OLYMPIAD SOLUTIONS

OC96. Let $a, b > 1$ be two relatively prime integers. We define $x_1 = a, x_2 = b$ and

$$x_n = \frac{{x_{n-1}}^2 + {x_{n-2}}^2}{x_{n-1} + x_{n-2}} \quad \forall n \geq 3\,.$$

Prove that x_n is not an integer for all $n \geq 3$.

Originally question 4 from Croatia, Team Selection Test 2011, Day 1.

There were no solutions received to this problem.

OC97. Let A be a set with 225 elements. Suppose that there are eleven subsets $A_1, \ldots, A_{11}$ of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \ldots \cup A_{11}| \geq 165$, and give an example for which equality holds.

Originally question 6 from 2011 USA Math Olympiad.

Solved by Oliver Geupel and we present his solution.

Without loss of generality we can assume that $A_1 \cup A_2 \cup \ldots \cup A_{11} = \{1, 2, \ldots, n\}$, where n is the cardinality of $A_1 \cup A_2 \cup \ldots \cup A_{11}$. We have to prove that $n \geq 165$.

For each $1 \leq i \leq n$ and $1 \leq j \leq 11$, we define

$$e_{ij} = \begin{cases} 0 & \text{if } i \notin A_j \\ 1 & \text{if } i \in A_j . \end{cases}$$

Then,

$$495 = 11 \cdot 45 = \sum_{j=1}^{11} |A_j| = \sum_{i=1}^{n} \sum_{j=1}^{11} e_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{11} e_{ij}^2 .$$

By the Arithmetic-Quadratic Means inequality, we have

$$\begin{aligned} \left(\sum_{i=1}^{n} \sum_{j=1}^{11} e_{ij} \right)^2 \leq n \sum_{i=1}^{n} \left(\sum_{j=1}^{11} e_{ij} \right)^2 &= n \left(\sum_{j=1}^{11} \sum_{i=1}^{n} e_{ij}^2 + 2 \sum_{1 \leq j < k \leq 11} \sum_{i=1}^{n} e_{ij} e_{ik} \right) \\ &= n \left(495 + 2 \sum_{1 \leq j < k \leq 11} |A_j \cap A_k| \right) \\ &= n \cdot \left(495 + 2 \binom{11}{2} \cdot 9 \right) = 1485n\,. \end{aligned}$$

We conclude $n \geq \dfrac{495^2}{1485} = 165$, which completes the proof.

It remains to give an example for which $n = 165$ holds. The set $\{1, 2, \ldots, 11\}$ has $\binom{11}{3} = 165$ subsets of cardinality 3. Let $B_1,\ B_2,\ \ldots,\ B_{165}$ be any list of these subsets. Define $A_1, \ldots, A_{11}$ by the condition

$$i \in A_j \Leftrightarrow j \in B_i, \qquad 1 \le i \le 165, \quad 1 \le j \le 11\,.$$

Every number $j \in \{1, 2, \ldots, 11\}$ is included in exactly $\binom{10}{2} = 45$ of the subsets B_i. Thus, $|A_j| = 45$. Moreover, for any fixed distinct values $j, k \in \{1, 2, \ldots, 11\}$, we have $i \in A_j \cap A_k$ if and only if $j, k \in B_i$, which is satisfied for nine subsets B_i. Therefore, $|A_j \cap A_k| = 9$. This completes our example.

OC98. Let ABC be a triangle with $\angle BAC = 60^\circ$. Let B_1 and C_1 be the feet of the bisectors from B and C. Let A_1 be the symmetrical of A with respect to the line B_1C_1. Prove that A_1, B and C are colinear.

Originally question 3 from Moldova's Team Selection Test 2011, Day 2.

Solved by Š. Arslanagić; M. Bataille; and T. Zvonaru and N. Stanciu. We give the solution of Šefket Arslanagić.

Let I denote the intersection of BB_1 and CC_1, that is I is the incenter of ABC. Let $\angle B = \beta, \angle C = \gamma$. We know that $\beta + \gamma = 120^\circ$. Then, we have

$$\angle B_1IC_1 = \angle BIC = 180^\circ - \frac{\beta}{2} - \frac{\gamma}{2} = 180^\circ - 60^\circ = 180^\circ - \angle A\,.$$

This implies that the quadrilateral AB_1IC_1 is cyclic, and hence

$$\angle AB_1C_1 = \angle AIC_1\,.$$

As AIC_1 is an exterior angle for the triangle AIC, it follows that

$$\angle AB_1C_1 = \angle AIC_1 = \angle IAC + \angle ICA = 30^\circ + \frac{\gamma}{2}\,.$$

As A_1 is the symmetric of A with respect to B_1C_1, we have $\Delta AB_1C_1 \equiv \Delta A_1B_1C_1$ and hence

$$\angle A_1B_1C_1 = \angle AB_1C_1 = 30^\circ + \frac{\gamma}{2},$$
$$\angle B_1A_1C_1 = \angle B_1AC = 60^\circ.$$

Since AB_1IC_1 is cyclic, we also get

$$\angle IB_1C_1 = \angle IAC_1 = 30^\circ,$$
$$\angle IC_1B_1 = \angle IAB_1 = 30^\circ.$$

As $\angle CB_1A_1 + \angle A_1B_1A = 180°$, it follows that

$$\angle C_1B_1A = 180° - \angle AB_1C_1 - \angle A_1B_1C_1 = 180° - 2(30° + \frac{\gamma}{2}) = \beta\,.$$

By the bisector theorem we have

$$\frac{BA}{BC} = \frac{AB_1}{B_1C} = \frac{A_1B_1}{B_1C}\,.$$

Therefore, as

$$\frac{BA}{BC} = \frac{A_1B_1}{B_1C} \quad \text{and} \quad \angle ABC = \angle A_1B_1C,$$

the triangles ΔBAC and ΔB_1A_1C are similar. This implies that

$$\angle B_1A_1C = \angle BAC = 60°\,.$$

Similarly, we can show that $\angle BA_1C_1 = 60°$. Therefore,

$$\angle BA_1C = \angle BA_1C_1 + \angle C_1A_1B_1 + \angle B_1A_1C = 60° + 60° + 60° = 180°\,.$$

This proves that B, A_1, C are colinear.

OC99. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ so that, for all $x \in \mathbb{Q}^+$ we have

$$f\left(\frac{x}{x+1}\right) = \frac{f(x)}{x+1} \text{ and } f\left(\frac{1}{x}\right) = \frac{f(x)}{x^3}\,.$$

Originally question 1 from Turkey Team Selection Test 2011.

Solved by Michel Bataille and we give his solution.

We show that the solutions are the functions $f_a : \mathbb{Q}^+ \to \mathbb{Q}^+$ (where $a \in \mathbb{Q}^+$) defined by

$$f_a(x) = a \cdot \frac{n^2}{d} \text{ if } x = \frac{n}{d} \text{ with } n, d \in \mathbb{N},\ \gcd(n,d) = 1\,.$$

Such a function f_a is a solution. Indeed, the equation $f_a\left(\frac{1}{x}\right) = \frac{f_a(x)}{x^3}$ is quickly verified and noticing that $\gcd(n, n+d) = 1$ if $\gcd(n,d) = 1$, we have

$$f_a\left(\frac{x}{x+1}\right) = f_a\left(\frac{n}{n+d}\right) = a \cdot \frac{n^2}{n+d} = \frac{a \cdot (n^2/d)}{1 + \frac{n}{d}} = \frac{f_a(x)}{x+1}\,.$$

whenever $x = \frac{n}{d}$ with $n, d \in \mathbb{N},\ \gcd(n,d) = 1$.

Conversely, let f be an arbitrary solution and let $a = f(1)$. We show that $f\left(\frac{n}{d}\right) = f_a\left(\frac{n}{d}\right)$ for all $n, d \in \mathbb{N}$, $\gcd(n, d) = 1$ by induction on the positive integer $\max(n, d)$.

If $\max(n, d) = 1$, then $n = d = 1$ and $f(1) = a = a \cdot \frac{1^2}{1} = f_a(1)$.

Let $k \geq 2$ and assume that $f\left(\frac{\nu}{\delta}\right) = f_a\left(\frac{\nu}{\delta}\right)$ whever $\nu, \delta \in \mathbb{N}$, $\gcd(\nu, \delta) = 1$ with $\max(\nu, \delta) < k$. Let $n, d \in \mathbb{N}$ $\gcd(n, d) = 1$ with $\max(n, d) = k$. Note that $n = d$ cannot occur, so we distinguish the two cases $n < d$ and $d < n$. In the former case, since $\gcd(n, d - n) = 1$ and $\max(n, d - n) < d = k$, we may write

$$f\left(\frac{n}{d}\right) = f\left(\frac{n/(d-n)}{1 + (n/(d-n))}\right) = \frac{f(n/(d-n))}{d/(d-n)} = \frac{an^2/(d-n)}{d/(d-n)} = a \cdot \frac{n^2}{d}.$$

In the latter case, using the previous case for the calculation of $f(d/n)$,

$$f\left(\frac{n}{d}\right) = f\left(\frac{1}{d/n}\right) = \frac{f(d/n)}{d^3/n^3} = a \cdot \frac{d^2}{n} \cdot \frac{n^3}{d^3} = a \cdot \frac{n^2}{d}\,.$$

In any event, $f\left(\frac{n}{d}\right) = a \cdot \frac{n^2}{d} = f_a\left(\frac{n}{d}\right)$ and the induction step is complete.

OC100. Let a_n be the sequence defined by

$$a_0 = 1,\, a_1 = -1, \text{ and } a_n = 6a_{n-1} + 5a_{n-2}\, \forall n \geq 2\,.$$

Prove that $a_{2012} - 2010$ is divisible by 2011.

Originally question 2 from Vietnam National Olympiad 2011, Day 2.

No solution was received to this problem. We give a solution by the editor .

Consider the quadratic equation

$$X^2 - 6X - 5 \equiv 0 \pmod{2011}\,.$$

The discriminant of this equation is $\Delta = 56$ which is a quadratic residue modulo 2011. [This can be seen either by calculating the Legendre symbol, or more easely by seeing that $56 \equiv 56 + 4 * 2011 = 8100 \pmod{2011}$.]

Therefore, as $p = 2011$ is a prime, the equation has two roots $s \not\equiv t \pmod{2011}$. Moreover, it is clear that neither of them is $0 \pmod{2011}$. It follows that s, t are invertible modulo 2011.

As the characteristic equation of the recursion $\pmod{2011}$ has two roots, and we work over a field, it follows that there exists some integers C_1, C_2 so that

$$a_n \equiv C_1 s^n + C_2 t^n \pmod{2011}\,;\, \forall n\,.$$

By Fermat Little Theorem, we have

$$s^{2010} \equiv t^{2010} \equiv 1 \pmod{2011}$$

Therefore

$$\begin{aligned} a_{2012} &\equiv C_1 s^{2012} + C_2 t^{2012} \equiv C_1 s^{2010} s^2 + C_2 t^{2010} t^2 \\ &\equiv C_1 s^2 + C_2 t^2 \equiv a_2 \pmod{2011} . \end{aligned}$$

As $a_2 = -6 + 5 = -1$, the claim of the problem follows.