2+(\sqrt{b})^2}{2}}+\sqrt{\frac{(\sqrt{b})^2+(\sqrt{c})^2}{2}}+\sqrt{\frac{(\sqrt{c})^2+(\sqrt{a})^2}{2}}\right)\\ &\geq \sqrt{2}\left(\frac{\sqrt{a}+\sqrt{b}}{2}+\frac{\sqrt{b}+\sqrt{c}}{2}+\frac{\sqrt{c}+\sqrt{a}}{2}\right)\\ &= \sqrt{2}(\sqrt{a}+\sqrt{b}+\sqrt{c}) = \sqrt{2}(1007\sqrt{2}) = 2014.\end{aligned}$$

Equality holds if and only if $a=b=c=2\left(\frac{1007}{3}\right)^2$, which implies that the required minimum value is 2014.

*Solution 2 by Itachi Uchiha.*




In the diagram $AB$ is the hypotenuse of the right triangle $ABC$, which shows that

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \geq AB = \sqrt{2}(\sqrt{a}+\sqrt{b}+\sqrt{c}) = 2014.$$

Equality holds if and only if the slopes of the thick lines are equal, namely $\frac{\sqrt{b}}{\sqrt{a}} = \frac{\sqrt{c}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{c}}$; that is, if and only if $a=b=c$, and then the desired minimum is 2014.

*Editor's Comment.* Most of the submitted solutions were quite similar to the first of the featured solutions. Due to a transcription error, the problem before the correction ([2012: 420, 422]) called for the maximum value of $\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}$. Several solvers observed that because $\sqrt{x+y}<\sqrt{x}+\sqrt{y}$ when $x$ and $y$ are both positive,

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}<2(\sqrt{a}+\sqrt{b}+\sqrt{c})=2014\sqrt{2}.$$

As the sum on the left equals $2014\sqrt{2}$ when $\sqrt{a}=1007\sqrt{2}$ while $b=c=0$, we deduce that $2014\sqrt{2}$ is the least upper bound (which cannot be attained using positive values for $a,b$, and $c$). Also many correspondents observed that the quantity $1007\sqrt{2}$ in the statement of the problem could be replaced by any constant $k$, in which case we would have

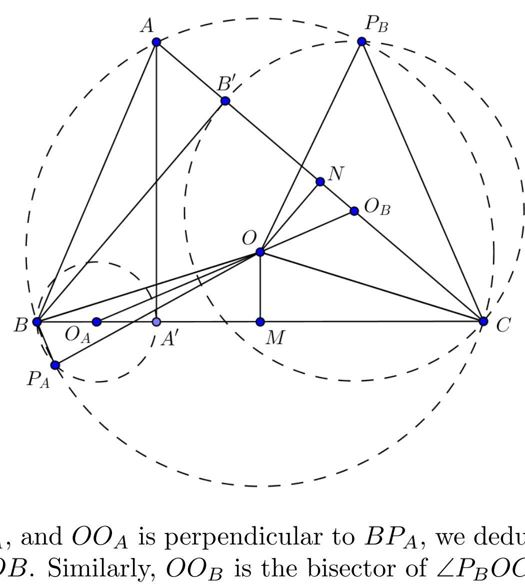
$$\sqrt{2}\,k\leq\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}<2k.$$

**3794**. [2012 : 420, 422] *Proposed by Václav Konečný.*

Let an acute triangle $ABC$ be inscribed in the circle $\Gamma$, and $A'$ and $B'$ be the feet of the altitudes from $A$ and $B$ respectively. Let the circle on diameter $BA'$ intersect $\Gamma$ again at $P_A$, and the circle on diameter $CB'$ intersect $\Gamma$ again at $P_B$. Determine the angle between the lines $AP_A$ and $BP_B$.

*Solved by T. Zvonaru; and the proposer. We present the solution by Titu Zvonaru.*

As usual, let $a=BC$, $b=CA$, and $c=AB$. Let $O$ be the center of $\Gamma$ and $R$ the radius of $\Gamma$. Let $M$ and $N$ be the midpoints of of $BC$ and $CA$, respectively, and let $O_A$ and $O_B$ be the midpoints of $BA'$ and $CB'$, respectively.




Since $OB=OP_A$, and $OO_A$ is perpendicular to $BP_A$, we deduce that $OO_A$ is the bisector of $\angle P_AOB$. Similarly, $OO_B$ is the bisector of $\angle P_BOC$.

Denote $x = \tan A$, $y = \tan B$, $z = \tan C$. It is known that $x + y + z = xyz$.

We have

$$O_A M = BM - O_A B = \frac{a - c\cos B}{2} = \frac{b\cos C}{2}$$

and $OM = R\cos A$; hence

$$\begin{aligned}\tan\angle O_A OM &= \frac{O_A M}{OM} = \frac{b\cos C}{2R\cos A} = \frac{\sin B\cos C}{\cos A} = \frac{\sin B\cos C}{-\cos(B+C)}\\ &= \frac{\sin B\cos C}{\sin B\sin C - \cos B\cos C} = \frac{\tan B}{\tan B\tan C - 1} = \frac{y}{yz-1}.\end{aligned}$$

Similarly $\tan\angle O_B ON = \frac{z}{zx-1}$. Since $\angle BOM = A$, it follows that

$$\tan\angle BOO_A = \tan(A - \angle O_A OM) = \frac{x - \frac{y}{yz-1}}{1 + \frac{xy}{yz-1}} = \frac{xyz - x - y}{xy + yz - 1} = \frac{z}{xy + yz - 1}.$$

We denote by $\phi$ the angle between the lines $AP_A$ and $BP_B$. We have

$$\begin{aligned}\phi &= \frac{BP_A + AP_B}{2} = \frac{BP_A}{2} + \frac{AC}{2} - \frac{CP_B}{2} = \angle BOO_A + B - \angle COO_B\\ &= \angle BOO_A + B - (B - \angle O_B ON) = \angle BOO_A + \angle O_B ON.\end{aligned}$$

Then we have:

$$\tan\phi = \frac{\frac{z}{xy+yz-1} + \frac{z}{zx-1}}{1 - \frac{z^2}{(xy+yz-1)(zx-1)}} = \frac{z(xy + yz + zx - 2)}{(xy + yz - 1)(zx - 1) - z^2}.$$

Since

$$\begin{aligned}&(xy + yz - 1)(zx - 1) - z^2\\ &= x^2yz - xy + xyz^2 - yz - xz + 1 - z^2\\ &= x(x + y + z) + z(x + y + z) - xy - yz - zx + 1 - z^2\\ &= 1 + zx + x^2,\end{aligned}$$

it follows that

$$\tan\phi = \frac{z(xy + yz + zx - 2)}{1 + zx + x^2}.$$

**3795**. [2012 : 421, 422] *Proposed by José Luis Díaz-Barrero.*

Let $a$, $b$, $c$ be the lengths of the sides of a triangle $ABC$ with altitudes $h_a$, $h_b$, $h_c$ and circumradius $R$. Prove that

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(R + \frac{h_a + h_b + h_c}{6}\right) > 3\ .$$

*Solved by AN-anduud Problem Solving Group; G. Apostolopoulos; Š. Arslanagić; M. Bataille; C. Curtis; O. Geupel; K. Lau; S. Malikić; P. Perfetti; I. Uchiha;*

*P. Y. Woo; T. Zvonaru; D. Văcaru; and the proposer. We present 2 solutions and an extension.*

*Solution 1 by Itachi Uchiha.*

Let $\triangle$ denote the area of the triangle. By the well-known formulae

$$\triangle = \frac{abc}{4R} = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c.$$

Now, using this fact, together with the AM-GM Inequality, we have

$$\begin{aligned}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(R+\frac{h_a+h_b+h_c}{6}\right) &= \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{abc}{4\triangle}+\frac{\triangle}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right)\\ &\geq 3\frac{1}{\sqrt[3]{abc}}\left(\frac{abc}{4\triangle}+\frac{\triangle}{\sqrt[3]{abc}}\right)\\ &\geq 3\frac{1}{\sqrt[3]{abc}}\sqrt[3]{abc}=3.\end{aligned}$$

Equality holds if and only if $a=b=c$ and $\dfrac{abc}{4\triangle} = \dfrac{\triangle}{\sqrt[3]{abc}}$, or in other words, when $4\triangle^2 = (abc)^{\frac{4}{3}}$ or $\triangle = \frac{1}{2}(abc)^{\frac{2}{3}} = \frac{1}{2}a^2$. However, when $a=b=c$, we have $\triangle = \frac{\sqrt{3}}{4}a^2$. Hence, the inequality is strict.

*Solution 2 by P. Y. Woo modified slightly by the editor.*

Since $c = 2R\sin C$, we have $\frac{R}{c} = \frac{1}{2\sin C}$ and, similarly, $\frac{R}{a} = \frac{1}{2\sin A}$ and $\frac{R}{b} = \frac{1}{2\sin B}$. Also, $h_a = c\sin B = 2R\sin B\sin C$ and, similarly, $h_b = 2R\sin C\sin A$ and $h_c = 2R\sin A\sin B$.

Let $k = (\sin A\sin B\sin C)^{\frac{1}{3}}$ and let $L$ denote the left side of the given inequality. Then using the AM-GM Inequality, we have

$$\begin{aligned}L &= \frac{1}{2}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\left(1+\frac{\sin B\sin C+\sin C\sin A+\sin A\sin B}{3}\right)\\ &\geq \frac{1}{2}\left(\frac{1}{k}\right)\left(1+k^2\right)\\ &= \frac{3}{2}\left(\frac{1}{k}+k\right)\geq 3.\end{aligned}$$

If equality holds, then $\frac{1}{k} = k$ implies $k=1$, so $\sin A = \sin B = \sin C = 1$, which is clearly impossible. Hence the inequality is strict.

*Extension by AN-anduud Problem Solving Group.*

Let $L$ be as in Solution 2 above. We prove the stronger result that $L \geq \frac{7\sqrt{3}}{4}$.

Since $ab = 2Rh_c$, $bc = 2Rh_a$, $ca = 2Rh_b$ and $a+b+c \leq 3\sqrt{3}R$, by applying the AM-GM Inequality several times, we then have

$$\begin{aligned}
L &= \frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(4R+\frac{2}{3}h_a+\frac{2}{3}h_b+\frac{2}{3}h_c\right)\\
&\geq \frac{1}{4}\cdot 3\cdot\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}\cdot 7\sqrt[7]{R^4\left(\frac{2}{3}h_a\right)\left(\frac{2}{3}h_b\right)\left(\frac{2}{3}h_c\right)}\\
&= \frac{7}{4}\cdot\frac{3}{\sqrt[3]{abc}}\cdot\sqrt[7]{\frac{R}{27}(abc)^2}\\
&\geq \frac{7}{4}\cdot\frac{3}{\sqrt[3]{abc}}\cdot\sqrt[7]{\frac{1}{27}\cdot\frac{a+b+c}{3\sqrt{3}}\cdot(abc)^2}\\
&\geq \frac{7}{4}\cdot\frac{3}{\sqrt[3]{abc}}\cdot\sqrt[7]{\frac{1}{27}\cdot\frac{1}{\sqrt{3}}(abc)^{\frac{7}{3}}}\\
&= \frac{7}{4}\sqrt{3},
\end{aligned}$$

and we are done. Note that this is in fact sharp, as equality holds for the equilateral triangle.

**3796**. [2012 : 421, 423] *Proposed by Michel Bataille.*

Show that

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{(2k)^{\binom{2n}{2k-1}}}{(2k+1)^{\binom{2n}{2k}}} = 1\ .$$

*Solved by A. Kotronis; O. Kouba; and the proposer. We present two solutions.*

*Solution 1 by Omran Kouba.*

We will use the following lemma:

**Lemma 1** *For $a > 0$,*

$$\int_0^\infty \frac{e^{-t}-e^{-at}}{t}dt = \ln a.$$

**Proof.** For $0 < \epsilon < X$, we have that

$$\begin{aligned}
\int_\epsilon^X \frac{e^{-t}-e^{-at}}{t}dt &= \int_\epsilon^X \frac{e^{-t}}{t}dt - \int_{a\epsilon}^{aX}\frac{e^{-u}}{u}du\\
&= \int_\epsilon^{a\epsilon}\frac{e^{-t}}{t}dt - \int_X^{aX}\frac{e^{-t}}{t}dt\\
&= \ln a - \int_\epsilon^\infty \frac{1-e^{-t}}{t}dt - \int_X^{aX}\frac{e^{-t}}{t}dt.
\end{aligned}$$

Since the integrals $\int_0^1(1-e^{-t})t^{-1}dt$ and $\int_1^\infty e^{-t}t^{-1}dt$ are convergent, we can let $\epsilon$ tend to 0 and $X$ tend to infinity to obtain the result. ■

Let $u_n$ denote the logarithm of the product in the statement of the problem. Then

$$\begin{aligned} u_n &= \sum_{k=0}^{2n} (-1)^{k-1} \binom{2n}{k} \ln(k+1) \\ &= \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \int_0^\infty \frac{e^{-(k+1)t} - e^{-t}}{t} dt \\ &= \int_0^\infty \frac{e^{-t}}{t} \left( \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} e^{-kt} - \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \right) dt \\ &= \int_0^\infty \frac{e^{-t}}{t} \left( (1-e^{-t})^{2n} - 0 \right) dt = \int_0^\infty \frac{(1-e^{-t})^{2n} e^{-t}}{t} dt. \end{aligned}$$

For $t > 0$, the integrand decreases to 0 with $n$, so that

$$\lim_{n\to\infty} u_n = \int_0^\infty \lim_{n\to 0} \left( \frac{(1-e^{-t})^{2n} e^{-t}}{t} \right) dt = 0,$$

from which we see that the desired limit is 1.

*Solution 2 by Michel Bataille.*

Define $u_n$ as in the first solution. We first show that $u_n = (2n)! I_{2n}$, where

$$I_n = \int_0^\infty \frac{dx}{(x+1)(x+2)\cdots(x+n+1)}$$

for $n \geq 2$. From the partial fraction decomposition

$$\frac{1}{(x+1)(x+2)\cdots(x+n+1)} = \sum_{k=0}^{n} (-1)^k \frac{1}{n!} \binom{n}{k} \frac{1}{x+k+1},$$

we have, for $X > 0$ and $n \geq 2$,

$$\begin{aligned} &n! \int_0^X \frac{dx}{(x+1)(x+2)\cdots(x+n+1)} \\ &= \left[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(x+k+1) \right]_0^X \\ &= (\ln X) \sum_{k=0}^{n} (-1)^k \binom{n}{k} + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln\left(1 + \frac{k+1}{X}\right) - \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(k+1). \end{aligned}$$

Since $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$ and $\lim_{X\to\infty} \ln(1 + (k+1)X^{-1}) = 0$, it follows that

$$(n!) I_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \ln(k+1).$$

It suffices to show that $\lim_{n\to\infty}(n!)I_n = 0$. We make use of the following inequality: *if $a_i$ and $b_i$ $(1 \le i \le m)$ are nonnegative, then*

$$(a_1+b_1)(a_2+b_2)\cdots(a_m+b_m) \ge ((a_1a_2\cdots a_m)^{1/m} + (b_1b_2\cdots b_m)^{1/m})^m.$$

(To see this, note that the $\binom{m}{j}$ terms on the left side with exactly $j$ factors $a_i$ and $m-j$ factors $b_i$ together involve each $a_i$ recurring $(j/m)\binom{m}{j}$ times and each $b_i$ recurring $(m-j)/m\binom{m}{j}$ times; by the arithmetic-geometric means inequality, their sum is not less than $\binom{m}{j}(a_1a_2\cdots a_m)^{j/m}(b_1b_2\cdots b_m)^{(m-j)/m}$.) Thus

$$(x+1)(x+2)\cdots(x+n+1) \ge (x+\sqrt[n+1]{(n+1)!})^{n+1},$$

so that

$$0 \le (n!)I_n \le n!\int_0^\infty \frac{dx}{\left(x+\sqrt[n+1]{(n+1)!}\right)^{n+1}} = \frac{\sqrt[n+1]{(n+1)!}}{n(n+1)}.$$

Since $\sqrt[n]{n} \sim n/e$ as $n \to \infty$, then the right side of this inequality tends to 0 and with it $(n!)I_n$. The desired result follows.

**3797**. [2012 : 421, 423] *Proposed by Panagiote Ligouras.*

Let $m_a$, $m_b$, $m_c$ be the medians and $k_a$, $k_b$, $k_c$ be the symmedians of a triangle $ABC$. If $n$ is a positive integer, prove that

$$\left(\frac{m_a}{k_a}\right)^n + \left(\frac{m_b}{k_b}\right)^n + \left(\frac{m_c}{k_c}\right)^n \ge 3\ .$$

*Solved by A. Alt; AN-Anduud Problem Solving Group; G. Apostolopoulos; Š. Arslanagić; M. Bataille; M. Amengual Covas; C. Curtis; O. Geupel; O. Kouba; S. Malikić; C. M. Quang; P. Y. Woo; T. Zvonaru; and the proposer. We present the solution that is a composite of nearly all received solutions.*

By standard formulas,

$$m_a^2 = \frac{2b^2+2c^2-a^2}{4} \text{ and } k_a^2 = \frac{b^2c^2\left(2b^2+2c^2-a^2\right)}{\left(b^2+c^2\right)^2},$$

so that

$$\frac{m_a}{k_a} = \frac{b^2+c^2}{2bc} \ge 1,$$

by the AM-GM inequality. Similarly, $\frac{m_b}{k_b} \ge 1$ and $\frac{m_c}{k_c} \ge 1$. Thus,

$$\sum_{\text{cyclic}}\left(\frac{m_a}{k_a}\right)^n \ge \sum_{\text{cyclic}}\frac{m_a}{k_a} \ge 3.$$

**3798**. [2012 : 421, 423] *Proposed by Albert Stadler.*

Let $n$ be a nonnegative integer. Prove that

$$\sum_{k=0}^{\infty} k^n \left(k+1-\frac{1}{k!}\int_1^{\infty} e^{-t}t^{k+1}dt\right) = \sum_{k=0}^{n} \frac{S(n,k)}{k+2},$$

where $k^n$ is taken to be 1 for $k = n = 0$ and $S(n,k)$ are the Stirling numbers of the second kind that are defined by the recursion

$$S(n,m) = S(n-1,m-1) + mS(n-1,m), S(n,0) = \delta_{0,n}, S(n,n) = 1\,.$$

*Editor's Comment.* Note, this is the corrected version of problem 3687.

*Solved by AN-Anduud Problem Solving Group; M. Bataille; O. Kouba; and the proposer. We present two solutions.*

*Solution 1 by Omran Kouba.*

We will use the following lemma:

**Lemma 1** *Let $f : (0,\infty) \longrightarrow \mathbb{R}$ be an $n-$times differentiable function, and let $g : \mathbb{R} \to \mathbb{R}$ be defined by the formula $g(t) = f(e^t)$. Then*

$$g^{(n)}(t) = \sum_{k=0}^{n} S(n,k) f^{(k)}(e^t)e^{kt}.$$

**Proof.** The result holds for $n = 0$ and $n = 1$. Assume that it holds for $n-1 \geq 1$ and consider an $n-$times differentiable function $f$. Since

$$g^{(n-1)}(t) = \sum_{k=0}^{n-1} S(n-1,k) f^{(k)}(e^t)e^{kt},$$

we obtain that

$$\begin{aligned} g^{(n)}(t) &= \sum_{k=0}^{n-1} S(n-1,k)(f^{(k+1)}(e^t)e^{(k+1)t} + kf^{(k)}(e^t)e^{kt}) \\ &= \sum_{k=1}^{n} S(n-1,k-1) f^{(k)}(e^t)e^{kt} + \sum_{k=0}^{n-1} kS(n-1,k)f^{(k)}(e^t)e^{kt} \\ &= \sum_{k=1}^{n} (S(n-1,k-1) + kS(n-1,k)) f^{(k)}(e^t)e^{kt} \\ &= \sum_{k=1}^{n} S(n,k) f^{(k)}(e^t)e^{kt}. \ \blacksquare \end{aligned}$$

To solve the problem let $f(x) = e^x$. From the lemma

$$(\exp(e^t))^{(n)} = \exp(e^t) \sum_{k=0}^{n} S(n,k) e^{kt}.$$

Since $\exp(e^t) = \sum_{k=0}^{\infty} e^{kt}/k!$, by taking the $n$th derivative, we obtain that

$$\exp(-e^t) \sum_{k=0}^{\infty} \frac{k^n e^{kt}}{k!} = \sum_{k=0}^{n} S(n,k) e^{kt}.$$

Set $t = \ln x$ to obtain for $x > 0$ that

$$e^{-x} \sum_{k=0}^{\infty} \frac{k^n x^k}{k!} = \sum_{k=0}^{n} S(n,k) x^k.$$

Since $m! = \int_0^\infty e^{-t} t^m dt$ for each nonnegative integer $m$, we see that

$$(k+1)! - \int_1^\infty e^{-t} t^{k+1} dt = \int_0^1 e^{-t} t^{k+1} dt.$$

Thus

$$\begin{aligned}
\sum_{k=0}^{\infty} k^n \left( k+1 - \frac{1}{k!} \int_1^\infty e^{-t} t^{k+1} dt \right) &= \sum_{k=0}^{\infty} \frac{k^n}{k!} \int_0^1 e^{-t} t^{k+1} dt \\
&= \int_0^1 t e^{-t} \left( \sum_{k=0}^{\infty} \frac{k^n t^k}{k!} \right) dt \\
&= \int_0^1 \left( \sum_{k=0}^{n} S(n,k) t^{k+1} \right) dt \\
&= \sum_{k=0}^{n} \frac{S(n,k)}{k+2}
\end{aligned}$$

as desired.

*Solution 2 by Michel Bataille.*

First, we show that for all nonnegative integers $k$, we have that

$$k + 1 - \frac{1}{k!} \int_1^\infty e^{-t} t^{k+1} dt = \frac{(k+1) R_{k+1}}{e}$$

where $R_m = \sum_{j \geq m+1} 1/j!$.

Let $I_k = \int_1^\infty e^{-t} t^k dt$. Then $I_0 = 1/e$. Integrating by parts, we find that $I_{k+1} = (1/e) + (k+1) I_k$, so that

$$\frac{I_{k+1}}{(k+1)!} - \frac{I_k}{k!} = \frac{1}{e} \cdot \frac{1}{(k+1)!}$$

for $k \geq 0$. Therefore

$$\frac{I_{k+1}}{(k+1)!} - I_0 = \sum_{j=0}^{k} \left( \frac{I_{j+1}}{(j+1)!} - \frac{I_j}{j!} \right) = \frac{1}{e} \sum_{j=0}^{k} \frac{1}{(j+1)!}.$$

Finally, we obtain that

$$\begin{aligned} k+1 - \frac{1}{k!} I_{k+1} &= (k+1)\left(1 - \frac{I_{k+1}}{(k+1)!}\right) \\ &= \frac{k+1}{e}\left(e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(k+1)!}\right)\right) = \frac{(k+1)R_{k+1}}{e}. \end{aligned}$$

Recall that the factorial powers are defined by $x^{(0)} = 1$ and $x^{(m)} = x(x-1)\cdots(x-m+1)$ for positive integer $m$ and that $x^n = \sum_{k=0}^{n} S(n,k) x^{(k)}$. The left side $L$ of the desired equality satisfies

$$L = \sum_{k=0}^{\infty} \frac{k^n (k+1) R_{k+1}}{e} = \frac{1}{e} \sum_{i=0}^{n} S(n,i) \sum_{k=0}^{\infty} (k+1)^{(i+1)} R_{k+1}.$$

Since the summands are nonnegative, we have that

$$\begin{aligned} \sum_{k=0}^{\infty} (k+1)^{(i+1)} R_{k+1} &= \sum_{k=i}^{\infty} \sum_{j=k+2}^{\infty} \frac{(k+1)^{(i+1)}}{j!} = \sum_{j=1}^{\infty} \frac{1}{(i+j+1)!} \sum_{l=1}^{j} (i+l)^{(i+1)} \\ &= \sum_{j=1}^{\infty} \frac{1}{(i+j+1)!} \sum_{l=1}^{j} l(l+1)\cdots(l+i) \\ &= \sum_{j=1}^{\infty} \frac{1}{(i+j+1)!} \left( \frac{(j+i+1)!}{(i+2)(j-1)!} \right) \\ &= \sum_{j=1}^{\infty} \frac{1}{(i+2)(j-1)!} = \frac{e}{i+2}. \end{aligned}$$

Therefore,

$$L = \frac{1}{e} \sum_{i=0}^{n} S(n,i) \frac{e}{i+2} = \sum_{k=0}^{n} \frac{S(n,k)}{k+2}.$$

**3799**. [2012 : 421, 423] *Proposed by Constantin Mateescu.*

Let $ABC$ be a triangle with circumradius $R$, inradius $r$ and semiperimeter $s$ for which we denote $K = \sum_{\text{cyclic}} \sin \frac{A}{2}$. Prove that

$$s^2 = 4R(K-1)^2 \left[ R(K+1)^2 + r \right].$$

*Solved by A. Alt; M. Bataille; S. Malikić; P. Y. Woo; T. Zvonaru; and the proposer. We present the solution by Michel Bataille.*

First, we recall some well-known formulas involving the elements of a triangle:

$$\begin{aligned}
&\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\\
&\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R},\\
&ab+bc+ca = s^2+r^2+4rR,\\
&\cos A+\cos B+\cos C = 1-\frac{r}{R}.
\end{aligned}$$

From the latter, we have

$$\sum_{\text{cyclic}} \sin^2\frac{A}{2} = \sum_{\text{cyclic}} \frac{1-\cos A}{2} = 1-\frac{r}{2R}. \qquad (1)$$

We will also prove the following formula

$$\sum_{\text{cyclic}} \sin^2\frac{A}{2}\sin^2\frac{B}{2} = \frac{s^2+r^2-8rR}{16R^2}. \qquad (2)$$

**Proof.** We have

$$\begin{aligned}
\sin^2\frac{A}{2}\sin^2\frac{B}{2} &= \frac{(s-b)(s-c)}{bc}\cdot\frac{(s-c)(s-a)}{ca}\\
&= \frac{s-c}{c}\cdot\frac{F^2}{s}\cdot\frac{1}{abc} = \frac{s-c}{c}\cdot r^2 s\cdot\frac{1}{4Rrs} = \frac{s-c}{c}\cdot\frac{r}{4R}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{\text{cyclic}} \sin^2\frac{A}{2}\sin^2\frac{B}{2} &= \frac{r}{4R}\sum_{\text{cyclic}}\frac{s-c}{c}\\
&= \frac{r}{4R}\left(\frac{s(s^2+r^2+4rR)}{4rsR}-3\right) = \frac{s^2+r^2-8rR}{16R^2}. \blacksquare
\end{aligned}$$

Turning back to the problem, we need to prove

$$4R^2(K^2-1)^2+4rR(K-1)^2 = s^2. \qquad (3)$$

Let $Q = \sum_{\text{cyclic}} \frac{1}{\sin\frac{A}{2}}$. Then, using (1), we have

$$K^2 = \sum_{\text{cyclic}} \sin^2\frac{A}{2} + 2\sum_{\text{cyclic}} \sin\frac{A}{2}\sin\frac{B}{2} = 1-\frac{r}{2R}+2\cdot\frac{r}{4R}\cdot Q = 1+\frac{r}{2R}(Q-1).$$

Therefore,

$$K^2-1 = \frac{r}{2R}(Q-1)$$

and

$$(K-1)^2 = 1 + \frac{r}{2R}(Q-1) - 2K + 1 = \frac{r}{2R}(Q-1) - 2(K-1).$$

As a result,

$$\begin{aligned} 4R^2(K^2-1)^2 + 4rR(K-1)^2 &= r^2(Q-1)^2 + 2r^2(Q-1) - 8rR(K-1) \\ &= (r^2Q^2 - 8rRK) - r^2 + 8rR \qquad\qquad (4). \end{aligned}$$

Now,

$$Q^2 = \sum_{\text{cyclic}} \frac{1}{\sin^2 \frac{A}{2}} + 2\sum_{\text{cyclic}} \frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}} = \left(\frac{4R}{r}\right)^2 \sum_{\text{cyclic}} \sin^2\frac{A}{2}\sin^2\frac{B}{2} + 2\cdot\frac{4R}{r}\cdot K.$$

Using (2), we have $r^2Q^2 = s^2 + r^2 - 8rR + 8rRK$. Back to (4), we finally get

$$4R^2(K^2-1)^2 + 4rR(K-1)^2 = s^2 + r^2 - 8rR - r^2 + 8rR$$

and (3) follows.

**3800**. [2012 : 422, 423] *Proposed by Ovidiu Furdui.*

Let $n \geq 2$ be an integer. Calculate

$$\int_0^\infty \int_0^\infty \left(\frac{e^{-x} - e^{-y}}{x-y}\right)^n dxdy \ .$$

*Solved by M. Bataille; O. Kouba; and the proposer. We present the solution by Omran Kouba.*

Let the considered integral be denoted by $I_n$. We will prove that

$$I_n = \frac{2}{n!}\sum_{k=2}^{n}(-1)^k\binom{n}{k}k^{n+1}\ln k.$$

The function $t \mapsto e^{-t}$ is decreasing, so $\frac{e^{-y}-e^{-x}}{x-y} > 0$ for every $x \neq y$. Consequently, the integrand in the following integral is positive, and we can make the change of variables $(x,y) \leftarrow (t+u, t-u)$, as follows:

$$
\begin{aligned}
(-1)^n I_n &= \int_0^\infty \int_0^\infty \left(\frac{e^{-y}-e^{-x}}{x-y}\right)^n dxdy \\
&= 2\int_{-\infty}^\infty \int_{|u|}^\infty \left(\frac{e^{-t+u}-e^{-t-u}}{2u}\right)^n dtdu \\
&= 2\int_{-\infty}^\infty \left(\frac{e^{u}-e^{-u}}{2u}\right)^n \left(\int_{|u|}^\infty e^{-nt}dt\right) du \\
&= \frac{4}{n}\int_0^\infty \left(\frac{e^{u}-e^{-u}}{2u}\right)^n e^{-nu}du \\
&= \frac{4}{n}\int_0^\infty \left(\frac{1-e^{-2u}}{2u}\right)^n du = \frac{2}{n}J_n, \qquad (1)
\end{aligned}
$$

where

$$
J_n = \int_0^\infty \left(\frac{1-e^{-v}}{v}\right)^n dv \qquad (2)
$$

Now, the integral $J_n$ was calculated in Solution 2 of Problem 3670 by Mohammed Aassila [see ***Crux Mathematicorum***, Vol 38, No 7 (2013), p.300-301]. It was proven that

$$
J_n = \frac{(-1)^n}{(n-1)!}\sum_{k=2}^{n}(-1)^k \binom{n}{k} k^{n-1}\ln k. \qquad (3)
$$

The announced result follows immediately from (1), (2), and (3).